AdS/QCD and the Holographic Light-Front Representation

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$$\langle \zeta | H_{LF} | \phi \rangle = \mathcal{M}^2 \langle \zeta | \phi \rangle$$

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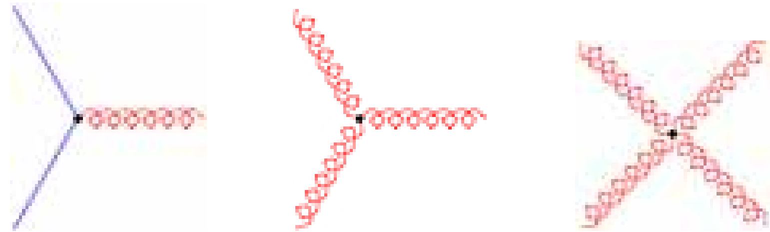
1 Introduction



- Quark-gluon dynamics described by the Yang-Mills $SU(3)$ color QCD Lagrangian $\mathcal{L}_{\rm QCD}$

$$S_{\rm QCD} = \int d^4x \, \mathcal{L}_{\rm QCD}(x),$$

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4g^2} {\rm Tr} \left(G^{\mu\nu} G_{\mu\nu}\right) + \sum_{f=1}^{n_f} i \bar{q}_f D_\mu \gamma^\mu q_f + \sum_{f=1}^{n_f} m_f \bar{q} q.$$

- Dimensionless coupling renormalizable theory with asymptotic freedom and color confinement.
- Most challenging problem of strong interaction dynamics: How the fundamental constituents in the QCD Lagrangian appear in the physical spectrum as colorless states, mesons and baryons.
- Recent developments using the AdS/CFT correspondence between string states in AdS space and conformal field theories in physical space-time have renewed the hope of finding an analytical approximation to describe the confining dynamics of QCD, at least in its strongly coupling regime.

The Original Holographic Correspondence

- Holographic duality requires a higher dimensional warped space. Space with negative curvature and a 4-dim boundary: AdS$_5$.
- Original correspondence between $\mathcal{N} = 4$ SYM at large N_C and the low energy supergravity approximation to Type IIB string on AdS$_5 \times S^5$ Maldacena, hep-th/9711200.

$$\begin{array}{ccc} \textbf{Warped higher dim space} & & \textbf{Conformal d = 4 spacetime boundary} \\ \text{Type IIB} \ \ (\text{AdS}_5 \times \text{S}^5) & \leftrightarrow & \mathcal{N} = 4\, \text{SYM}\, (\text{SO}(4,2) \otimes \text{SU}(4)) \\ ? & \leftrightarrow & \text{QCD} \end{array}$$

- The group of conformal transformations $SO(4, 2)$ is also the group of isometries of AdS$_5$, and S^5 corresponds to the global $SU(4) \sim SO(6)$ group which rotates the particles in the SYM multiplet.
- Description of strongly coupled gauge theory using a dual gravity description!
- QCD is fundamentally different from SYM theories where all matter is in the adjoint rep of $SU(N_C)$, and is non-conformal. Is there a dual string theory to QCD?

Strongly Coupled QCD and AdS/CFT

- Effective gravity description of strongly coupled quasi-conformal QCD.
- Semi-classical correspondence as a first approximation to QCD (strongly coupled at all scales).
- Strings describe extended objects (no quarks). QCD degrees of freedom are pointlike particles: how can they be related? How can we map string states into partons?
- Precise mapping of string amplitudes to light-front wavefuntions of hadrons in the light-front for strongly coupled QCD in the conformal limit.
- Eigenvalues of normalizable modes inside AdS give the hadronic spectrum. AdS modes represent also the probability amplitude for distribution of quarks at a given scale.
- To each state of the gauge theory should correspond a normalized mode in AdS. The lowest stable mode should correspond to the lowest state of the QCD Hamiltonian.
- Non-normalizable modes are related to external currents: they probe the cavity interior. Also couple to boundary QCD interpolating operators.

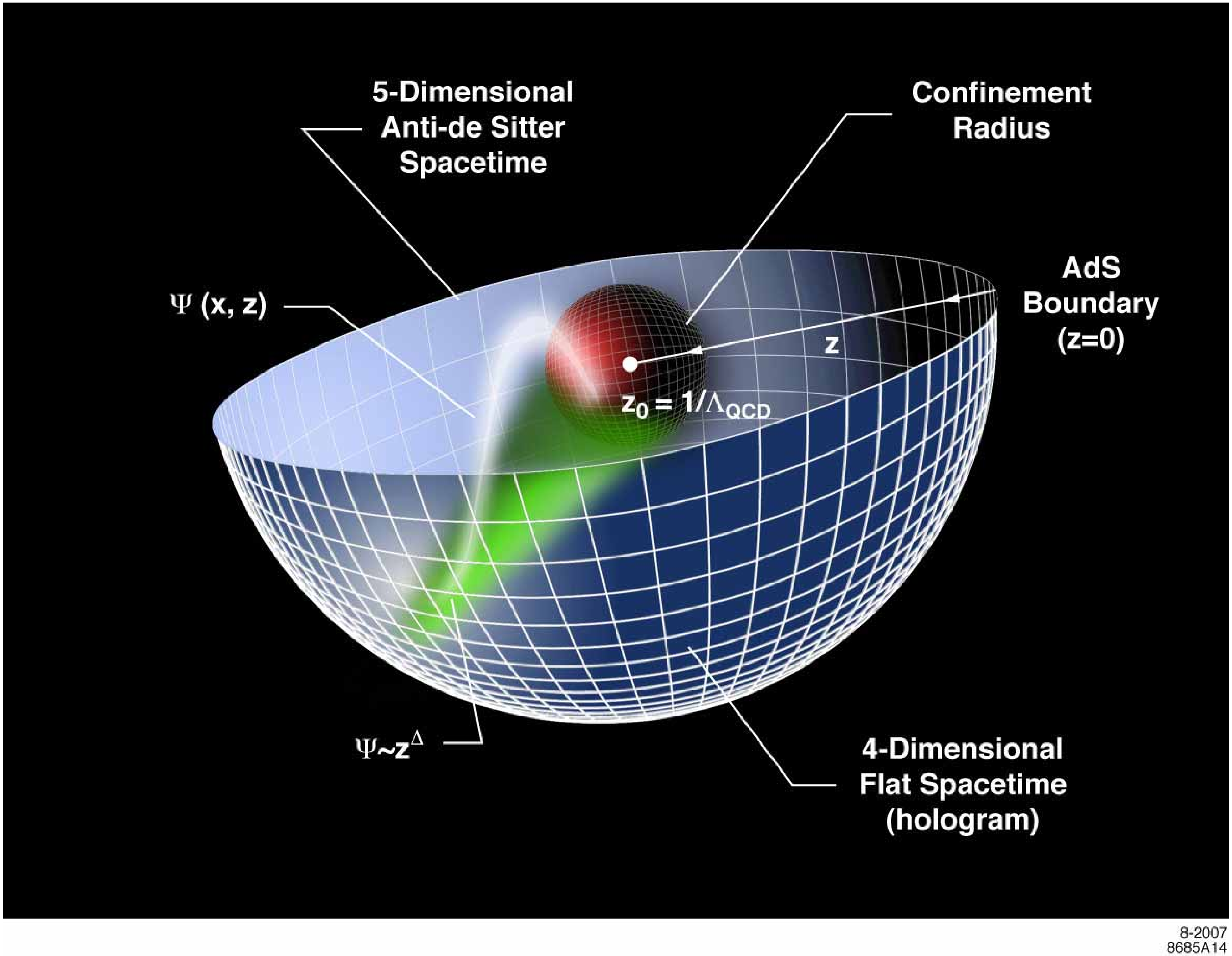
- Isomorphism of $SO(4,2)$ of conformal QCD with the group of isometries of AdS space

$$ds^2 = \frac{R^2}{z^2}(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2).$$

- $x^\mu \to \lambda x^\mu, \ z \to \lambda z,$ maps scale transformations into the holographic coordinate z.
- A distance $L_{\rm AdS}$ shrinks by a warp factor as observed in Minkowski space $(dz = 0)$:

$$L_{Minkowski} \sim \frac{z}{R} \, L_{\rm AdS}.$$

- Different values of z correspond to different scales at which the hadron is examined: AdS boundary at $z \to 0$ correspond to the $Q \to \infty$, UV zero separation limit.
- There is a maximum separation of quarks and a maximum value of z at the IR boundary
- Truncated AdS/CFT model: cut-off at $z_0 = 1/\Lambda_{\rm QCD}$ breaks conformal invariance and allows the introduction of the QCD scale (Hard-Wall Model) Polchinski and Strassler (2001).
- Smooth cutoff: introduction of a background dilaton field $\varphi(z)$ usual Regge dependence can be obtained (Soft-Wall Model) Karch, Katz, Son and Stephanov (2006).


5-Dimensional
Anti-de Sitter
Spacetime
Confinement
Radius
AdS
Boundary
(z=0)
Ψ (x, z)
z
$z_0 = 1/\Lambda_{QCD}$
Ψ~z^Δ
4-Dimensional
Flat Spacetime
(hologram)
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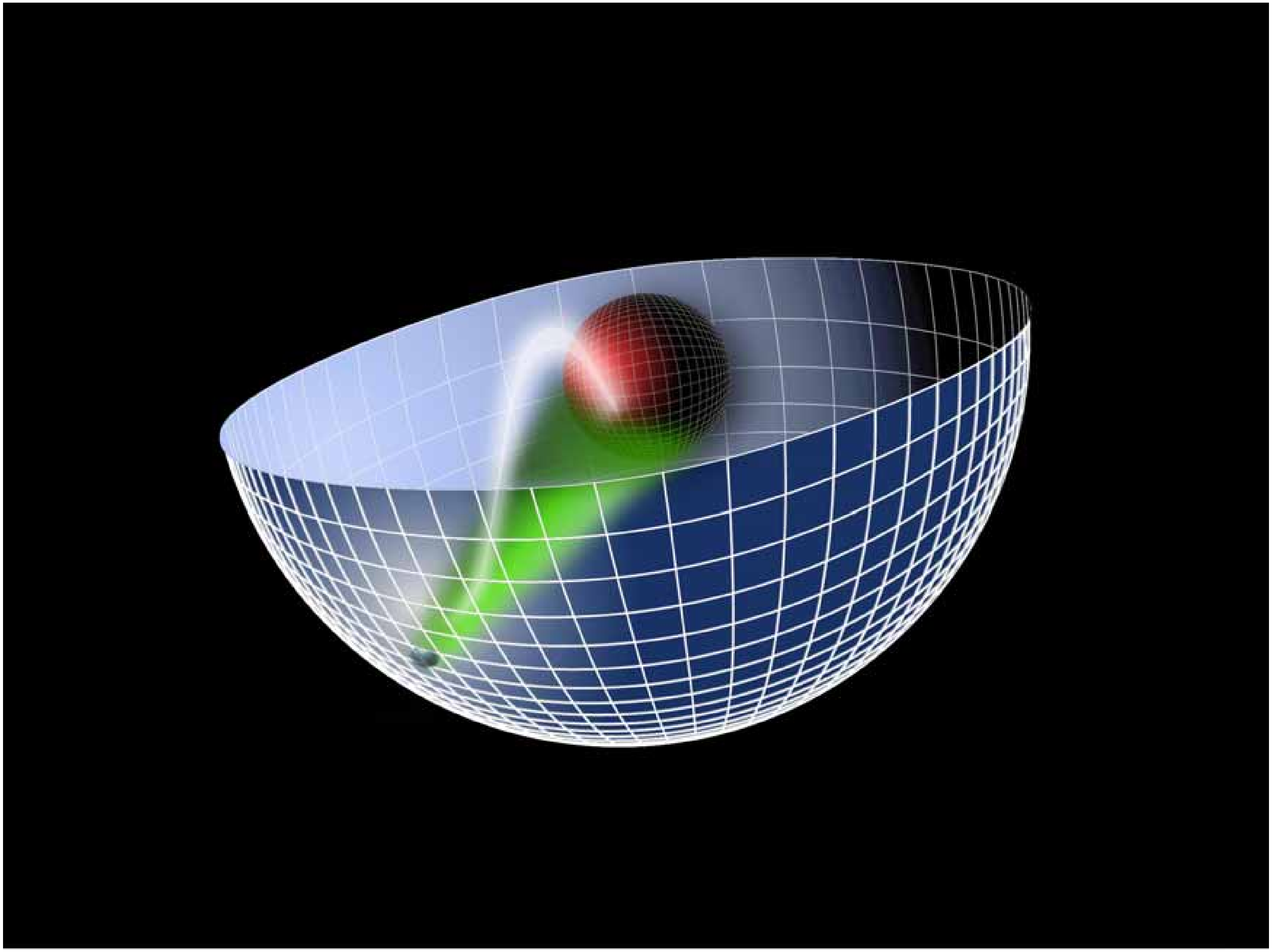
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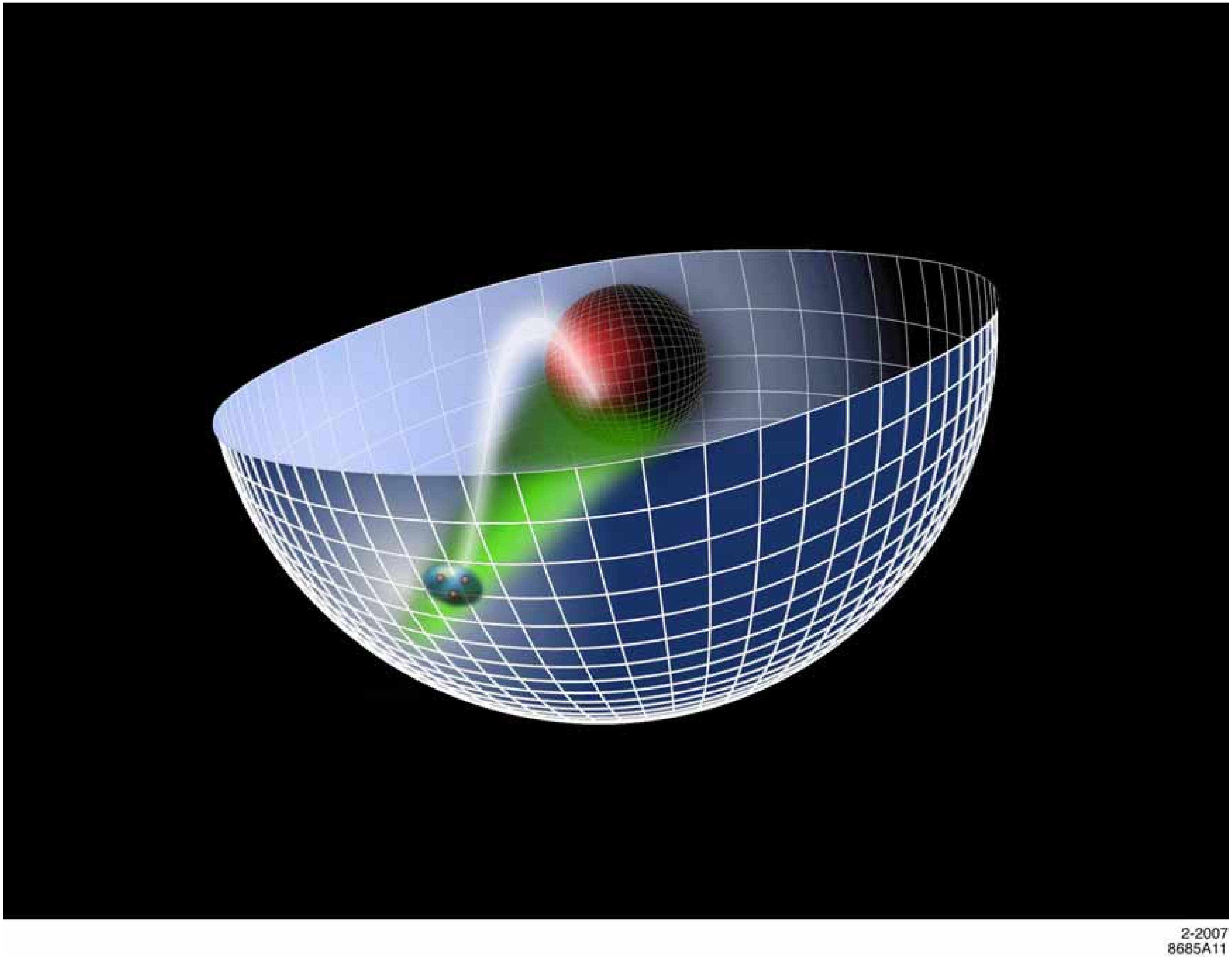
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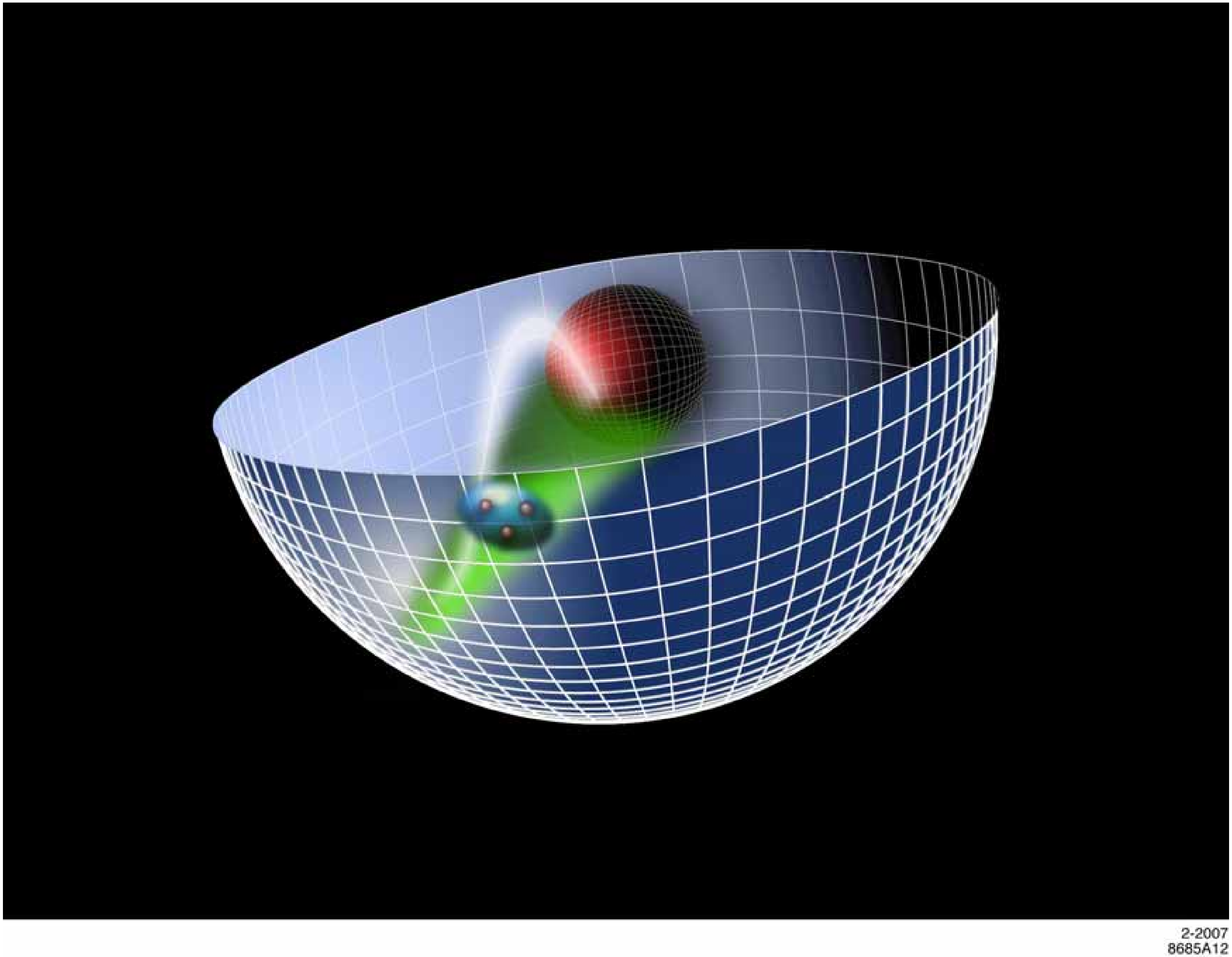
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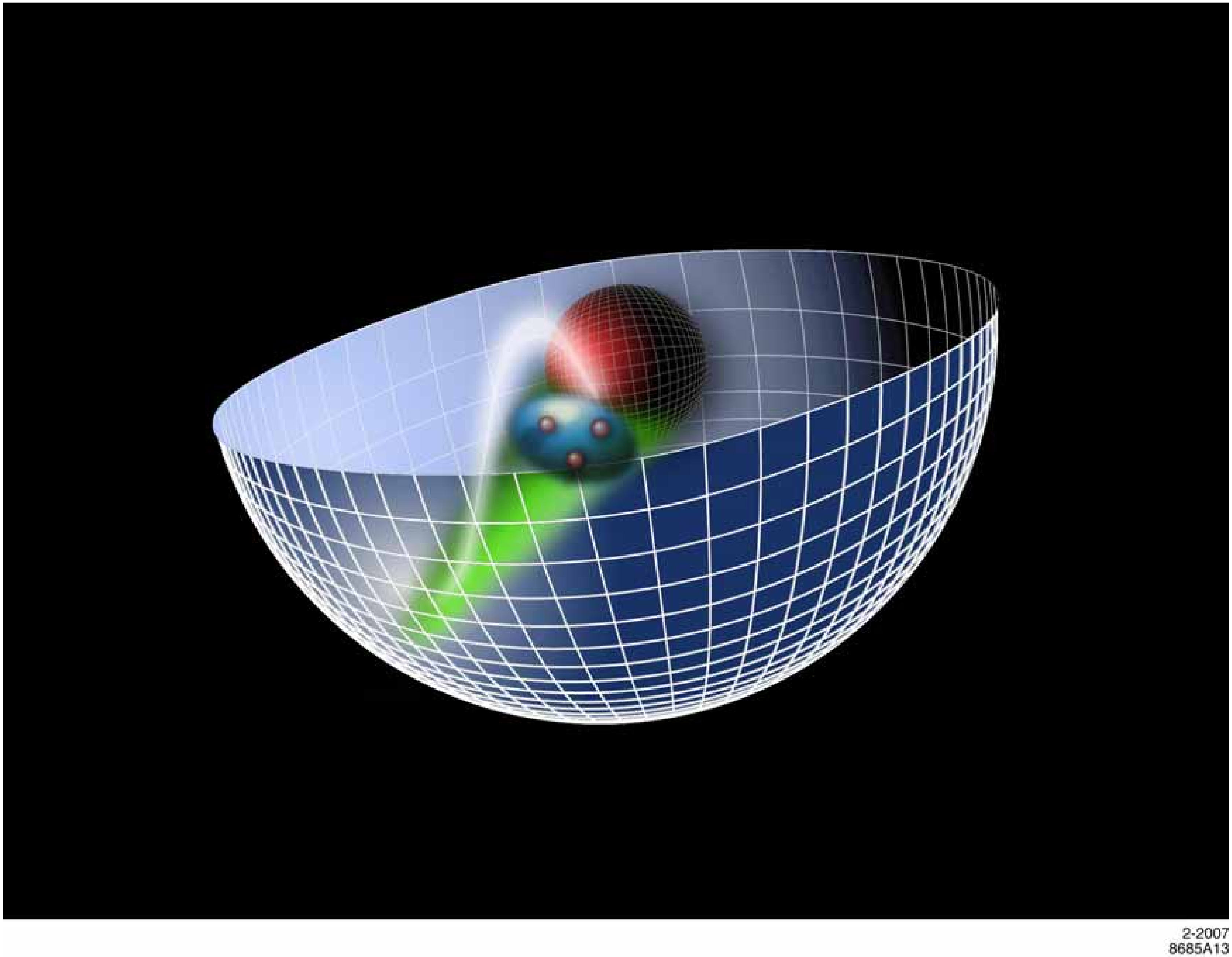
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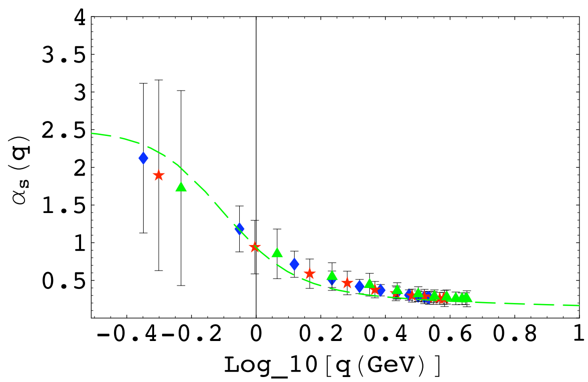
Conformal QCD Window in Exclusive Processes

- In the semi-classical approximation to QCD with massless quarks and no quantum loops the β function is zero and the approximate theory is scale and conformal invariant.
- Does α_s develop an IR fixed point? D-S Equation Alkofer, Fischer, LLanes-Estrada, Deur ...
- Recent lattice simulations: evidence that α_s becomes constant and not small in the infrared Furui and Nakajima, hep-lat/0612009 (Green dashed curve: DSE).

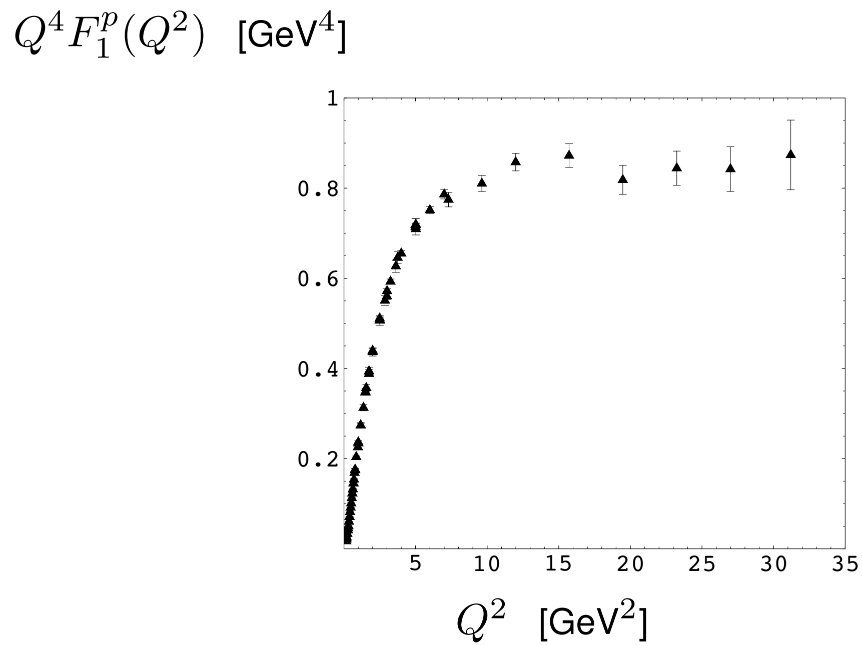


- Phenomenological success of dimensional scaling laws for exclusive processes

$$d\sigma/dt \sim 1/s^{n-2}, \quad n = n_A + n_B + n_C + n_D,$$

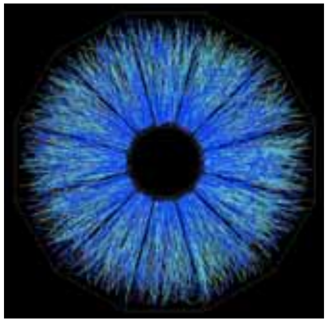
implies QCD is a strongly coupled conformal theory at moderate but not asymptotic energies Brodsky and Farrar (1973); Matveev *et al.* (1973).

- Derivation of counting rules for gauge theories with mass gap dual to string theories in warped space (hard behavior instead of soft behavior characteristic of strings) Polchinski and Strassler (2001).
- Example: Dirac proton form factor: $F_1(Q^2) \sim \left[1/Q^2\right]^{n-1}, \; n = 3$



From: M. Diehl *et al.* Eur. Phys. J. C **39**, 1 (2005).

Finite Temperature Gauge Theory and AdS/CFT



- Perfect quark-gluon liquid observed at RHIC: strongly coupled state.
- Conjectured lower bound:

$$\frac{\eta}{S} \geq \frac{1}{4\pi}$$

- Quantum mechanical result derived from the AdS/CFT correspondence for all relativistic quantum field theories at finite temperature and zero chemical potential.
- Hydrodynamic gauge theory properties of AdS/CFT:

  Policastro, Son and Starinets (2001); Kovtun, Son and Starinets (2005).

2 The Holographic Correspondence and Interpolating Operators

Precise statement of duality between a gravity theory in AdS$_{d+1}$ and the strong coupling limit of a conformal field theory at the $z = 0$ boundary Gubser, Klebanov and Polyakov (1998); Witten (1998) :

- $d + 1$-dim gravity partition function for scalar field in AdS$_{d+1}$: $\Phi(x, z)$

$$Z_{grav}[\Phi(x,z)] = e^{iS_{eff}[\Phi]} = \int \mathcal{D}[\Phi] e^{iS[\Phi]} \,.$$

- d-dim generating functional in presence of external source Φ_0

$$Z_{\rm QCD}[\Phi_0(x)] = e^{iW_{\rm QCD}[\Phi_0]} = \int \mathcal{D}[\psi, \overline{\psi}, A] \exp\left\{ iS_{\rm QCD} + i \int d^d x \Phi_0 \mathcal{O} \right\}.$$

  with $\mathcal{O}$ a hadronic local interpolating operator ($\mathcal{O} = G^a_{\mu\nu} G^{a\mu\nu}, \overline{q}\gamma_5 q, \cdots$)

- Boundary condition:

$$Z_{grav}\left[\Phi(x, z = 0) = \Phi_0(x)\right] = Z_{\rm QCD}\left[\Phi_0\right].$$

- Semi-Classical Approximation

$$W_{\rm QCD}\left[\phi_0\right] = S_{eff}\left[\Phi(x,z)|_{z=0} = \Phi_0(x)\right]_{\rm on-shell}.$$

- Near the boundary of AdS_{d+1} space $z \to 0$:

$$\Phi(x,z) \to z^\Delta \Phi_+(x) + z^{d-\Delta}\Phi_-(x).$$

- $\Phi_-(x)$ is the boundary limit of non-normalizable mode (source): $\Phi_- = \Phi_0$
- $\Phi_+(x)$ is the boundary limit of the normalizable mode (physical states)
- Using the equations of motion AdS action reduces to a UV surface term

$$S_{eff} = \frac{R^{d-1}}{4} \lim_{z \to 0} \int d^d x \, \frac{1}{z^{d-1}} \, \Phi \partial_z \Phi,$$

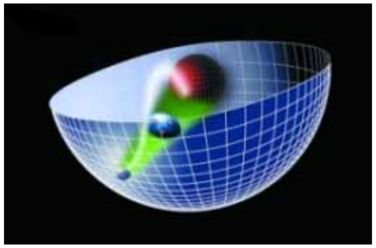
- S_{eff} is identified with the boundary functional W_{CFT}

$$\langle \mathcal{O} \rangle_{\Phi_0} = \frac{\delta W_{CFT}}{\delta \Phi_0} = \frac{\delta S_{\rm eff}}{\delta \Phi_0} \sim \Phi_+(x),$$

Balasubramanian *et. al.* (1998), Klebanov and Witten (1999).

- Physical AdS modes $\Phi_P(x,z) \sim e^{-iP \cdot x} \, \Phi(z)$ are plane waves along the Poincaré coordinates with four-momentum P^μ and hadronic invariant mass states $P_\mu P^\mu = \mathcal{M}^2$.
- For small-z $\Phi(z) \sim z^\Delta$. The scaling dimension Δ of a normalizable string mode, is the same dimension of the interpolating operator $\mathcal{O}$ which creates a hadron out of the vacuum: $\langle P | \mathcal{O} | 0 \rangle \neq 0$.

3 Bosonic Modes



- Conformal metric $x^\ell = (x^\mu, z)$:

$$\begin{aligned} ds^2 &= g_{\ell m} dx^\ell dx^m \\ &= \frac{R^2}{z^2}(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2). \end{aligned}$$

- Action for massive scalar modes on AdS$_{d+1}$:

$$S[\Phi] = \frac{1}{2}\int d^{d+1}x \sqrt{g}\, \tfrac{1}{2}\left[g^{\ell m}\partial_\ell \Phi \partial_m \Phi - \mu^2 \Phi^2\right],$$

with $\sqrt{g} \to (R/z)^{d+1}$ in the conformal limit.

- Equation of motion:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\ell}\left(\sqrt{g}\, g^{\ell m}\frac{\partial}{\partial x^m}\Phi\right) + \mu^2 \Phi = 0.$$

- Factor out dependence of string mode $\Phi_P(x,z)$ along x^μ-coordinates

$$\Phi_P(x,z) = e^{-iP\cdot x}\,\Phi(z),$$

with $P_\mu P^\mu = \mathcal{M}^2$.

- Find AdS equation of motion

$$\left[z^2\partial_z^2 - (d-1)z\,\partial_z + z^2\mathcal{M}^2 - (\mu R)^2\right]\Phi(z) = 0.$$

- Solution:

$$\Phi(x,z) = C z^{\frac{d}{2}} J_{\Delta-\frac{d}{2}}\left(z\mathcal{M}\right),$$

- Conformal dimension $\Phi(z) \to z^\Delta$ as $z \to 0$,

$$\Delta = \tfrac{1}{2}\left(d + \sqrt{d^2 + 4\mu^2 R^2}\,\right).$$

- Normalization

$$R^{d-1}\int_0^{\Lambda_{\rm QCD}^{-1}} \frac{dz}{z^{d-1}}\,\Phi_{S=0}^2(z) = 1.$$

Holographic Light-Front Representation (Hard Wall Model)

- We can represent the EOM in AdS space in light-front Lorentz invariant Hamiltonian form in physical 3+1 space-time at fixed LC time $\tau = t + z/c$

$$H_{LF}|\phi\rangle = \mathcal{M}^2|\phi\rangle.$$

- Write the AdS metric in terms of light front coordinates $x^{\pm} = x^0 \pm x^3$

$$ds^2 = \frac{R^2}{z^2}\left(dx^+dx^- - d\mathbf{x}_\perp^2 - dz^2\right).$$

- The AdS metric ds^2 is invariant if $\mathbf{x}_\perp^2 \to \lambda^2 \mathbf{x}_\perp^2$ and $z \to \lambda z$ at equal light-front time τ.

  Small z related to small transverse dimensions !

- We can identify the light-front variable ζ in 3+1 space with the fifth dimension z of AdS space, $\zeta = z$.

  The LF variable ζ represents the invariant transverse separation between pointlike constituents.

  SJB and GdT, Phys. Rev. Lett. **96**, 201601 (2006) **(See: Section 6 of the Talk)**

- Substitute in AdS EOM: $\phi(\zeta) \sim \zeta^{-3/2}\Phi(\zeta), \quad (\mu R)^2 = -4 + \nu^2$

$$\left[-\frac{d^2}{d\zeta^2} - \frac{1-4\nu^2}{4\zeta^2}\right]\phi(\zeta) = \mathcal{M}^2\phi(\zeta).$$

- Transverse impact holographic representation with light-front wavefunctions $\phi(\zeta) = \langle\zeta|\phi\rangle$

$$\langle\zeta|H_{LF}|\phi\rangle = \mathcal{M}^2\langle\zeta|\phi\rangle,$$

with

$$\langle\zeta|H_{LF}|\phi\rangle = \left[-\frac{d^2}{d\zeta^2} - \frac{1-4\nu^2}{4\zeta^2}\right]\langle\zeta|\phi\rangle,$$

in the conformal limit.

- Holographic light-front wave functions $\phi(\zeta) = \langle\zeta|\phi\rangle$ are normalized by

$$\langle\phi|\phi\rangle = \int d\zeta\, |\langle\zeta|\phi\rangle|^2 = 1,$$

and represent the probability amplitude to find n-partons at transverse impact separation $\zeta = z$. Its eigenmodes determine the hadronic mass spectrum.

- AdS/CFT equation as an effective light-front Schrödinger equation: relativistic, covariant and analytically tractable.

Algebraic Structure , Integrability and Stability Conditions (HW Model)

- If $\nu^2 > 0$ the LF Hamiltonian, H_{LF}, is written as a bilinear form

$$H^{\nu}_{LF}(\zeta) = \Pi^{\dagger}_{\nu}(\zeta)\Pi_{\nu}(\zeta), \quad \nu^2 \geq 0,$$

  in terms of the operator

$$\Pi_{\nu}(\zeta) = -i\left(\frac{d}{d\zeta} - \frac{\nu + \frac{1}{2}}{\zeta}\right),$$

  and its adjoint

$$\Pi^{\dagger}_{\nu}(\zeta) = -i\left(\frac{d}{d\zeta} + \frac{\nu + \frac{1}{2}}{\zeta}\right),$$

  with commutation relations

$$\left[\Pi_{\nu}(\zeta), \Pi^{\dagger}_{\nu}(\zeta)\right] = \frac{2\nu + 1}{\zeta^2}.$$

- For $\nu^2 \geq 0$ the Hamiltonian is positive definite

$$\langle \phi \,| H^{\nu}_{LF} |\, \phi \rangle = \int d\zeta \, |\Pi_{\nu}\phi(z)|^2 \geq 0$$

  and thus $\mathcal{M}^2 \geq 0$.

- For $\nu^2 < 0$ the Hamiltonian cannot be written as a bilinear form, but

$$\langle \phi \,| H_{LF}^{\nu} |\, \phi \rangle \geq 2\nu^2 \int d\zeta \frac{|\phi|^2}{\zeta^2},$$

  and the Hamiltonian is not bounded from below (“Fall-to-the-center” problem in Q.M.)

- Critical value of the potential corresponds to $\nu = 0$ with potential

$$V_{crit}(\zeta) = \frac{1}{4\zeta^2}.$$

- The Q.M. stability conditions are equivalent to the Breitenlohner-Freedman stability conditions

$$(\mu R)^2 \geq -\frac{d^4}{4}.$$

  For $d = 4$, $(\mu R)^2 = -4 + \nu^2$, and thus $\nu = 0$ correspond to the lowest stable solution.

Ladder Construction of Orbital States

- Orbital excitations constructed by the L-th application of the raising operator $a_L^\dagger = -i\Pi_L$ on the ground state:

$$a^\dagger|L\rangle = c_L|L+1\rangle.$$

- In the light-front ζ-representation

$$\begin{aligned}\phi_L(\zeta) &= \langle\zeta|L\rangle = C_L\sqrt{\zeta}\,(-\zeta)^L\left(\frac{1}{\zeta}\frac{d}{d\zeta}\right)^L J_0(\zeta\mathcal{M})\\ &= C_L\sqrt{\zeta}J_L\left(\zeta\mathcal{M}\right).\end{aligned}$$

- The solutions ϕ_L are solutions of the light-front equation $(L=0,\pm1,\pm2,\cdots)$

$$\left[-\frac{d^2}{d\zeta^2}-\frac{1-L^2}{4\zeta^2}\right]\phi(\zeta)=\mathcal{M}^2\phi(\zeta), \qquad \nu = L$$

- Mode spectrum from boundary conditions $\phi\left(\zeta = 1/\Lambda_{\rm QCD}\right)=0$, thus $\mathcal{M}^2=\beta_{Lk}\Lambda_{\rm QCD}$.

- The effective wave equation in the two-dim transverse LF plane has the Casimir representation L^2 corresponding to the $SO(2)$ rotation group [The Casimir for $SO(N)\sim S^{N-1}$ is $L(L+N-2)$].

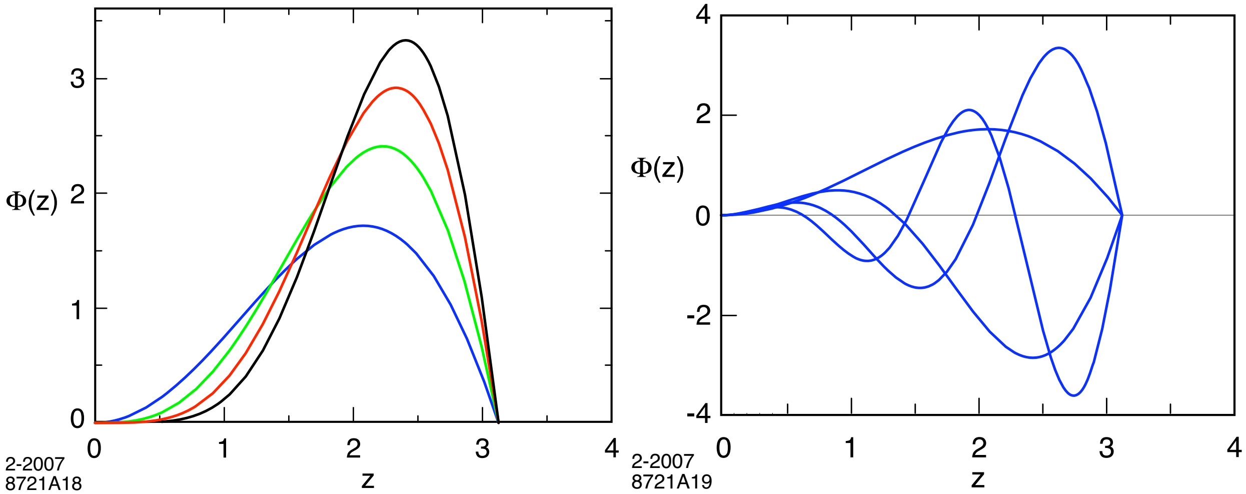


Fig: Orbital and radial AdS modes in the hard wall model for $\Lambda_{\rm QCD}$ = 0.32 GeV .

Higher Spin Bosonic Modes HW

- Each hadronic state of integer spin $S \le 2$ is dual to a normalizable string mode

$$\Phi(x,z)_{\mu_1\mu_2\cdots\mu_S} = \epsilon_{\mu_1\mu_2\cdots\mu_S}\, e^{-iP\cdot x}\, \Phi_S(z).$$

  with four-momentum P_μ and spin polarization indices along the 3+1 physical coordinates.

- Wave equation for spin S-mode W. S. I'Yi, Phys. Lett. B **448**, 218 (1999)

$$\left[z^2\partial_z^2 - (d+1-2S)z\,\partial_z + z^2\mathcal{M}^2 - (\mu R)^2\right]\Phi_S(z) = 0,$$

- Solution

$$\widetilde{\Phi}(z)_S = \left(\frac{z}{R}\right)^S \Phi(z)_S = Ce^{-iP\cdot x} z^{\frac{d}{2}} J_{\Delta-\frac{d}{2}}(z\mathcal{M})\,\epsilon(P)_{\mu_1\mu_2\cdots\mu_S},$$

- We can identify the conformal dimension:

$$\Delta = \frac{1}{2}\left(d + \sqrt{(d-2S)^2 + 4\mu^2R^2}\right).$$

- Normalization:

$$R^{d-2S-1}\int_0^{\Lambda_{\rm QCD}^{-1}} \frac{dz}{z^{d-2S-1}}\,\Phi_S^2(z) = 1.$$

- Upon the substitution $\phi(\zeta)_S \sim \zeta^{-3/2+S}\,\Phi(\zeta)_S$ in the spin-S AdS wave equation ($d=4$)

$$\left[-\frac{d^2}{d\zeta^2} - \frac{1-4\nu^2}{4\zeta^2}\right]\phi_S = \mathcal{M}^2\phi_S,$$

  where $(\mu R)^2 = -(2-S)^2 + \nu^2$.

- Solution

$$\phi(\zeta)_S = \epsilon_{\mu_1\mu_2\cdots\mu_S}\phi(\zeta),$$

  where the profile function $\phi(\zeta)$ is the solution for the scalar mode !

- Stable solutions satisfy a generalized B-F bound $\;(d=4)$

$$(\mu R)^2 \geq -\frac{(d-2S)^2}{4}.$$

- For the ground state $\Delta = 2$, independent of S. Higher excitations created by the ladder operators, thus $\nu = L$ and $\Delta = 2 + L$. Lowest stable solution corresponds to $L = 0$ for every spin mode S.

- AdS shifted field $\widetilde{\Phi}$ couples to the interpolating operator $\mathcal{O}^S$ with scaling dimensions

$$\left[\mathcal{O}^S\right] = d - \left[\widetilde{\Phi}_S\right] = 2 + L.$$

State	I	J^P	L	S	$\mathcal{O}$
$\pi(140)$	1	0^-	0	0	$\overline{q}\gamma_5 \frac{1}{2}\vec{\tau}q$
$b_1(1235)$	1	1^+	1	0	$-i\overline{q}\gamma_5 \vec{\partial}\frac{1}{2}\vec{\tau}q$
$\pi_2(1670)$	1	2^+	2	0	$-\overline{q}\gamma_5 \frac{1}{2}(3\partial_i\partial_j - \delta_{ij}\vec{\partial}^2)\frac{1}{2}\vec{\tau}q$
$\dots$					
$\rho(770)$	1	1^-	0	1	$q^\dagger\vec{\alpha}\frac{1}{2}\vec{\tau}q$
$\omega(782)$	0	1^-	0	1	$q^\dagger\vec{\alpha}\, q$
$a_1(1260)$	1	1^+	1	1	$-iq^\dagger(\vec{\alpha}\times\vec{\partial})\frac{1}{2}\tau q$
$f_2(1270)$	0	2^+	1	1	$-iq^\dagger[\frac{3}{2}(\alpha_i\partial_j + \alpha_j\partial_i) - \vec{\alpha}\cdot\vec{\partial}\delta_{ij}]q$
$f_1(1285)$	0	1^+	1	1	$-iq^\dagger(\vec{\alpha}\times\vec{\partial})q$
$a_2(1320)$	1	2^+	1	1	$-iq^\dagger[\frac{3}{2}(\alpha_i\partial_j + \alpha_j\partial_i) - \vec{\alpha}\cdot\vec{\partial}\delta_{ij}]\frac{1}{2}\vec{\tau}q$
$a_0(1450)$	1	0^+	1	1	$-iq^\dagger\vec{\alpha}\cdot\vec{\partial}\frac{1}{2}\vec{\tau}q$
$\dots$					

Tensor decomposition of total angular momentum interpolating operators $\mathcal{O}$, $[\mathcal{O}] = 2 + L$

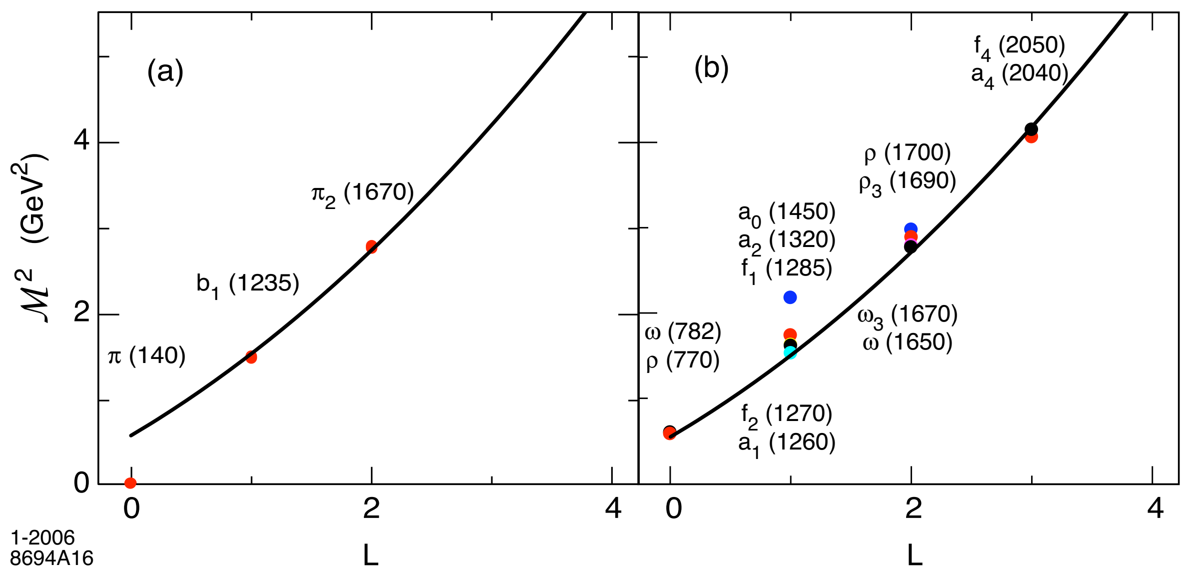


Fig: Light meson and vector meson orbital spectrum $\Lambda_{QCD} = 0.32 \text{ GeV}$

Non-Conformal Extension of Algebraic Integrability (SW Model)

- Soft-wall model [Karch, Katz, Son and Stephanov (2006)] retain conformal AdS metrics but introduce smooth cutoff wich depends on the profile of a dilaton background field $\varphi(z) = \kappa^2 z^2$.

$$S = \int d^4x\, dz\, \sqrt{g}\, e^{-\varphi(z)} \mathcal{L}. \quad (1)$$

- SW model can be constructed by extension of the conformal operator algebra. Consider the generator

$$\Pi_\nu(\zeta) = -i\left(\frac{d}{d\zeta} - \frac{\nu + \frac{1}{2}}{\zeta} - \kappa^2 \zeta\right),$$

and its adjoint

$$\Pi_\nu^\dagger(\zeta) = -i\left(\frac{d}{d\zeta} + \frac{\nu + \frac{1}{2}}{\zeta} + \kappa^2 \zeta\right),$$

with commutation relations

$$\left[\Pi_\nu(\zeta), \Pi_\nu^\dagger(\zeta)\right] = \frac{2\nu + 1}{\zeta^2} - 2\kappa^2.$$

- The LF Hamiltonian

$$H_{LF} = \Pi_\nu^\dagger \Pi_\nu + C$$

is positive definite $\langle\phi|H_{LF}|\phi\rangle \geq 0$ for $\nu^2 \geq 0$, and $C \geq -4\kappa^2$.

- Identify the zero mode ($C = -4\kappa^2$) with the pion.
- Orbital and radial excited states are constructed from the ladder operators from the $\nu = 0$ state.
- Light-front Hamiltonian equation

$$H_{LF}|\phi\rangle = \mathcal{M}^2|\phi\rangle,$$

leads to effective LF Schrödinger wave equation (KKSS)

$$\left[-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2} + \kappa^4\zeta^2 + 2\kappa^2(L-1)\right]\phi(\zeta) = \mathcal{M}^2\phi(\zeta)$$

with eigenvalues $\mathcal{M}^2 = 4\kappa^2(n+L)$ and eigenfunctions

$$\phi_L(\zeta) = \kappa^{1+L}\sqrt{\frac{2n!}{(n+L)!}}\zeta^{1/2+L}e^{-\kappa^2\zeta^2/2}L_n^L\left(\kappa^2\zeta^2\right).$$

- Transverse oscillator in the LF plane with $SO(2)$ rotation subgroup has Casimir L^2 representing rotations in the transverse LF plane.
- SW model is a remarkable example of integrability to a non-conformal extension of AdS/CFT (Chim and Zamolodchikov (1992) - Potts Model.)

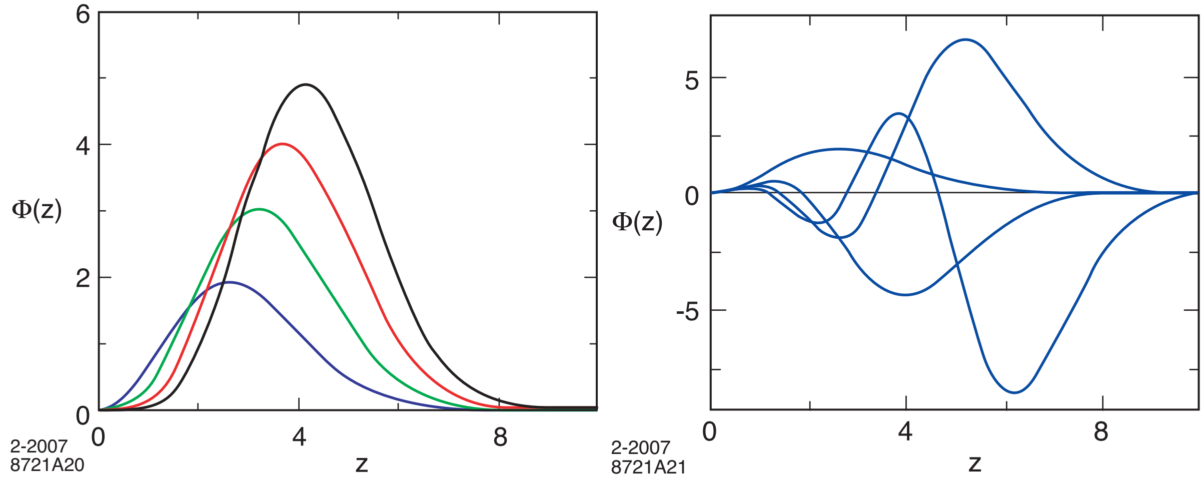
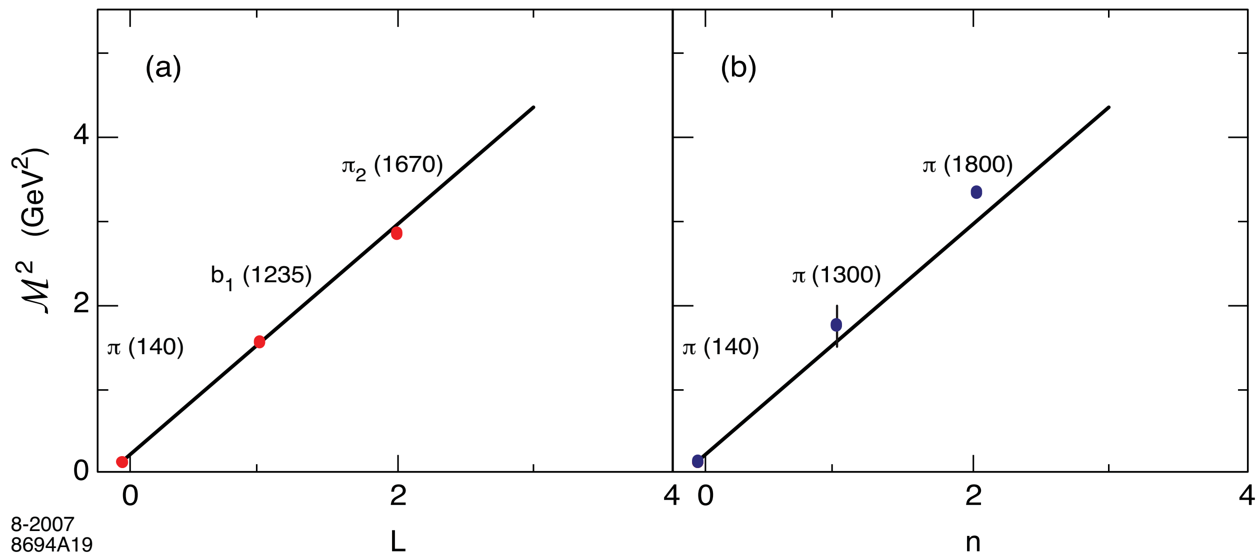


Fig: Orbital and radial AdS modes in the soft wall model for κ = 0.6 GeV .



Light meson orbital (a) and radial (b) spectrum for $\kappa = 0.6$ GeV.

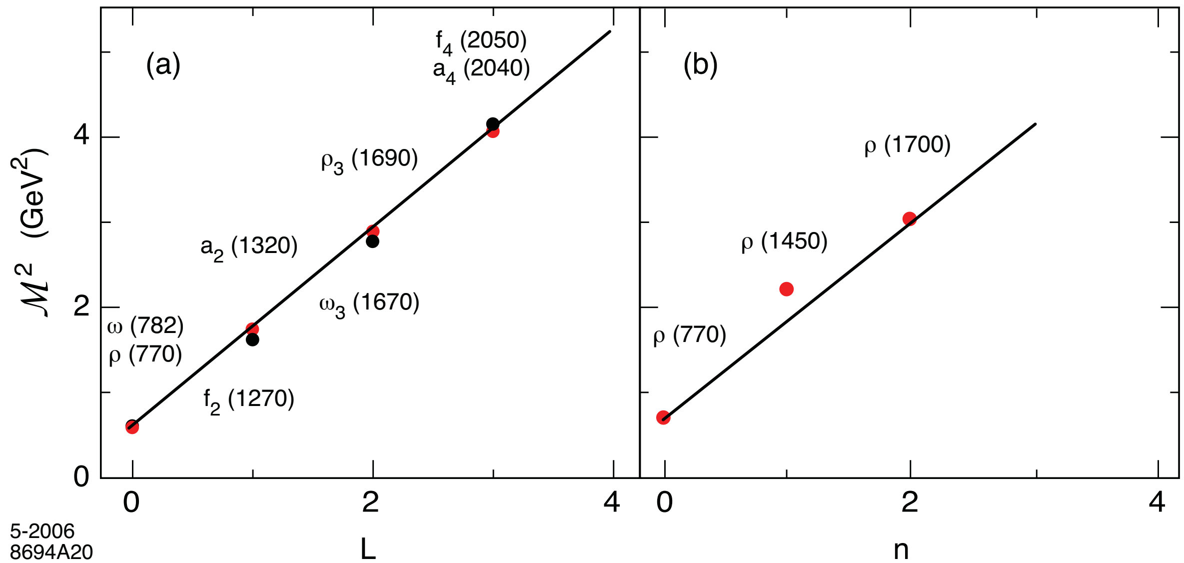
Higher Spin Bosonic Modes SW

- Effective LF Schrödinger wave equation

$$\left[-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2} + \kappa^4\zeta^2 + 2\kappa^2(L+S-1)\right]\phi_S(\zeta) = \mathcal{M}^2\phi_S(\zeta)$$

with eigenvalues $\mathcal{M}^2 = 2\kappa^2(2n + 2L + S)$.

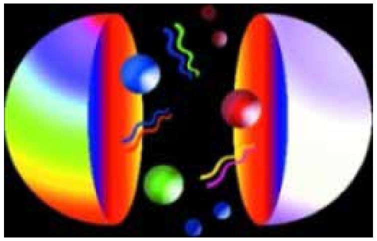
- Compare with Nambu string result (rotating flux tube): $M_n^2(L) = 2\pi\sigma\,(n + L + 1/2)$.



Vector mesons orbital (a) and radial (b) spectrum for $\kappa = 0.54$ GeV.

- Glueballs in the bottom-up approach: (HW) Boschi-Filho, Braga and Carrion (2005); (SW) Colangelo, De Facio, Jugeau and Nicotri(2007).

4 Fermionic Modes



From Nick Evans

- Baryons Spectrum in ”bottom-up” holographic QCD
  GdT and Brodsky: hep-th/0409074, hep-th/0501022.

- Conformal metric $x^\ell = (x^\mu, z)$:

$$\begin{aligned} ds^2 &= g_{\ell m} dx^\ell dx^m \\ &= \frac{R^2}{z^2}(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2). \end{aligned}$$

- Action for massive fermionic modes on AdS$_{d+1}$:

$$S[\overline{\Psi}, \Psi] = \int d^{d+1}x \, \sqrt{g} \, \overline{\Psi}(x,z) \left(i\Gamma^\ell D_\ell - \mu \right) \Psi(x,z).$$

- Equation of motion: $\left(i\Gamma^\ell D_\ell - \mu \right) \Psi(x,z) = 0$

$$\left[i \left(z\eta^{\ell m} \Gamma_\ell \partial_m + \frac{d}{2}\Gamma_z \right) + \mu R \right] \Psi(x^\ell) = 0.$$

Holographic Light-Front Integrable Form and Spectrum

- In the conformal limit fermionic spin-$\frac{1}{2}$ modes $\psi(z)$ and spin-$\frac{3}{2}$ modes $\psi_\mu(z)$ are solutions of the Dirac light-front equation

$$H_{LF}|\psi\rangle = \mathcal{M}|\psi\rangle,$$

with

$$H_{LF} = \alpha\,\Pi.$$

The operator

$$\Pi_\nu(\zeta) = -i\left(\frac{d}{d\zeta} - \frac{\nu + \frac{1}{2}}{\zeta}\gamma_5\right),$$

and its adjoint $\Pi^\dagger_\nu(\zeta)$ satisfy the commutation relations

$$\left[\Pi_\nu(\zeta), \Pi^\dagger_\nu(\zeta)\right] = \frac{2\nu + 1}{\zeta^2}\gamma_5.$$

- Supersymmetric QM between bosonic and fermionic modes in AdS?

- Note: in the Weyl representation ($i\alpha = \gamma_5 \beta$)

$$i\alpha = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

- Baryon: twist-dimension $3 + L \quad (\nu = L + 1)$

$$\mathcal{O}_{3+L} = \psi D_{\{\ell_1} \dots D_{\ell_q} \psi D_{\ell_{q+1}} \dots D_{\ell_m\}} \psi, \qquad L = \sum_{i=1}^{m} \ell_i.$$

- Solution to Dirac eigenvalue equation with UV matching boundary conditions

$$\psi(\zeta) = C\sqrt{\zeta} \left[J_{L+1}(\zeta \mathcal{M}) u_+ + J_{L+2}(\zeta \mathcal{M}) u_- \right].$$

  Baryonic modes propagating in AdS space have two components: orbital L and $L + 1$.

- Hadronic mass spectrum determined from IR boundary conditions

$$\psi_\pm \left(\zeta = 1/\Lambda_{\rm QCD} \right) = 0,$$

  given by

$$\mathcal{M}^+_{\nu,k} = \beta_{\nu,k} \Lambda_{\rm QCD}, \quad \mathcal{M}^-_{\nu,k} = \beta_{\nu+1,k} \Lambda_{\rm QCD},$$

  with a scale independent mass ratio.

$SU(6)$	S	L	Baryon State
56	$\frac{1}{2}$	0	$N\frac{1}{2}^{+}(939)$
	$\frac{3}{2}$	0	$\Delta\frac{3}{2}^{+}(1232)$
70	$\frac{1}{2}$	1	$N\frac{1}{2}^{-}(1535)$ $N\frac{3}{2}^{-}(1520)$
	$\frac{3}{2}$	1	$N\frac{1}{2}^{-}(1650)$ $N\frac{3}{2}^{-}(1700)$ $N\frac{5}{2}^{-}(1675)$
	$\frac{1}{2}$	1	$\Delta\frac{1}{2}^{-}(1620)$ $\Delta\frac{3}{2}^{-}(1700)$
56	$\frac{1}{2}$	2	$N\frac{3}{2}^{+}(1720)$ $N\frac{5}{2}^{+}(1680)$
	$\frac{3}{2}$	2	$\Delta\frac{1}{2}^{+}(1910)$ $\Delta\frac{3}{2}^{+}(1920)$ $\Delta\frac{5}{2}^{+}(1905)$ $\Delta\frac{7}{2}^{+}(1950)$
70	$\frac{1}{2}$	3	$N\frac{5}{2}^{-}$ $N\frac{7}{2}^{-}$
	$\frac{3}{2}$	3	$N\frac{3}{2}^{-}$ $N\frac{5}{2}^{-}$ $N\frac{7}{2}^{-}(2190)$ $N\frac{9}{2}^{-}(2250)$
	$\frac{1}{2}$	3	$\Delta\frac{5}{2}^{-}(1930)$ $\Delta\frac{7}{2}^{-}$
56	$\frac{1}{2}$	4	$N\frac{7}{2}^{+}$ $N\frac{9}{2}^{+}(2220)$
	$\frac{3}{2}$	4	$\Delta\frac{5}{2}^{+}$ $\Delta\frac{7}{2}^{+}$ $\Delta\frac{9}{2}^{+}$ $\Delta\frac{11}{2}^{+}(2420)$
70	$\frac{1}{2}$	5	$N\frac{9}{2}^{-}$ $N\frac{11}{2}^{-}(2600)$
	$\frac{3}{2}$	5	$N\frac{7}{2}^{-}$ $N\frac{9}{2}^{-}$ $N\frac{11}{2}^{-}$ $N\frac{13}{2}^{-}$

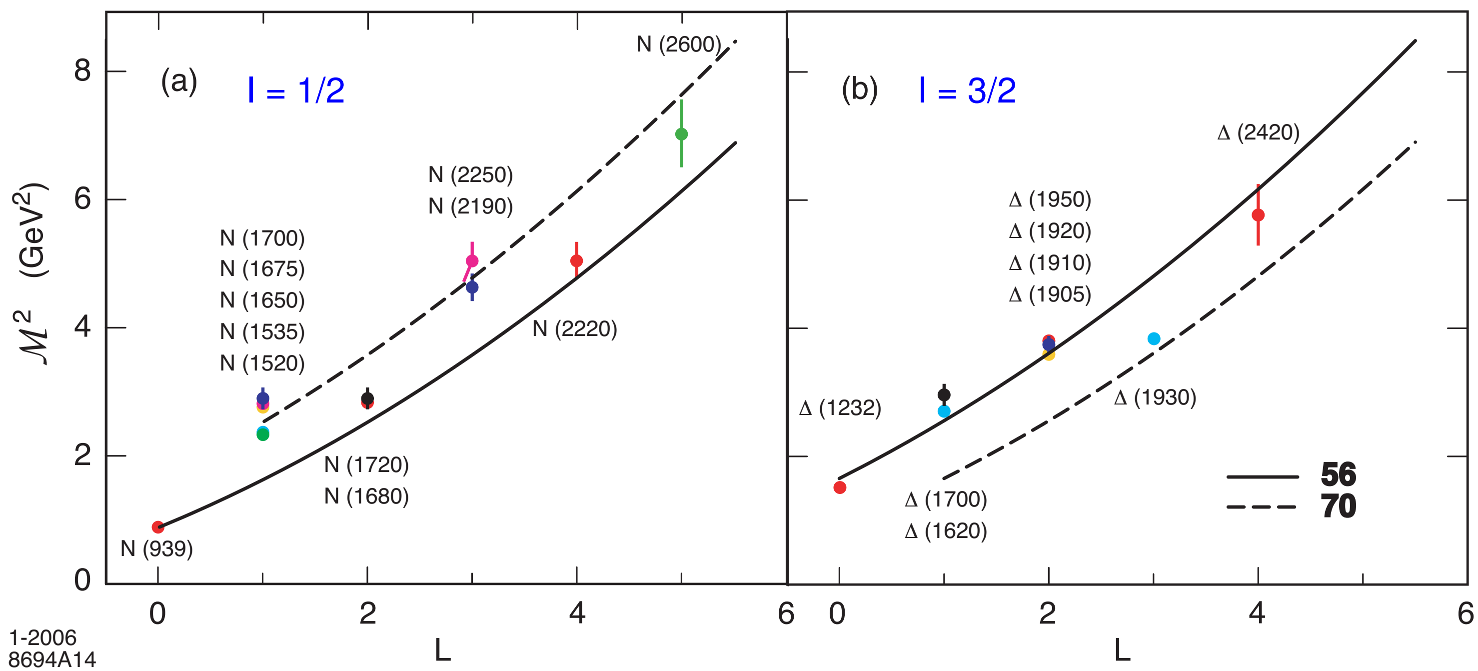


Fig: Light baryon orbital spectrum for Λ_{QCD} = 0.25 GeV in the HW model. The $\mathbf{56}$ trajectory corresponds to L even $P = +$ states, and the $\mathbf{70}$ to L odd $P = -$ states.

- New analysis: $\Delta(1930)$ as a $L = 1, N = 1, J = \frac{3}{2}$ state I. Horn *et. al.* arXiv: 0711.1138

Non-Conformal Extension of Algebraic Structure (Soft Wall Model)

- We write the Dirac equation

$$(\alpha\Pi(\zeta) - \mathcal{M})\,\psi(\zeta) = 0,$$

in terms of the matrix-valued operator Π_ν

$$\Pi_\nu(\zeta) = -i\left(\frac{d}{d\zeta} - \frac{\nu + \frac{1}{2}}{\zeta}\gamma_5 - \kappa^2\zeta\gamma_5\right),$$

- Commutation relations for fermionic generators

$$\left[\Pi_\nu(\zeta), \Pi_\nu^\dagger(\zeta)\right] = \left(\frac{2\nu+1}{\zeta^2} - 2\kappa^2\right)\gamma_5.$$

- Solutions to the Dirac equation

$$\begin{aligned}\psi_+(\zeta) &\sim z^{\frac{1}{2}+\nu} e^{-\kappa^2\zeta^2/2} L_n^\nu(\kappa^2\zeta^2),\\ \psi_-(\zeta) &\sim z^{\frac{3}{2}+\nu} e^{-\kappa^2\zeta^2/2} L_n^{\nu+1}(\kappa^2\zeta^2).\end{aligned}$$

- Eigenvalues

$$\mathcal{M}^2 = 4\kappa^2(n+\nu+1).$$

Linear Holographic Confinement

- Compare with usual Dirac equation in AdS space $\left(x^\ell = (x^\mu, z)\right)$

$$\left[i\left(z\eta^{\ell m}\Gamma_\ell \partial_m + \frac{d}{2}\Gamma_z\right) + \mu R + V(z)\right]\Psi(x^\ell) = 0.$$

in presence of a linear confining potential $V(z) = \kappa^2 z.$

- Upon substitution $\Psi(x, z) = e^{-iP\cdot x} z^2 \psi(z),\ z \to \zeta$ we find

$$\alpha \Pi(\zeta)\psi(\zeta) = \mathcal{M}\,\psi(\zeta)$$

with

$$\Pi_\nu(\zeta) = -i\left(\frac{d}{d\zeta} - \frac{\nu + \frac{1}{2}}{\zeta}\gamma_5 - \kappa^2\zeta\gamma_5\right), \quad \mu R = \nu + \frac{1}{2},$$

our previous result.

- Soft-wall model for baryons corresponds to a linear confining potential in the LF transverse variable ζ!

- Baryon: twist-dimension $3 + L \quad (\nu = L + 1)$

$$\mathcal{O}_{3+L} = \psi D_{\{\ell_1} \dots D_{\ell_q} \psi D_{\ell_{q+1}} \dots D_{\ell_m\}} \psi, \qquad L = \sum_{i=1}^{m} \ell_i.$$

- Define the zero point energy (identical as in the meson case) $\mathcal{M}^2 \to \mathcal{M}^2 - 4\kappa^2$:

$$\mathcal{M}^2 = 4\kappa^2(n + L + 1).$$

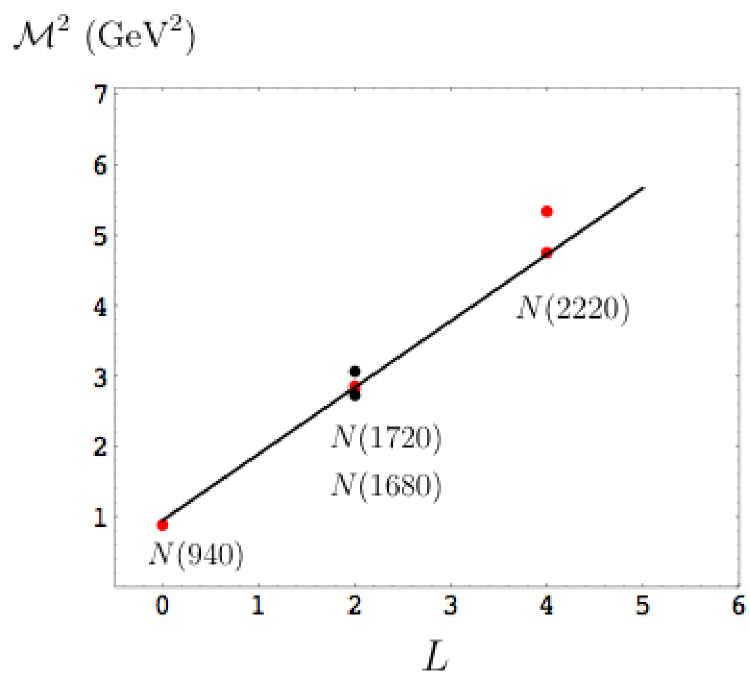


Fig: Proton Regge Trajectory $\quad \kappa = 0.49$ GeV

Excitation Spectrum of Baryons

- Hard wall holographic model: $\mathcal{M}_n(L) \sim L + 2n$
- Soft wall holographic model: $\mathcal{M}_n^2(L) \sim L + n$
- Quark models (shell structure of excitations): $\mathcal{M}_n^2(L) \sim L + 2n$
- Observed same multiplicity of states for mesons and baryons!
- Natural feature of AdS/QCD and the quantized Nambu String Baker and Steinke (2002).

  See: E. Klempt (2007)

5 Current Matrix Elements in AdS Space: The Form Factor

- Hadronic matrix element for EM coupling with string mode $\Phi(x^\ell),\ x^\ell = (x^\mu, z)$

$$ig_5 \int d^4x\, dz\, \sqrt{g}\, A^\ell(x,z) \Phi^*_{P'}(x,z) \overleftrightarrow{\partial}_\ell \Phi_P(x,z).$$

- Electromagnetic probe polarized along Minkowski coordinates $(Q^2 = -q^2 > 0)$

$$A(x,z)_\mu = \epsilon_\mu e^{-iQ\cdot x} J(Q,z), \quad A_z = 0.$$

- Propagation of external current inside AdS space described by the AdS wave equation

$$\left[z^2 \partial_z^2 - z\,\partial_z - z^2 Q^2\right] J(Q,z) = 0,$$

subject to boundary conditions $J(Q=0,z) = J(Q,z=0) = 1.$

- Solution

$$J(Q,z) = zQK_1(zQ).$$

- Substitute hadronic modes $\Phi(x,z)$ in the AdS EM matrix element

$$\Phi_P(x,z) = e^{-iP\cdot x}\,\Phi(z), \quad \Phi(z) \to z^\Delta, \ \ z \to 0.$$

- Find form factor in AdS as overlap of normalizable modes dual to the in and out hadrons Φ_P and $\Phi_{P'}$, with the non-normalizable mode $J(Q,z)$ dual to external source [Polchinski and Strassler (2002)].

$$F(Q^2) = R^3 \int_0^{\Lambda_{\rm QCD}^{-1}} \frac{dz}{z^3}\, \Phi(z) J(Q,z) \Phi(z).$$

- Since $K_n(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$, the external source is suppressed inside AdS for large Q. Important contribution to the integral is from $z \sim 1/Q$, where $\Phi \sim z^\Delta$.

- For large Q^2

$$F(Q^2) \to \left[\frac{1}{Q^2}\right]^{\Delta - 1},$$

and the power-law ultraviolet point-like scaling is recovered [Polchinski and Susskind (2001)]

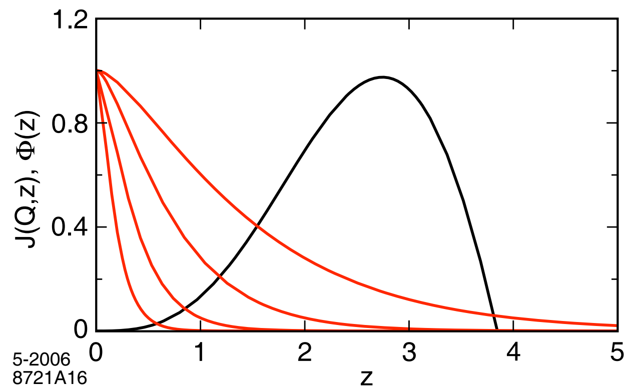


Fig: Suppression of external modes for large Q inside AdS. Red curves: $J(Q,z)$, black: $\Phi(z)$.

Current Propagation in the SW Model

- Propagation of external current inside AdS space described by the AdS wave equation

$$\left[z^2\partial_z^2 - z\left(1 + 2\kappa^2 z^2\right)\partial_z - Q^2 z^2\right] J_\kappa(Q,z) = 0.$$

- Solution: bulk-to-boundary propagator

$$J_\kappa(Q,z) = \Gamma\left(1 + \frac{Q^2}{4\kappa^2}\right) U\left(\frac{Q^2}{4\kappa^2}, 0, \kappa^2 z^2\right),$$

  where $U(a,b,c)$ is the confluent hypergeometric function

$$\Gamma(a)U(a,b,z) = \int_0^\infty e^{-zt} t^{a-1}(1+t)^{b-a-1} dt.$$

- Form factor in presence of the dilaton background $\varphi = \kappa^2 z^2$

$$F(Q^2) = R^3 \int \frac{dz}{z^3}\, e^{-\kappa^2 z^2} \Phi(z) J_\kappa(Q,z) \Phi(z).$$

- For large $Q^2 \gg 4\kappa^2$

$$J_\kappa(Q,z) \to zQK_1(zQ) = J(Q,z),$$

  the external current decouples from the dilaton field.

Space and Time-Like Pion Form Factor

- Hadronic string modes $\Phi_\pi(z) \to z^2$ as $z \to 0$ (twist $\tau = 2$)

$$\Phi_\pi^{HW}(z) = \frac{\sqrt{2}\Lambda_{QCD}}{R^{3/2} J_1(\beta_{0,1})} z^2 J_0\left(z\beta_{0,1}\Lambda_{QCD}\right),$$

$$\Phi_\pi^{SW}(z) = \frac{\sqrt{2}\kappa}{R^{3/2}} z^2.$$

- F_π has analytical solution in the SW model $F_\pi(Q^2) = \frac{4\kappa^2}{4\kappa^2+Q^2}$.

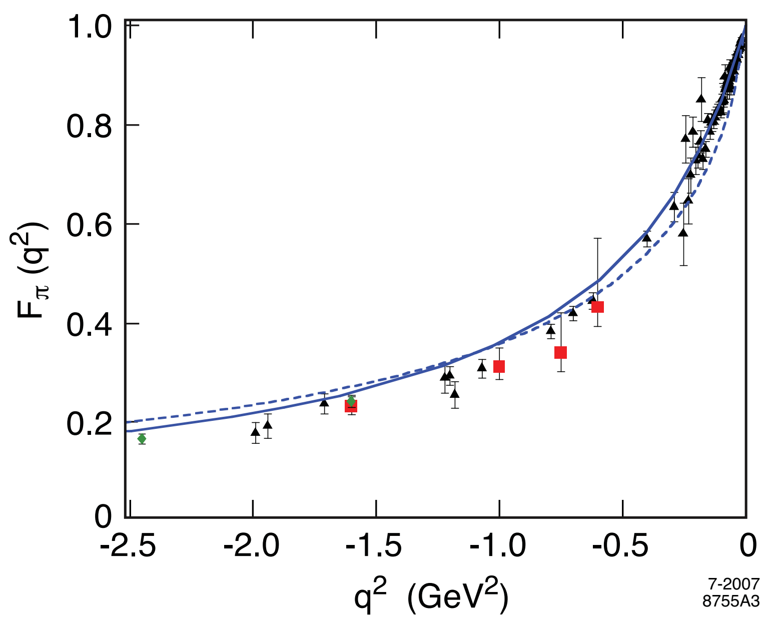


Fig: $F_\pi(q^2)$ for $\kappa = 0.375$ GeV and $\Lambda_{QCD} = 0.22$ GeV. Continuous line: SW, dashed line: HW.

- Scaling behavior for large Q^2: $Q^2 F_\pi(Q^2) \to \text{constant}$ $\boxed{\text{Pion } \tau = 2}$

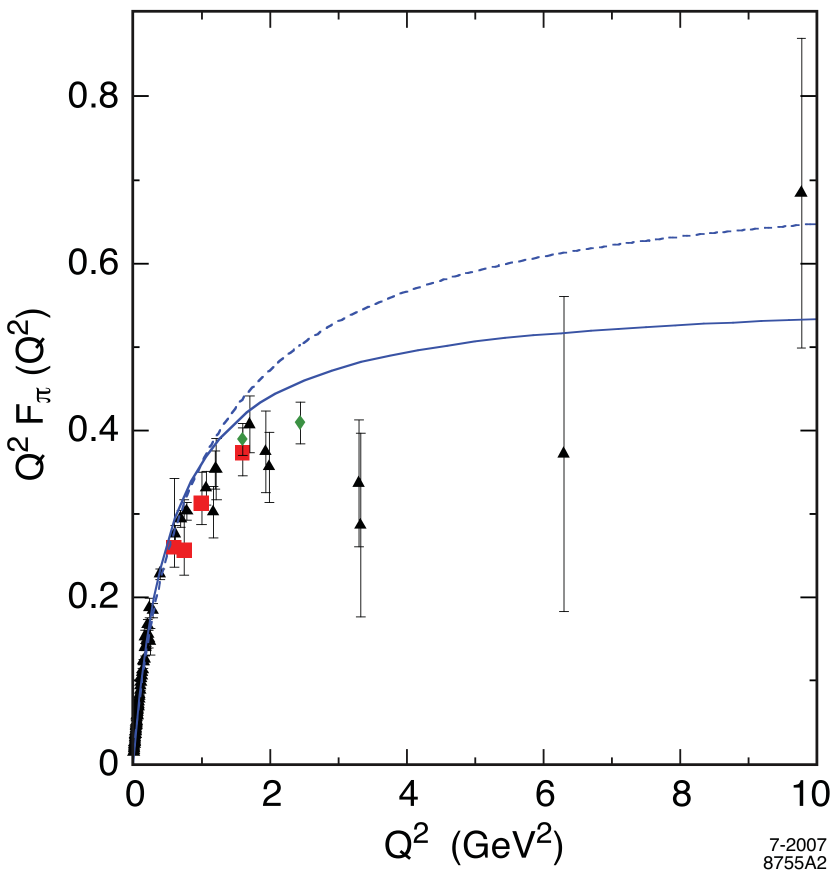
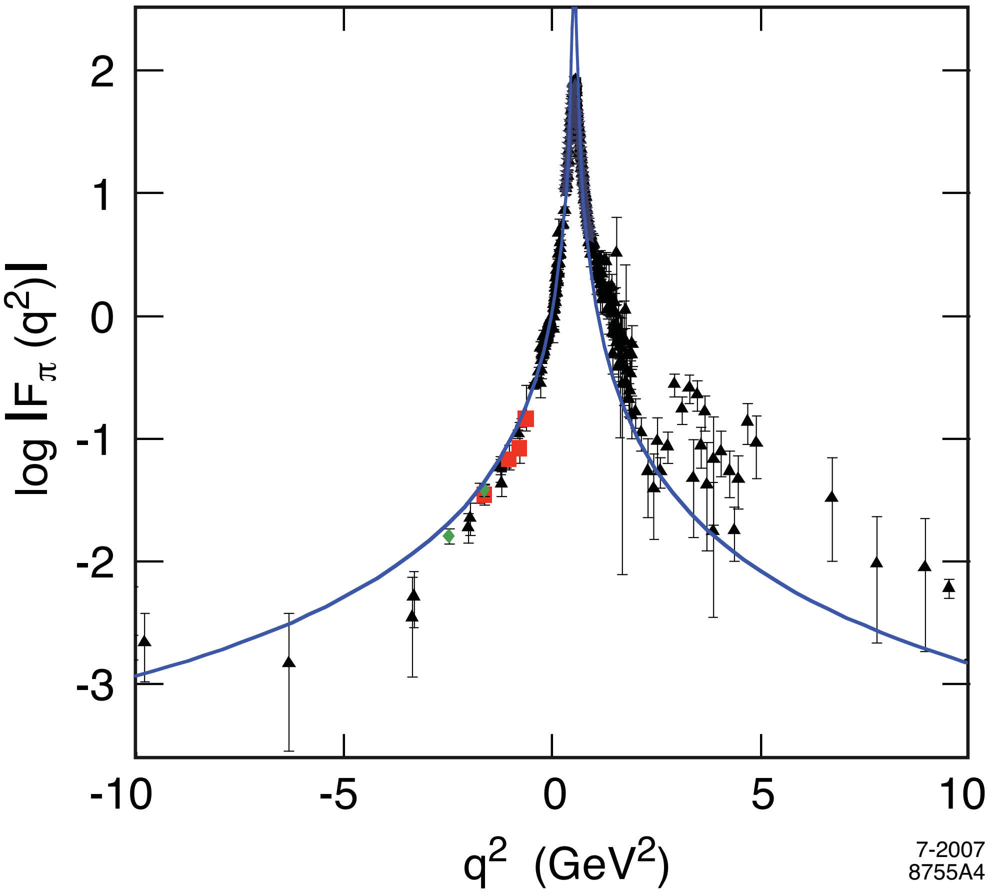


Fig: Continuous line: SW model for $\kappa = 0.375$ GeV. Dashed line: HW model for $\Lambda_{QCD} = 0.22$ GeV.

- Analytical continuation to time-like region $q^2 \to -q^2 \quad (M_\rho = 4\kappa^2 = 750$ MeV)
- Strongly coupled semiclassical gauge/gravity limit hadrons have zero widths (stable).



Space and time-like pion form factor for $\kappa = 0.375$ GeV in the SW model.

- Vector Mesons: Hong, Yoon and Strassler (2004); Grigoryan and Radyushkin (2007).

- Extract the value of the mean pion charge radius

$$F_\pi(Q^2) = 1 - \frac{1}{6}\langle r_\pi^2\rangle\, Q^2 + \mathcal{O}(Q^4), \qquad \langle r_\pi^2\rangle = -6\frac{dF_\pi(Q^2)}{dQ^2}\Big|_{Q^2=0}.$$

- Soft wall model

$$\langle r_\pi^2\rangle_{SW} = \frac{3}{2\kappa^2} \simeq 0.42\,\mathrm{fm}^2,$$

compared with the PDG value $\langle r_\pi^2\rangle = 0.45(1)\,\mathrm{fm}^2$.

- Hard-wall model with non-confined electromagnetic current expand $J(Q^2, z)$ for small values of Q^2

$$J(Q^2, z) = 1 + \frac{z^2Q^2}{4}\left[2\gamma - 1 + \ln\left(\frac{z^2Q^2}{4}\right)\right] + \mathcal{O}^4,$$

where $\gamma = 0.5772\ldots$. Since there is no scale in $J(Q^2, z)$, value of $\langle r_\pi^2\rangle$ diverges logarithmically.

- Problem in defining $\langle r_\pi^2\rangle$ does not appear if one uses Neumann boundary conditions for the HW model:

$$\langle r_\pi^2\rangle_{HW} \sim 1/\Lambda_{\mathrm{QCD}}^2.$$

(Grigoryan and Radyushkin (2007))

Note: Analytical Form of Hadronic Form Factor for Arbitrary Twist

- Form factor for a string mode with scaling dimension τ, Φ_τ in the SW model

$$F(Q^2) = \Gamma(\tau) \frac{\Gamma\left(1+\frac{Q^2}{4\kappa^2}\right)}{\Gamma\left(\tau+\frac{Q^2}{4\kappa^2}\right)}.$$

- For $\tau = N, \quad \Gamma(N+z) = (N-1+z)(N-2+z)\ldots(1+z)\Gamma(1+z)$.

- Form factor expressed as $N-1$ product of poles

$$\begin{aligned}
F(Q^2) &= \frac{1}{1+\frac{Q^2}{4\kappa^2}}, \quad N=2, \\
F(Q^2) &= \frac{2}{\left(1+\frac{Q^2}{4\kappa^2}\right)\left(2+\frac{Q^2}{4\kappa^2}\right)}, \quad N=3, \\
&\cdots \\
F(Q^2) &= \frac{(N-1)!}{\left(1+\frac{Q^2}{4\kappa^2}\right)\left(2+\frac{Q^2}{4\kappa^2}\right)\cdots\left(N-1+\frac{Q^2}{4\kappa^2}\right)}, \quad N.
\end{aligned}$$

- For large Q^2:

$$F(Q^2) \to (N-1)!\left[\frac{4\kappa^2}{Q^2}\right]^{(N-1)}.$$

Space-Like Dirac Proton and Neutron Form Factors

- Consider the spin non-flip form factors

$$
\begin{aligned}
F_+(Q^2) &= g_+ \int d\zeta \, J(Q,\zeta) |\psi_+(\zeta)|^2, \\
F_-(Q^2) &= g_- \int d\zeta \, J(Q,\zeta) |\psi_-(\zeta)|^2,
\end{aligned}
$$

  where the effective charges g_+ and g_- are determined from the spin-flavor structure of the theory.

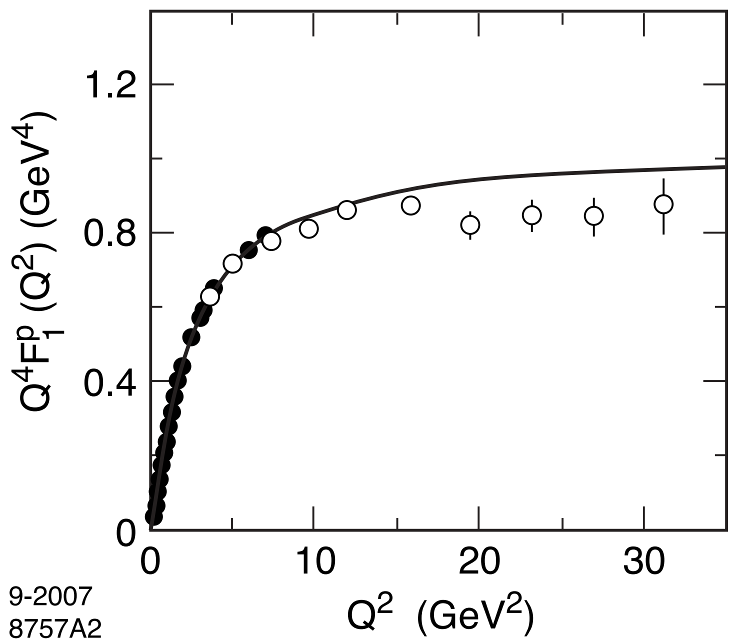
- Choose the struck quark to have $S^z = +1/2$. The two AdS solutions $\psi_+(\zeta)$ and $\psi_-(\zeta)$ correspond to nucleons with $J^z = +1/2$ and $-1/2$.

- For $SU(6)$ spin-flavor symmetry

$$
\begin{aligned}
F_1^p(Q^2) &= \int d\zeta \, J(Q,\zeta) |\psi_+(\zeta)|^2, \\
F_1^n(Q^2) &= -\frac{1}{3} \int d\zeta \, J(Q,\zeta) \left[|\psi_+(\zeta)|^2 - |\psi_-(\zeta)|^2 \right],
\end{aligned}
$$

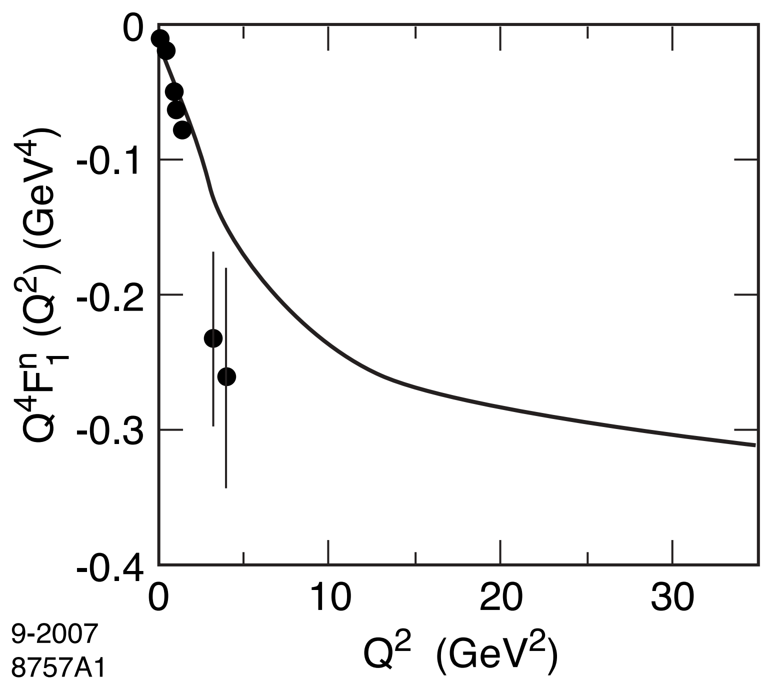
  where $F_1^p(0) = 1, \; F_1^n(0) = 0$.

- Scaling behavior for large Q^2: $Q^4 F_1^p(Q^2) \to \text{constant}$ $\boxed{\text{Proton } \tau = 3}$



SW model predictions for $\kappa = 0.424$ GeV. Data analysis from: M. Diehl *et al.* Eur. Phys. J. C **39**, 1 (2005).

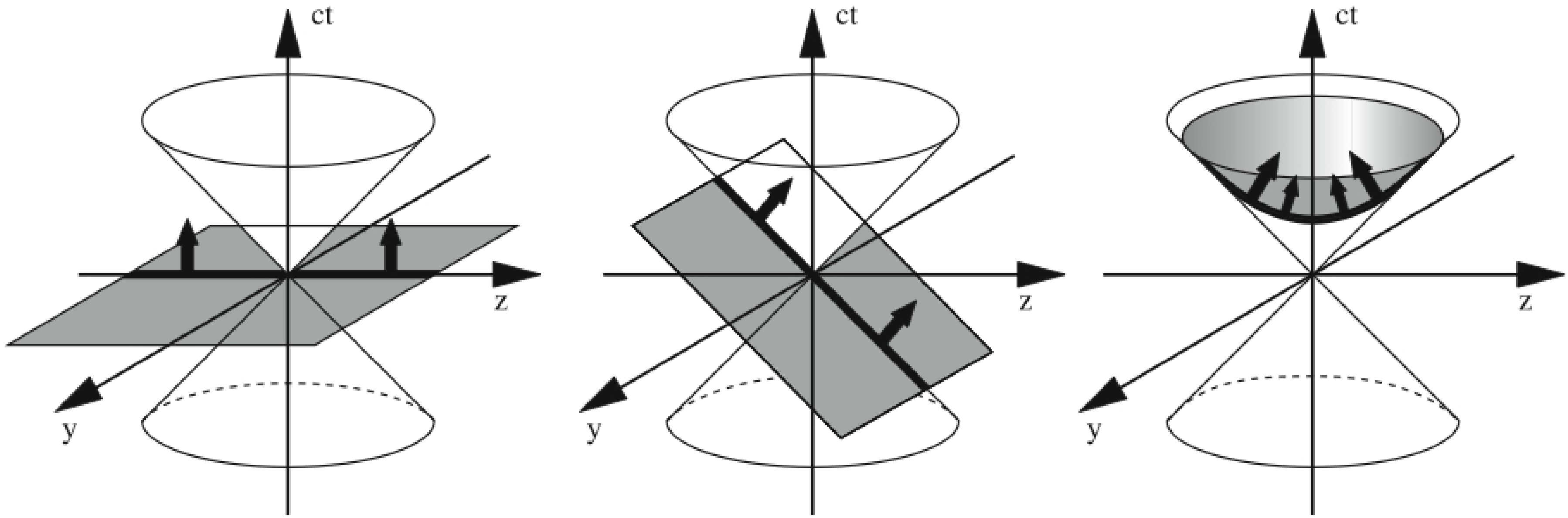
- Scaling behavior for large Q^2: $Q^4 F_1^n(Q^2) \rightarrow \text{constant}$ $\boxed{\text{Neutron } \tau = 3}$



SW model predictions for $\kappa = 0.424$ GeV. Data analysis from M. Diehl *et al.* Eur. Phys. J. C **39**, 1 (2005).

6 Light Front Dynamics

- Different possibilities to parametrize space-time in terms of general coordinates $\overline{x}(x)$ (excluding all related by a Lorentz transformation).
- According to Dirac there are no more than three different parametrization of space-time, the *instant form*, the *front form* and the *point form*, Dirac (1949).
- Parametrizations differ by the hypersurface on which the initial conditions are specified. Each evolve with different ”times” and has its own Hamiltonian, but should give the same physical results.
- *Instant form*: hypersurface defined by $t = 0$, the familiar one.
- *Front form*: hypersurface is tangent to the light cone.
- *Point form*: hypersurface is an hyperboloid



The instant form

$$\tilde{x}^0 = ct,\quad \tilde{x}^1 = x,\quad \tilde{x}^2 = y,\quad \tilde{x}^3 = z$$

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The front form

$$\tilde{x}^0 = ct+z,\quad \tilde{x}^1 = x,\quad \tilde{x}^2 = y,\quad \tilde{x}^3 = ct-z$$

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$

The point form

$$\tilde{x}^0 = \tau \quad , \quad ct = \tau \cosh\omega$$
$$\tilde{x}^1 = \omega \quad , \quad x = \tau \sinh\omega \sin\theta \cos\phi$$
$$\tilde{x}^2 = \theta \quad , \quad y = \tau \sinh\omega \sin\theta \sin\phi$$
$$\tilde{x}^3 = \phi \quad , \quad x = \tau \sinh\omega \cos\theta$$

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\tau^2 & 0 & 0 \\ 0 & 0 & -\tau^2 \sinh^2\omega & 0 \\ 0 & 0 & 0 & -\tau^2 \sinh^2\omega \sin^2\theta \end{pmatrix}$$

Light-Front Fock Representation

- Light-front expansion constructed by quantizing QCD at fixed light-cone time $\tau = t + z/c$ and forming the invariant light-front Hamiltonian (Brodsky, Pauli and Pinski, Phys. Rept. **301** 299 (1998)) :

$$H_{LF} = P^+ P^- - \vec{P}_\perp^2$$

where $P^\pm = P^0 \pm P^z$.

- Momentum generators P^+ and $\vec{P}_\perp$ are kinematical (independent of the interactions) and $P^- = i\frac{d}{d\tau}$ generates light-front time translations.

- Eigenvalues of H_{LF} give the mass spectrum of the color-singlet hadron states:

$$H_{LF} \, | \psi_h \rangle = \mathcal{M}_h^2 \, | \psi_h \rangle \, .$$

- State $| \psi_h \rangle$ is an expansion in multi-particle Fock eigenstates $| n \rangle$ of the free light-front Hamiltonian:

$$|\psi_h\rangle = \sum_n \psi_{n/h} |n\rangle .$$

- Example: $|P\rangle = |uud\rangle + |uudg\rangle + |uud\bar{q}q\rangle \ldots$

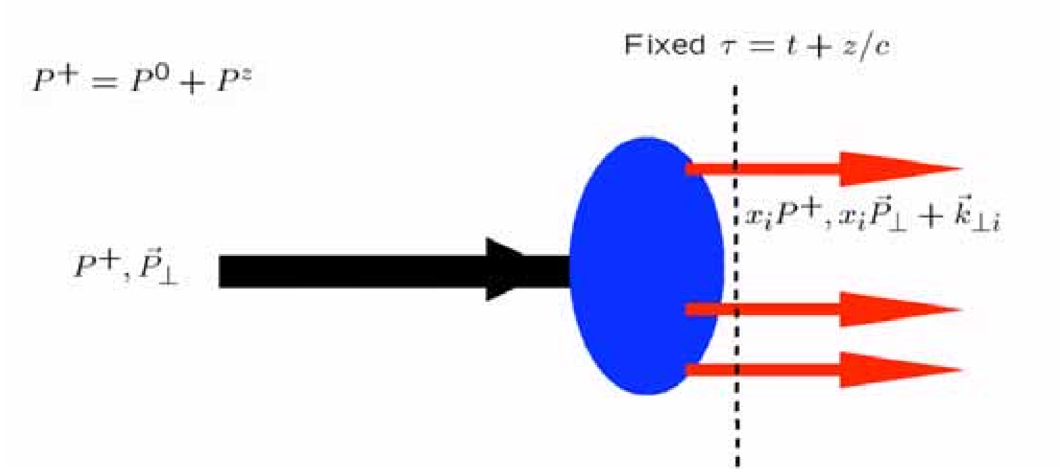
- Coefficients of the Fock expansion $\psi_{n/h}$ are independent of the total momentum P^+ and $\mathbf{P}_\perp$ of the hadron and depend only on the relative partonic coordinates of parton i in Fock-state n: momentum fraction $x_i = k_i^+/P^+$ and $\mathbf{k}_{\perp i}$

$$\sum_{i=1}^{n} x_i = 1 \quad \sum_{i=1}^{n} \mathbf{k}_{\perp i} = 0.$$

- Fock components

$$\psi_{n/h}(x_i, \mathbf{k}_{\perp i}) = \left\langle n;\, x_i, \mathbf{k}_{\perp i} \middle| \psi_h(P^+, \mathbf{P}_\perp) \right\rangle ,$$

frame independent and encode hadron properties in high momentum-transfer collisions.



Current Matrix Elements in the QCD Light-Front Frame

- Electromagnetic form factor $(P' = P + q)$

$$\left\langle P' \left| J^+(0) \right| P \right\rangle = 2 \left(P + P' \right)^+ F(Q^2).$$

- Drell-Yan-West (DYW) expression for meson form factor integrated over phase-space momentum

$$F(q^2) = \sum_n \int \left[dx_i \right] \left[d^2 \mathbf{k}_{\perp i} \right] \sum_j e_j \psi^*_{n/P'}(x_i, \mathbf{k}'_{\perp i}, \lambda_i) \psi_{n/P}(x_i, \mathbf{k}_{\perp i}, \lambda_i),$$

where $\mathbf{k}'_{\perp i} = \mathbf{k}_{\perp i} + (1 - x_i)\, \mathbf{q}_\perp$ for a struck quark and $\mathbf{k}'_{\perp i} = \mathbf{k}_{\perp i} - x_i\, \mathbf{q}_\perp$ for each spectator. The formula is exact if the sum is over all Fock states n.

- Normalization of LFWFs

$$\sum_n \int \left[dx_i \right] \left[d^2 \mathbf{k}_{\perp i} \right] \left| \psi_{n/h}(x_i, \mathbf{k}_{\perp i}) \right|^2 = 1,$$

- Transverse position coordinates $x_i \mathbf{r}_{\perp i} = x_i \mathbf{R}_\perp + \mathbf{b}_{\perp i}$

$$\sum_{i=1}^{n} \mathbf{b}_{\perp i} = 0, \quad \sum_{i=1}^{n} x_i \mathbf{r}_{\perp i} = \mathbf{R}_\perp .$$

- LFWF $\psi_n(x_j, \mathbf{k}_{\perp j})$ expanded in terms of $n-1$ independent coordinates $\mathbf{b}_{\perp j}$, $j = 1, 2, \ldots, n-1$

$$\psi_n(x_j, \mathbf{k}_{\perp j}) = (4\pi)^{\frac{n-1}{2}} \prod_{j=1}^{n-1} \int d^2\mathbf{b}_{\perp j} \exp\Big(i \sum_{j=1}^{n-1} \mathbf{b}_{\perp j} \cdot \mathbf{k}_{\perp j}\Big) \widetilde{\psi}_n(x_j, \mathbf{b}_{\perp j}).$$

- Normalization

$$\sum_n \prod_{j=1}^{n-1} \int dx_j d^2\mathbf{b}_{\perp j} |\widetilde{\psi}_n(x_j, \mathbf{b}_{\perp j})|^2 = 1.$$

- The form factor has the exact representation (DYW)

$$F(q^2) = \sum_n \prod_{j=1}^{n-1} \int dx_j d^2\mathbf{b}_{\perp j} \exp\big(i \mathbf{q}_\perp \cdot \sum_{j=1}^{n-1} x_j \mathbf{b}_{\perp j}\big) |\widetilde{\psi}_n(x_j, \mathbf{b}_{\perp j})|^2,$$

corresponding to a change of transverse momentum $x_j \mathbf{q}_\perp$ for each of the $n-1$ spectators and elementary coupling to the struck parton.

- Define effective single particle transverse density (Soper '77)

$$F(q^2) = \int_0^1 dx \rho(x, \vec{q}_\perp)$$

with

$$\rho(x, \vec{q}_\perp) = \int d^2\vec{\eta}_\perp e^{i\vec{\eta}_\perp \cdot \vec{q}_\perp} \tilde{\rho}(x, \vec{\eta}_\perp).$$

- From DYW expression for FF in transverse position space:

$$\tilde{\rho}(x, \vec{\eta}_\perp) = \sum_n \prod_{j=1}^{n-1} \int dx_j d^2\mathbf{b}_{\perp j} \, \delta\big(1 - x - \sum_{j=1}^{n-1} x_j\big) \delta^{(2)}\big(\sum_{j=1}^{n-1} x_j \mathbf{b}_{\perp j} - \vec{\eta}_\perp\big) \left|\widetilde{\psi}_n(x_j, \mathbf{b}_{\perp j})\right|^2$$

- Integration over the $n-1$ spectator partons, and $x = x_n$ is the coordinate of the active quark.

- $\vec{\eta}_\perp = \sum_{j=1}^{n-1} x_j \mathbf{b}_{\perp j}$ is the x-weighted transverse position coordinate of the $n-1$ spectators.

Holographic Mapping of AdS Modes to QCD LFWFs

- Integrate Soper formula over angles:

$$F(q^2) = 2\pi \int_0^1 dx \, \frac{(1-x)}{x} \int \zeta d\zeta J_0\left(\zeta q \sqrt{\frac{1-x}{x}}\right) \tilde{\rho}(x,\zeta),$$

  with $\widetilde{\rho}(x,\zeta)$ QCD effective transverse charge density.

- Transversality variable

$$\zeta = \sqrt{\frac{x}{1-x}} \, \Big\vert \sum_{j=1}^{n-1} x_j \mathbf{b}_{\perp j} \Big\vert .$$

- Compare AdS and QCD expressions of FFs for arbitrary Q using identity:

$$\int_0^1 dx J_0\left(\zeta Q \sqrt{\frac{1-x}{x}}\right) = \zeta Q K_1(\zeta Q),$$

  the solution for $J(Q,\zeta) = \zeta Q K_1(\zeta Q)$!

- Hadronic QCD transverse density $\tilde{\rho}$ is identified with the string mode density $|\Phi|^2$ in AdS space!

$$\boxed{\tilde{\rho}(x,\zeta) = \frac{R^3}{2\pi} \, \frac{x}{1-x} \, \frac{|\Phi(\zeta)|^2}{\zeta^4}}$$

SJB and GdT (2006)

- Transverse variable ζ represents the invariant separation between point-like constituents and is also the holographic variable: $\zeta = z$.

- For two-partons $\zeta^2 = x(1-x)\mathbf{b}_\perp^2$

$$\tilde{\rho}(x,\zeta) = \frac{1}{(1-x)^2} \left|\widetilde{\psi}\,(x,\zeta)\right|^2.$$

- Two-parton holographic bound state LFWF

$$\left|\widetilde{\psi}(x,\zeta)\right|^2 = \frac{R^3}{2\pi} \, x(1-x) \, \frac{|\Phi(\zeta)|^2}{\zeta^4}.$$

- AdS_3 dual to $1+1$ large N_C QCD (t'Hooft Model). Mapping between parton-x and radial AdS_3 coordinate: Katz and Okui (2007).

Example: Pion LFWF

- Two parton LFWF bound state:

$$\widetilde{\psi}^{HW}_{\overline{q}q/\pi}(x,\mathbf{b}_\perp) = \frac{\Lambda_{\rm QCD}\sqrt{x(1-x)}}{\sqrt{\pi}J_{1+L}(\beta_{L,k})} J_L\left(\sqrt{x(1-x)}\,|\mathbf{b}_\perp|\beta_{L,k}\Lambda_{\rm QCD}\right)\theta\left(\mathbf{b}_\perp^2 \le \frac{\Lambda^{-2}_{\rm QCD}}{x(1-x)}\right),$$

$$\widetilde{\psi}^{SW}_{\overline{q}q/\pi}(x,\mathbf{b}_\perp) = \frac{\kappa^{L+1}}{\sqrt{\pi}}\sqrt{\frac{n!}{(n+L)!}}\,[x(1-x)]^{\frac{1}{2}+L}|\mathbf{b}_\perp|^L e^{-\frac{1}{2}\kappa^2 x(1-x)\mathbf{b}_\perp^2} L_n^L\big(\kappa^2 x(1-x)\mathbf{b}_\perp^2\big).$$

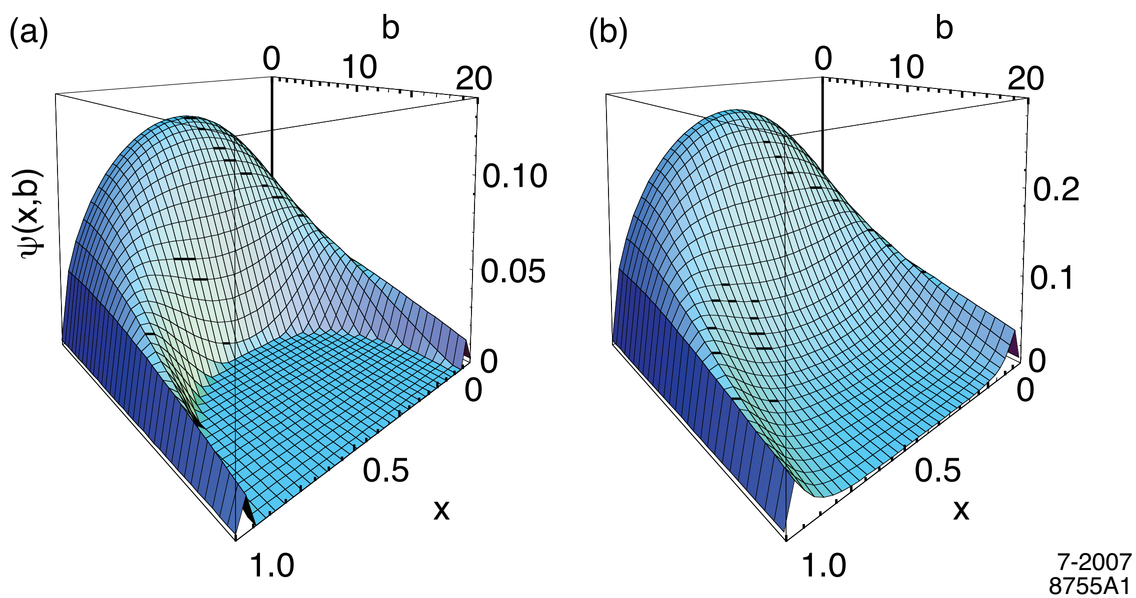


Fig: Ground state pion LFWF in impact space. (a) HW model $\Lambda_{\rm QCD} = 0.32$ GeV, (b) SW model $\kappa = 0.375$ GeV.

Example: Evaluation of QCD Matrix Elements

- Pion decay constant f_π defined by the matrix element of EW current J^+_W:

$$\left\langle 0 \left| \overline{\psi}_u \gamma^+ \tfrac{1}{2}(1-\gamma_5)\psi_d \right| \pi^- \right\rangle = i \frac{P^+ f_\pi}{\sqrt{2}}$$

with

$$\left|\pi^-\right\rangle = |d\overline{u}\rangle = \frac{1}{\sqrt{N_C}} \frac{1}{\sqrt{2}} \sum_{c=1}^{N_C} \left(b^\dagger_{c\, d\downarrow} d^\dagger_{c\, u\uparrow} - b^\dagger_{c\, d\uparrow} d^\dagger_{c\, u\downarrow} \right) \left|0\right\rangle.$$

- Find light-front expression (Lepage and Brodsky ’80):

$$f_\pi = 2\sqrt{N_C} \int_0^1 dx \int \frac{d^2 \vec{k}_\perp}{16\pi^3} \, \psi_{\overline{q}q/\pi}(x, k_\perp).$$

- Using relation between AdS modes and QCD LFWF in the $\zeta \to 0$ limit

$$f_\pi = \frac{1}{8}\sqrt{\frac{3}{2}}\, R^{3/2} \lim_{\zeta \to 0} \frac{\Phi(\zeta)}{\zeta^2}.$$

- Holographic result ($\Lambda_{\rm QCD} = 0.22$ GeV and $\kappa = 0.375$ GeV from pion FF data): Exp: $f_\pi = 92.4$ MeV

$$f_\pi^{HW} = \frac{\sqrt{3}}{8 J_1(\beta_{0,k})} \Lambda_{\rm QCD} = 91.7 \,{\rm MeV}, \quad f_\pi^{SW} = \frac{\sqrt{3}}{8} \kappa = 81.2 \,{\rm MeV},$$

Perturbative CFT vs Non-Perturbative AdS Results

- Heavy quark potential: $g_s N_C \to \sqrt{g_s N_C}$ Maldacena (1998), Rey and Yee (1998).
- Distribution amplitude $\phi_M(x,Q) \sim \int^{Q^2} d^2\mathbf{k}_\perp\, \psi_{\overline{q}q/M}(x,\mathbf{k}_\perp)$.
- Second Moment of the Distribution ($\xi = 1-2x$)

$$\langle \xi^2 \rangle = \int_{-1}^{1} d\xi\, \xi^2 \phi(\xi) \Big/ \int_{-1}^{1} d\xi\, \phi(\xi)$$

$$\langle \xi^2 \rangle = 1/5 \qquad \phi_{\rm PQCD} \sim x(1-x),$$

$$\langle \xi^2 \rangle = 1/4 \qquad \phi_{\rm AdS/QCD} \sim \sqrt{x(1-x)}.$$

- Sachrajda lattice result: $\langle \xi^2 \rangle = 0.28 \pm 0.02$

Introduction of Heavy Quark Masses

- Introduction of light quark masses in AdS/QCD involve complex dynamics, as the evolution from current to constituent quark masses should be included.
- Assume the momentum space LFWF is a function of the invariant (off-energy shell)

$$\mathcal{M}^2 - \mathcal{E} = \sum_{i=1}^{n} \left(\mathbf{k}_{\perp i}^2 + m_i^2\right)/x_i.$$

- Soft-Wall LFWF ansatz for bound state with massive constituents:

$$\psi(x, \mathbf{k}_\perp) \sim \frac{4\pi}{\kappa\sqrt{x(1-x)}}\, e^{-\frac{1}{2\kappa^2}\left(\frac{\mathbf{k}_\perp^2}{x(1-x)} + \frac{m_1^2}{x} + \frac{m_2^2}{1-x}\right)},$$

- Fourier transform to impact space

$$\widetilde{\psi}(x, \mathbf{b}_\perp) \sim \frac{\kappa}{\sqrt{\pi}} \sqrt{x(1-x)}\, e^{-\frac{1}{2}\kappa^2 x(1-x)\mathbf{b}_\perp^2 + \frac{1}{2\kappa^2}\left[\frac{m_1^2}{x} - \frac{m_2^2}{1-x}\right]}.$$

$$z \to \zeta \xrightarrow{m_i} \chi$$

$$\chi^2 = x(1-x)\mathbf{b}_\perp^2 + \frac{1}{\kappa^4}\left[\frac{m_1^2}{x} + \frac{m_2^2}{1-x}\right].$$

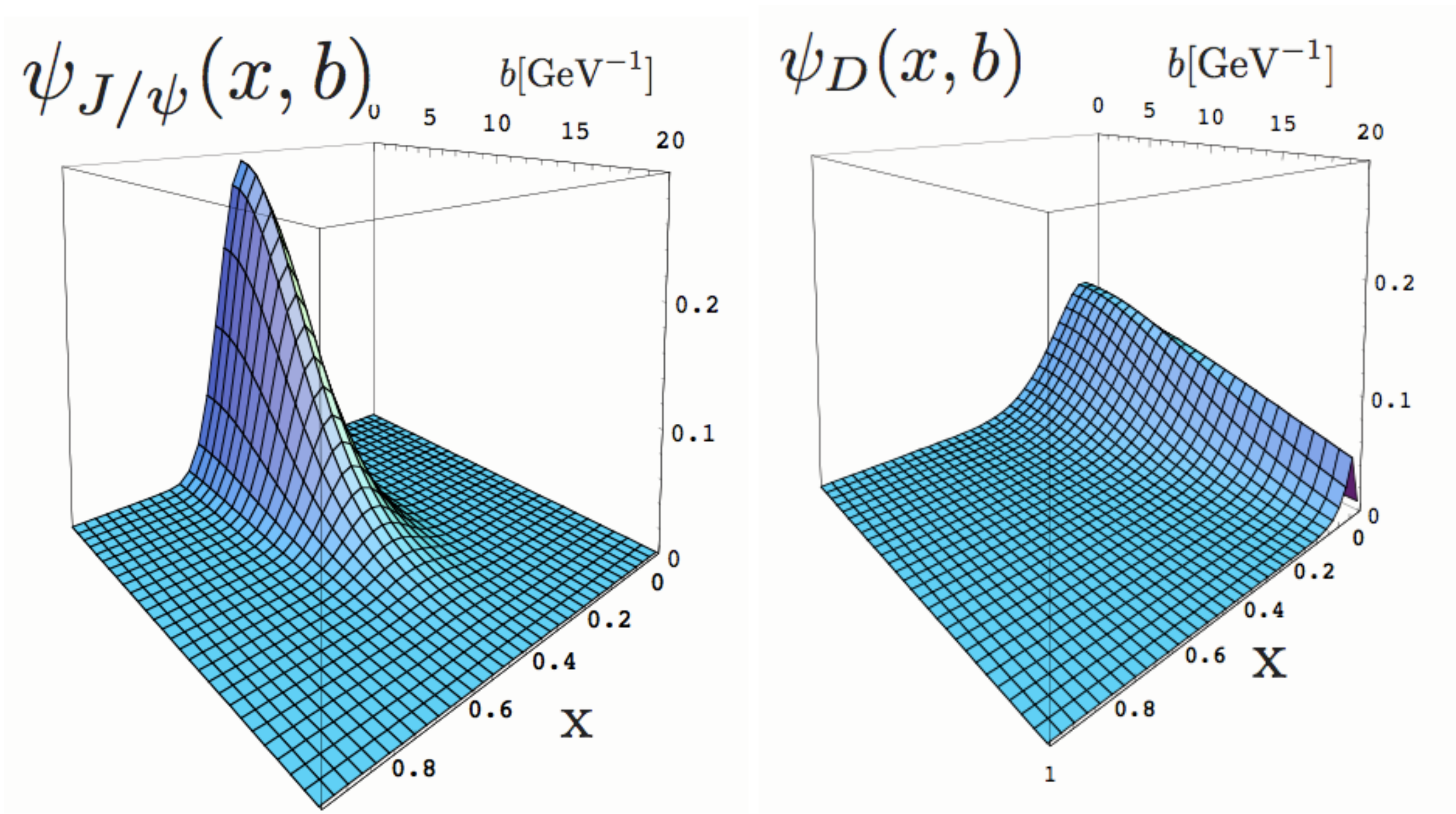


Fig: LFWF in impact space: J/Ψ meson: $m_1 = m_2 = m_c$; D meson: $m_1 = 0,\ m_2 = m_c,\ m_c = 1.25$ GeV .