AdS/QCD, Light-Front Holography, and Hadron Dynamics

\[ \Psi_n(x_i, \vec{k}_\perp i, \lambda_i) \]

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Oberwölz, August 31, 2010

QCD and Strings: Elements of a Universal Theory
Goal: an analytic first approximation to QCD

- As Simple as Schrödinger Theory in Atomic Physics
- Relativistic, Frame-Independent, Color-Confining
- QCD Coupling at all scales
- Hadron Spectroscopy
- Light-Front Wavefunctions
- Form Factors, Hadronic Observables, Constituent Counting Rules
- Insight into QCD Condensates
- Systematically improvable

de Teramond, Deur, Shrock, Roberts, Tandy
Light-Front Wavefunctions: rigorous representation of composite systems in quantum field theory

\[ x = \frac{k^+}{P^+} = \frac{k^0 + k^3}{P^0 + P^3} \]

Fixed \( \tau = t + z/c \)

\[ x_i P^+, x_i \vec{P}^\perp + \vec{k}^\perp_i \]

Process Independent
Direct Link to QCD Lagrangian!

\[ \Psi_n(x_i, \vec{k}^\perp_i, \lambda_i) \]

Invaraint under boosts! Independent of \( P \)

\[ \sum_i^n x_i = 1 \]

\[ \sum_i^n \vec{k}^\perp_i = \vec{0}^\perp \]
QCD and the LF Hadron Wavefunctions

- AdS/QCD Light-Front Holography LF Schrödinger Eqn
- Initial and Final State Rescattering DDIS, DDIS, T-Odd Non-Universal Antishadowing
- Baryon Excitations
- Gluonic properties DGLAP
- Heavy Quark Fock States Intrinsic Charm
- Coordinate space representation
- Quark & Flavor Structure
- J=0 Fixed Pole DVCS, GPDs, TMDs LF Overlap, incl ERBL
- Hadronization at Amplitude Level
- Nuclear Modifications Baryon Anomaly Color Transparency
- Orbital Angular Momentum Spin, Chiral Properties Crewther Relation
- Hard Exclusive Amplitudes Form Factors Counting Rules
- Distribution amplitude ERBL Evolution \( \phi_p(x_1, x_2, Q^2) \)
- Baryon Decay

\[ \Psi_n(x_i, \vec{k}_i, \lambda_i) \]
Goal:
Use AdS/QCD duality to construct
a first approximation to QCD

Hadron Spectrum
Light-Front Wavefunctions,
Running coupling in IR

Central problem for strongly-coupled gauge theories

in collaboration with
Guy de Teramond and Alexandre Deur
Applications of AdS/CFT to QCD

Changes in physical length scale mapped to evolution in the 5th dimension $z$

in collaboration with Guy de Teramond

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AdS/QCD and Light-Front Holography

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Conformal Theories are invariant under the Poincare and conformal transformations with

\[ M^{\mu \nu}, \, P^\mu, \, D, \, K^\mu, \]

the generators of \( SO(4,2) \)

\( SO(4,2) \) has a mathematical representation on \( AdS_5 \)
**AdS/CFT:** Anti-de Sitter Space / Conformal Field Theory

Maldacena:

*Map \( AdS_5 \times S_5 \) to conformal \( N=4 \) SUSY*

- **QCD is not conformal:** however, it has manifestations of a scale-invariant theory: Bjorken scaling, dimensional counting for hard exclusive processes

- **Conformal window:** \( \alpha_s(Q^2) \approx \text{const} \) at small \( Q^2 \)

- Use mathematical mapping of the conformal group \( SO(4,2) \) to \( AdS_5 \) space
Deur, Korsch, et al.

Cornwall: pinch scheme; effective gluon mass

Cornwall

Bhagwat et al.

Maris-Tandy

Furii et al.

DSE gluon couplings

Lattice QCD

Gluon couplings

α_{s,g1}/π JLab

GDH limit

Fit

pQCD evol. eq.

Burkert-Ioffe

Bloch et al.

Godfrey-Isgur

Q (GeV)

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AdS/QCD and Light-Front Holography

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Maximal Wavelength of Confined Fields

- Colored fields confined to finite domain
- All perturbative calculations regulated in IR
- High momentum calculations unaffected
- Bound-state Dyson-Schwinger Equation
- Analogous to Bethe’s Lamb Shift Calculation

\[(x - y)^2 < \Lambda_{QCD}^{-2}\]

Shrock, sjb

Quark and Gluon vacuum polarization insertions decouple: IR fixed Point

A strictly-perturbative space-time region can be defined as one which has the property that any straight-line segment lying entirely within the region has an invariant length small compared to the confinement scale (whether or not the segment is spacelike or timelike).
Scale Transformations

- Isomorphism of $SO(4, 2)$ of conformal QCD with the group of isometries of AdS space

\[ ds^2 = \frac{R^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2), \]

where $x^\mu \rightarrow \lambda x^\mu$, $z \rightarrow \lambda z$, maps scale transformations into the holographic coordinate $z$.

- AdS mode in $z$ is the extension of the hadron wf into the fifth dimension.

- Different values of $z$ correspond to different scales at which the hadron is examined.

\[ x^2 \rightarrow \lambda^2 x^2, \quad z \rightarrow \lambda z. \]

$x^2 = x_\mu x^\mu$: invariant separation between quarks

- The AdS boundary at $z \rightarrow 0$ correspond to the $Q \rightarrow \infty$, UV zero separation limit.
• Truncated AdS/CFT (Hard-Wall) model: cut-off at $z_0 = 1/\Lambda_{QCD}$ breaks conformal invariance and allows the introduction of the QCD scale (Hard-Wall Model) Polchinski and Strassler (2001).

• Smooth cutoff: introduction of a background dilaton field $\varphi(z)$ – usual linear Regge dependence can be obtained (Soft-Wall Model) Karch, Katz, Son and Stephanov (2006).
Bosonic Solutions: Hard Wall Model

- Conformal metric: \( ds^2 = g_{\ell m} dx^\ell dx^m \). \( x^\ell = (x^\mu, z) \), \( g_{\ell m} \rightarrow (R^2/z^2) \eta_{\ell m} \).

- Action for massive scalar modes on AdS\(_{d+1}\):
  \[
  S[\Phi] = \frac{1}{2} \int d^{d+1}x \sqrt{g} \frac{1}{2} \left[ g_{\ell m} \partial_\ell \Phi \partial_m \Phi - \mu^2 \Phi^2 \right], \quad \sqrt{g} \rightarrow (R/z)^{d+1}.
  \]

- Equation of motion
  \[
  \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\ell} \left( \sqrt{g} g_{\ell m} \frac{\partial}{\partial x^m} \Phi \right) + \mu^2 \Phi = 0.
  \]

- Factor out dependence along \( x^\mu \)-coordinates, \( \Phi_P(x, z) = e^{-iP \cdot x} \Phi(z) \), \( P_\mu P^\mu = M^2 \):
  \[
  [z^2 \partial_z^2 - (d - 1)z \partial_z + z^2 M^2 - (\mu R)^2] \Phi(z) = 0.
  \]

- Solution: \( \Phi(z) \rightarrow z^\Delta \) as \( z \rightarrow 0 \),
  \[
  \Phi(z) = C z^{d/2} J_{\Delta-d/2}(zM) \quad \Delta = \frac{1}{2} \left( d + \sqrt{d^2 + 4\mu^2 R^2} \right).
  \]

\[
\Delta = 2 + L \quad d = 4 \quad (\mu R)^2 = L^2 - 4
\]
Let $\Phi(z) = z^{3/2} \phi(z)$

**AdS Schrodinger Equation for bound state of two scalar constituents:**

$$\left[ -\frac{d^2}{dz^2} - \frac{1 - 4L^2}{4z^2} \right] \phi(z) = M^2 \phi(z)$$

$L$: light-front orbital angular momentum

**Derived from variation of Action in AdS$_5$**

**Hard wall model: truncated space**

$$\phi(z = z_0 = \frac{1}{\Lambda_c}) = 0.$$
**Match fall-off at small** \( z \) **to conformal twist-dimension at short distances**

- Pseudoscalar mesons: \( \mathcal{O}_{2+L} = \overline{\psi} \gamma_5 D_{\ell_1} \cdots D_{\ell_m} \psi \) (\( \Phi_\mu = 0 \) gauge). \( \Delta = 2 + L \)

- 4-\( d \) mass spectrum from boundary conditions on the normalizable string modes at \( z = z_0 \), \( \Phi(x, z_0) = 0 \), given by the zeros of Bessel functions \( \beta_{\alpha, k} \): \( M_{\alpha, k} = \beta_{\alpha, k} \Lambda_{QCD} \)

- Normalizable AdS modes \( \Phi(z) \)

\[ S' = 0 \quad \text{Meson orbital and radial AdS modes for} \quad \Lambda_{QCD} = 0.32 \text{ GeV.} \]
Fig: Orbital and radial AdS modes in the hard wall model for $\Lambda_{QCD} = 0.32$ GeV.

Fig: Light meson and vector meson orbital spectrum $\Lambda_{QCD} = 0.32$ GeV.
Soft-Wall Model

\[ S = \int d^4 x \, dz \, \sqrt{g} \, e^{\varphi(z)} \mathcal{L}, \quad \varphi(z) = \pm \kappa^2 z^2 \]

Retain conformal AdS metrics but introduce smooth cutoff which depends on the profile of a dilaton background field

Karch, Katz, Son and Stephanov (2006)

- Equation of motion for scalar field \( \mathcal{L} = \frac{1}{2} (g^{\ell m} \partial_\ell \Phi \partial_m \Phi - \mu^2 \Phi^2) \)

\[ \left[ z^2 \partial_z^2 - \left( 3 \mp 2\kappa^2 z^2 \right) z \partial_z + z^2 \mathcal{M}^2 - (\mu R)^2 \right] \Phi(z) = 0 \]

with \( (\mu R)^2 \geq -4 \).

- LH holography requires ‘plus dilaton’ \( \varphi = +\kappa^2 z^2 \). Lowest possible state \( (\mu R)^2 = -4 \)

\[ \mathcal{M}^2 = 0, \quad \Phi(z) \sim z^2 e^{-\kappa^2 z^2}, \quad \langle r^2 \rangle \sim \frac{1}{\kappa^2} \]

A chiral symmetric bound state of two massless quarks with scaling dimension 2:

Massless pion
$$ds^2 = e^{\kappa^2 z^2 \frac{R^2}{z^2}} (dx_0^2 - dx_1^2 - dx_3^2 - dx_3^2 - dz^2)$$

$$ds^2 = e^{A(y)} (-dx_0^2 + dx_1^2 + dx_3^2 + dx_3^2 + dy^2)$$
\[ ds^2 = e^{\kappa^2 z^2} \frac{R^2}{z^2} (dx_0^2 - dx_1^2 - dx_3^2 - dx_3^2 - dz^2) \]

\[ V(z) = m c^2 \sqrt{g_{00}} \]

\[ V(z) \sim \frac{R}{z} e^{+\kappa^2 z^2 / 2} \]

\[ V(z) \sim \frac{R}{z} e^{-\kappa^2 z^2 / 2} \]

Agrees with Klebanov and Maldacena for positive-sign exponent of dilaton
Fig: Orbital and radial AdS modes in the soft wall model for $\kappa = 0.6$ GeV.

Quark separation increases with $L$.

Pion has zero mass!

$m_q = 0$

Light meson orbital (a) and radial (b) spectrum for $\kappa = 0.6$ GeV.

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AdS/QCD and Light-Front Holography

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Higher-Spin Hadrons

- Obtain spin-$J$ mode $\Phi_{\mu_1\cdots\mu_J}$ with all indices along 3+1 coordinates from $\Phi$ by shifting dimensions

$$\Phi_J(z) = \left(\frac{z}{R}\right)^{-J} \Phi(z)$$

- Substituting in the AdS scalar wave equation for $\Phi$

$$\left[ z^2 \partial_z^2 - (3 - 2J - 2\kappa^2 z^2) z \partial_z + z^2 \mathcal{M}^2 - (\mu R)^2 \right] \Phi_J = 0$$

- Upon substitution $z \rightarrow \zeta$

$$\phi_J(\zeta) \sim \zeta^{-3/2+J} e^{\kappa^2 \zeta^2/2} \Phi_J(\zeta)$$

we find the LF wave equation

$$\left( - \frac{d^2}{d\zeta^2} - \frac{1 - 4L^2}{4\zeta^2} + \kappa^4 \zeta^2 + 2\kappa^2 (L + S - 1) \right) \phi_{\mu_1\cdots\mu_J} = \mathcal{M}^2 \phi_{\mu_1\cdots\mu_J}$$

with $(\mu R)^2 = -(2 - J)^2 + L^2$
AdS Soft-Wall Schrödinger Equation for bound state of two scalar constituents:

\[
\left[ -\frac{d^2}{dz^2} - \frac{1 - 4L^2}{4z^2} + U(z) \right] \phi(z) = M^2 \phi(z)
\]

\[
U(z) = \kappa^4 z^2 + 2\kappa^2 (L + S - 1)
\]

Derived from variation of Action Dilaton-Modified AdS$_5$

Positive-sign dilaton

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Quark separation increases with $L$

$\Phi(z)$

(a)

$\Phi(z)$

(b)

$S = 1$

$\rho_3 (1690)$

$f_4 (2050)$

$a_4 (2040)$

$\omega (782)$

$\omega_3 (1670)$

$\rho (770)$

$\rho_3 (1450)$

$\rho (1700)$

$\rho (770)$

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AdS/QCD and Light-Front Holography

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Higher Spin Bosonic Modes SW

• Effective LF Schrödinger wave equation

$$\left[ -\frac{d^2}{dz^2} - \frac{1 - 4L^2}{4z^2} + \kappa^4 z^2 + 2\kappa^2 (L + S - 1) \right] \phi_S(z) = M^2 \phi_S(z)$$

with eigenvalues $M^2 = 2\kappa^2 (2n + 2L + S)$.

• Compare with Nambu string result (rotating flux tube): $M_n^2(L) = 2\pi\sigma (n + L + 1/2)$.

Vector mesons orbital (a) and radial (b) spectrum for $\kappa = 0.54$ GeV.

• Glueballs in the bottom-up approach: (HW) Boschi-Filho, Braga and Carrion (2005); (SW) Colangelo, De Facio, Jugeau and Nicotri (2007).
Parent and daughter Regge trajectories for the $I = 1$ \( \rho \)-meson family (red) and the $I = 0$ \( \omega \)-meson family (black) for $\kappa = 0.54 \text{ GeV}$
Propagation of external perturbation suppressed inside AdS.

\[ J(Q, z) = zQ K_1(zQ) \]

\[ F(Q^2)_{I \rightarrow F} = \int \frac{dz}{z^3} \Phi_F(z) J(Q, z) \Phi_I(z) \]

Consider a specific AdS mode \( \Phi^{(n)} \) dual to an \( n \) partonic Fock state \( |n\rangle \). At small \( z \), \( \Phi \) scales as \( \Phi^{(n)} \sim z^{\Delta_n} \). Thus:

\[ F(Q^2) \rightarrow \left[ \frac{1}{Q^2} \right]^{\tau-1} \]

where \( \tau = \Delta_n - \sigma_n \), \( \sigma_n = \sum_{i=1}^{n} \sigma_i \). The twist is equal to the number of partons, \( \tau = n \).

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AdS/QCD and Light-Front Holography

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Spacelike pion form factor from AdS/CFT

\[ F_\pi(q^2) \]

Data Compilation
Baldini, Kloe and Volmer

de Teramond, sjb
See also: Radyushkin

One parameter - set by pion decay constant
Current Matrix Elements in AdS Space (SW)

- Propagation of external current inside AdS space described by the AdS wave equation

\[
\left[ z^2 \partial_z^2 - z \left( 1 + 2 \kappa^2 z^2 \right) \partial_z - Q^2 z^2 \right] J_\kappa(Q, z) = 0.
\]

- Solution bulk-to-boundary propagator

\[
J_\kappa(Q, z) = \Gamma \left( 1 + \frac{Q^2}{4 \kappa^2} \right) U \left( \frac{Q^2}{4 \kappa^2}, 0, \kappa^2 z^2 \right),
\]

where \( U(a, b, c) \) is the confluent hypergeometric function

\[
\Gamma(a) U(a, b, z) = \int_0^\infty e^{-zt} t^{a-1} (1 + t)^{b-a-1} dt.
\]

- Form factor in presence of the dilaton background \( \varphi = \kappa^2 z^2 \)

\[
F(Q^2) = R^3 \int \frac{dz}{z^3} e^{-\kappa^2 z^2} \Phi(z) J_\kappa(Q, z) \Phi(z).
\]

- For large \( Q^2 \gg 4 \kappa^2 \)

\[
J_\kappa(Q, z) \to zQK_1(zQ) = J(Q, z),
\]

the external current decouples from the dilaton field.
Dressed soft-wall current bring in higher Fock states and more vector meson poles

\[ e^+ + e^- \gamma^* \rightarrow \pi^+ + \pi^- \]
Form Factors in AdS/QCD

\[ F(Q^2) = \frac{1}{1 + \frac{Q^2}{M_{\rho}^2}}, \quad N = 2, \]

\[ F(Q^2) = \frac{1}{\left(1 + \frac{Q^2}{M_{\rho}^2}\right)\left(1 + \frac{Q^2}{M_{\rho'}^2}\right)}, \quad N = 3, \]

\[ \ldots \]

\[ F(Q^2) = \frac{1}{\left(1 + \frac{Q^2}{M_{\rho}^2}\right)\left(1 + \frac{Q^2}{M_{\rho'}^2}\right)\cdots\left(1 + \frac{Q^2}{M_{\rho N-2}^2}\right)}, \quad N, \]

Positive Dilaton Background \( \exp (+\kappa^2 z^2) \)

\[ M_n^2 = 4\kappa^2 \left(n + \frac{1}{2}\right) \]

\[ F(Q^2) \to (N - 1)! \left[\frac{4\kappa^2}{Q^2}\right]^{(N-1)} \quad Q^2 \to \infty \]

Constituent Counting
Space- and Time Like Pion Form-Factor (HFS)

$|\pi\rangle = \psi_{q\bar{q}/\pi}|q\bar{q}\rangle + \psi_{q\bar{q}q\bar{q}}/\pi|q\bar{q}q\bar{q}\rangle$

$M^2 \rightarrow 4\kappa^2(n + 1/2)$

$\kappa = 0.54 \text{ GeV}$

$\Gamma_{\rho} = 130, \Gamma_{\rho'} = 400, \Gamma_{\rho''} = 300 \text{ MeV}$

$P_{q\bar{q}q\bar{q}} = 13\%$
Light-Front Representation of Two-Body Meson Form Factor

- Drell-Yan-West form factor
  \[ F(q^2) = \sum_q e_q \int_0^1 dx \int \frac{d^2 k_\perp}{16\pi^3} \psi_{P'}^*(x, k_\perp - xq_\perp) \psi_P(x, k_\perp). \]

- Fourier transform to impact parameter space \( \vec{b}_\perp \)
  \[ \psi(x, \vec{k}_\perp) = \sqrt{4\pi} \int d^2 \vec{b}_\perp e^{i\vec{b}_\perp \cdot \vec{k}_\perp} \tilde{\psi}(x, \vec{b}_\perp) \]

- Find (\( b = |\vec{b}_\perp| \)):
  \[ F(q^2) = \int_0^1 dx \int d^2 \vec{b}_\perp e^{ix\vec{b}_\perp \cdot \vec{q}_\perp} |\tilde{\psi}(x, b)|^2 \]
  \[ = 2\pi \int_0^1 dx \int_0^\infty b \, db \, J_0(bqx) \, |\tilde{\psi}(x, b)|^2, \]

Soper
Holographic Mapping of AdS Modes to QCD LFWFs

- Integrate Soper formula over angles:

\[
F(q^2) = 2\pi \int_0^1 dx \frac{(1 - x)}{x} \int \zeta d\zeta J_0 \left( \zeta q \sqrt{\frac{1 - x}{x}} \right) \tilde{\rho}(x, \zeta),
\]

with \( \tilde{\rho}(x, \zeta) \) QCD effective transverse charge density.

- Transversality variable

\[
\zeta = \sqrt{x(1 - x) \vec{b}_\perp^2}
\]

- Compare AdS and QCD expressions of FFs for arbitrary \( Q \) using identity:

\[
\int_0^1 dx J_0 \left( \zeta Q \sqrt{\frac{1 - x}{x}} \right) = \zeta Q K_1(\zeta Q),
\]

the solution for \( J(Q, \zeta) = \zeta Q K_1(\zeta Q) \)!
$\psi(x, \vec{b}_\perp)$  \quad \leftrightarrow \quad \phi(z)$

$\zeta = \sqrt{x(1-x)}\vec{b}_\perp^2$

$\psi(x, \zeta) = \sqrt{x(1-x)\zeta^{-1/2}}\phi(\zeta)$

**Holography:** Unique mapping derived from equality of LF and AdS formula for current matrix elements
Light-Front Holography: Map AdS/CFT to 3+1 LF Theory

Relativistic LF radial equation

\[
\left[ -\frac{d^2}{d\zeta^2} + \frac{1 - 4L^2}{4\zeta^2} + U(\zeta) \right] \phi(\zeta) = M^2 \phi(\zeta)
\]

\[
\zeta^2 = x(1 - x)b_\perp^2.
\]

\[
U(\zeta) = \kappa^4 \zeta^2 + 2\kappa^2 (L + S - 1)
\]

G. de Teramond, sjb

soft wall confining potential:

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AdS/QCD and Light-Front Holography

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Gravitational Form Factor in AdS space

- Hadronic gravitational form-factor in AdS space

\[ A_{\pi}(Q^2) = R^3 \int \frac{dz}{z^3} H(Q^2, z) |\Phi_{\pi}(z)|^2, \]

where \( H(Q^2, z) = \frac{1}{2} Q^2 z^2 K_2(zQ) \)

- Use integral representation for \( H(Q^2, z) \)

\[ H(Q^2, z) = 2 \int_0^1 x \, dx \, J_0 \left( zQ \sqrt{\frac{1-x}{x}} \right) \]

- Write the AdS gravitational form-factor as

\[ A_{\pi}(Q^2) = 2R^3 \int_0^1 x \, dx \int \frac{dz}{z^3} \, J_0 \left( zQ \sqrt{\frac{1-x}{x}} \right) |\Phi_{\pi}(z)|^2 \]

- Compare with gravitational form-factor in light-front QCD for arbitrary \( Q \)

\[ \left| \tilde{\psi}_{qq/\pi}(x, \zeta) \right|^2 = \frac{R^3}{2\pi} x(1-x) \frac{|\Phi_{\pi}(\zeta)|^2}{\zeta^4}, \]

Identical to LF Holography obtained from electromagnetic current
LIGHT-FRONT SCHRODINGER EQUATION

\[
\left( M_\pi^2 - \sum_i \frac{\vec{k}_{\perp i}^2 + m_i^2}{x_i} \right) \begin{bmatrix} \psi_{q\bar{q}/\pi} \\ \psi_{q\bar{q}g/\pi} \\ \vdots \end{bmatrix} = \begin{bmatrix} \langle q\bar{q} | V | q\bar{q} \rangle & \langle q\bar{q} | V | q\bar{g}g \rangle & \cdots \\ \langle q\bar{g}g | V | q\bar{q} \rangle & \langle q\bar{g}g | V | q\bar{g}g \rangle & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \psi_{q\bar{q}/\pi} \\ \psi_{q\bar{q}g/\pi} \\ \vdots \end{bmatrix}
\]

\[A^+ = 0\]

G.P. Lepage, sjb

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AdS/QCD and Light-Front Holography

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Derivation of the Light-Front Radial Schrödinger Equation directly from LF QCD

\[ M^2 = \int_0^1 dx \int \frac{d^2 k_\perp}{16\pi^3} \frac{k_\perp^2}{x(1-x)} \left| \psi(x, k_\perp) \right|^2 + \text{interactions} \]

\[ = \int_0^1 dx \frac{1}{x(1-x)} \int d^2 b_\perp \psi^*(x, b_\perp) \left( -\nabla_{b_\perp e}^2 \right) \psi(x, b_\perp) + \text{interactions}. \]

**Change variables**

\[ (\tilde{\zeta}, \varphi), \quad \tilde{\zeta} = \sqrt{x(1-x)} b_\perp : \quad \nabla^2 = \frac{1}{\zeta} \frac{d}{d\zeta} \left( \frac{d}{d\zeta} \right) + \frac{1}{\zeta^2} \frac{\partial^2}{\partial \varphi^2} \]

\[ M^2 = \int d\zeta \phi^*(\zeta) \sqrt{\zeta} \left( -\frac{d^2}{d\zeta^2} - \frac{1}{\zeta} \frac{d}{d\zeta} + \frac{L^2}{\zeta^2} \right) \phi(\zeta) \]

\[ + \int d\zeta \phi^*(\zeta) U(\zeta) \phi(\zeta) \]

\[ = \int d\zeta \phi^*(\zeta) \left( -\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2} + U(\zeta) \right) \phi(\zeta) \]
\[ H_{QED} \]

\[ (H_0 + H_{int}) |\Psi> = E |\Psi> \]

\[ \left[ -\frac{\Delta^2}{2m_{red}} + V_{eff}(\vec{S}, \vec{r}) \right] \psi(\vec{r}) = E \psi(\vec{r}) \]

\[ \left[ -\frac{1}{2m_{red}} \frac{d^2}{dr^2} + \frac{1}{2m_{red}} \frac{\ell(\ell + 1)}{r^2} + V_{eff}(r, S, \ell) \right] \psi(r) = E \psi(r) \]

\[ V_{eff} \rightarrow V_C(r) = -\frac{\alpha}{r} \]

**QED atoms: positronium and muonium**

**Effective two-particle equation**

Includes Lamb Shift, quantum corrections

**Spherical Basis** \( r, \theta, \phi \)

**Coulomb potential**

Bohr Spectrum

**Semiclassical first approximation to QED**

**Coupled Fock states**
\( H_{QCD}^{LF} \)

Coupled Fock states

\[ (H_{LF}^0 + H_{LF}^I)\psi \geq M^2 \psi \]

Effective two-particle equation

\[ \left[ \frac{\vec{k}_\perp^2 + m^2}{x(1-x)} + V_{\text{eff}}^{LF} \right] \psi_{LF}(x, \vec{k}_\perp) = M^2 \psi_{LF}(x, \vec{k}_\perp) \]

Azimuthal Basis \( \zeta, \phi \)

\[ U(\zeta, S, L) = \kappa^2 \zeta^2 + \kappa^2 (L + S - 1/2) \]

Semiclassical first approximation to QCD

Confining AdS/QCD potential
Prediction from AdS/CFT: Meson LFWF

\[ \psi_M(x, k^2_\perp) \]

"Soft Wall" model

\[ \kappa = 0.375 \text{ GeV} \]

massless quarks

Note coupling \( k^2_\perp, x \)

\[ \psi_M(x, k^\perp) = \frac{4\pi}{\kappa \sqrt{x(1-x)}} e^{-\frac{k^2_\perp}{2\kappa^2 x(1-x)}} \]

Connection of Confinement to TMDs

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AdS/QCD and Light-Front Holography
Second Moment of Pion Distribution Amplitude

\[
< \xi^2 > = \int_{-1}^{1} d\xi \ \xi^2 \phi(\xi)
\]

\[
\xi = 1 - 2x
\]

\[
< \xi^2 >_\pi = 1/5 = 0.20 \quad \phi_{\text{asympt}} \propto x(1 - x)
\]

\[
< \xi^2 >_\pi = 1/4 = 0.25 \quad \phi_{\text{AdS/QCD}} \propto \sqrt{x(1 - x)}
\]

Lattice (I) \( < \xi^2 >_\pi = 0.28 \pm 0.03 \)

Lattice (II) \( < \xi^2 >_\pi = 0.269 \pm 0.039 \)

Donnellan et al.

Braun et al.
\[ \phi_{asympt} \sim x(1-x) \]

**AdS/CFT:**

\[ \phi(x, Q_0) \propto \sqrt{x(1-x)} \]

Increases PQCD leading twist prediction \( F_\pi(Q^2) \) by factor 16/9
ERBL Evolution of Pion Distribution Amplitude

\[ \frac{\phi(x, Q^2)}{f_\pi} = \frac{x(1-x)}{Q^2 = 100 \text{ GeV}^2} \]

\[ Q^2 = 2 \text{ GeV}^2 \]

\[ \sqrt{x(1-x)} \]

\[ \sqrt{x(1-x)} \]

F. Cao, GdT, sjb (preliminary)
Photon-to-pion transition form factor

\[ Q^2 F_{\gamma \rightarrow \pi^0}(Q^2) \]

F. Cao, GdT, sjb (preliminary)
Photon-to-pion transition form factor with ERBL evolution

\[ Q^2 F_{\gamma \rightarrow \pi^0}(Q^2) \]

NLO: Diehl, Kroll

F. Cao, GdT, sjb (preliminary)
Structure of Pion Transition Form Factor in AdS

- Consider the amplitude for $\gamma^* \to \pi^0 \gamma$

\[
M \sim \int d^4x \, dz \, \sqrt{g} \, \mathcal{E}^{\ell mnpq} \partial_\ell \Phi(x, z) \partial_m A_n(x, z) \partial_p A_q(x, z)
\]

where $\mathcal{E}^{\ell mnpq} = \frac{1}{\sqrt{g}} \epsilon^{\ell mnpq}$

- Integrating over $z$ and changing variables (soft-wall model)

\[
M \sim \frac{1}{Q^2} \int_0^1 dx \, x(1-x) \left(1 - e^{-Q^2x/2\kappa^2(1-x)}\right)
\]

- Identical with LF QCD result with DA $x(1-x)$! [Lepage and Brodsky (1980)]

  See also: Grigoryan and Radyushkin (2008)

- Mapping pion transition FF at fixed LF time: $\Phi_P(z) \Rightarrow |\psi(P)\rangle_A$
  (asymptotic component of LFWF is selected)

- Identical results for pion decay constant $f_\pi$: $\Phi_P(z) \Rightarrow |\psi(P)\rangle_A$
Baryons Spectrum in “bottom-up” holographic QCD

Baryons in Ads/CFT

• Action for massive fermionic modes on AdS₅:

\[ S[\overline{\Psi}, \Psi] = \int d^4x \, dz \, \sqrt{g} \, \overline{\Psi}(x, z) \left( i{\Gamma}^{\ell} D_{\ell} - \mu \right) \Psi(x, z) \]

• Equation of motion:

\[ \left( i{\Gamma}^{\ell} D_{\ell} - \mu \right) \Psi(x, z) = 0 \]

\[ \left[ i \left( z\eta^{\ell m} \Gamma_{\ell} \partial_m + \frac{d}{2} \Gamma_z \right) + \mu R \right] \Psi(x^{\ell}) = 0 \]

• Solution (\( \mu R = \nu + 1/2 \))

\[ \Psi(z) = Cz^{5/2} \left[ J_{\nu}(z\mathcal{M})u_+ + J_{\nu+1}(z\mathcal{M})u_- \right] \]

• Hadronic mass spectrum determined from IR boundary conditions \( \psi_{\pm}(z = 1/\Lambda_{\text{QCD}}) = 0 \)

\[ \mathcal{M}^+ = \beta_{\nu,k} \Lambda_{\text{QCD}}, \quad \mathcal{M}^- = \beta_{\nu+1,k} \Lambda_{\text{QCD}} \]

with scale independent mass ratio

• Obtain spin-\( J \) mode \( \Phi_{\mu_1 \cdots \mu_{J-1/2}} \), \( J > \frac{1}{2} \), with all indices along 3+1 from \( \Psi \) by shifting dimensions
In the conformal limit fermionic spin-$\frac{1}{2}$ modes $\psi(\zeta)$ and spin-$\frac{3}{2}$ modes $\psi_\mu(\zeta)$ are two-component spinor solutions of the Dirac light-front equation

$$\alpha \Pi(\zeta) \psi(\zeta) = M \psi(\zeta),$$

where $H_{LF} = \alpha \Pi$ and the operator

$$\Pi_L(\zeta) = -i \left( \frac{d}{d\zeta} - \frac{L + 1}{2} \frac{1}{\zeta} \gamma_5 \right),$$

and its adjoint $\Pi_L^\dagger(\zeta)$ satisfy the commutation relations

$$\left[ \Pi_L(\zeta), \Pi_L^\dagger(\zeta) \right] = \frac{2L + 1}{\zeta^2} \gamma_5.$$
Note: in the Weyl representation \((i\alpha = \gamma_5\beta)\)

\[
\begin{align*}
i\alpha &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, & \beta &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, & \gamma_5 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.
\end{align*}
\]

Baryon: twist-dimension \(3 + L\) \((\nu = L + 1)\)

\[
O_{3+L} = \psi D_{\ell_1} \cdots D_{\ell_q} \psi D_{\ell_{q+1}} \cdots D_{\ell_m} \psi, \quad L = \sum_{i=1}^{m} \ell_i.
\]

Solution to Dirac eigenvalue equation with UV matching boundary conditions

\[
\psi(\zeta) = C \sqrt{\zeta} \left[ J_{L+1}(\zeta \mathcal{M}) u_+ + J_{L+2}(\zeta \mathcal{M}) u_- \right].
\]

Baryonic modes propagating in AdS space have two components: orbital \(L\) and \(L + 1\).

Hadronic mass spectrum determined from IR boundary conditions

\[
\psi_\pm (\zeta = 1/\Lambda_{QCD}) = 0,
\]

given by

\[
\mathcal{M}^+_{\nu,k} = \beta_{\nu,k} \Lambda_{QCD}, \quad \mathcal{M}^-_{\nu,k} = \beta_{\nu+1,k} \Lambda_{QCD},
\]

with a scale independent mass ratio.
Fig: Light baryon orbital spectrum for $\Lambda_{QCD} = 0.25$ GeV in the HW model. The 56 trajectory corresponds to $L$ even $P = +$ states, and the 70 to $L$ odd $P = -$ states.
We write the Dirac equation

\[(\alpha \Pi(\zeta) - \mathcal{M}) \psi(\zeta) = 0,\]

in terms of the matrix-valued operator \(\Pi\)

\[\Pi_\nu(\zeta) = -i \left( \frac{d}{d\zeta} - \frac{\nu + \frac{1}{2}}{\zeta} \gamma_5 - \kappa^2 \zeta \gamma_5 \right),\]

and its adjoint \(\Pi^\dagger\), with commutation relations

\[\left[ \Pi_\nu(\zeta), \Pi^\dagger_\nu(\zeta) \right] = \left( \frac{2\nu + 1}{\zeta^2} - 2\kappa^2 \right) \gamma_5.\]

Solutions to the Dirac equation

\[\psi_+(\zeta) \sim z^{\frac{1}{2} + \nu} e^{-\kappa^2 \zeta^2 / 2} L_n^\nu(\kappa^2 \zeta^2),\]

\[\psi_-(\zeta) \sim z^{\frac{3}{2} + \nu} e^{-\kappa^2 \zeta^2 / 2} L_n^{\nu+1}(\kappa^2 \zeta^2).\]

Eigenvalues

\[\mathcal{M}^2 = 4\kappa^2 (n + \nu + 1).\]
\(4\kappa^2\) for \(\Delta n = 1\)
\(4\kappa^2\) for \(\Delta L = 1\)
\(2\kappa^2\) for \(\Delta S = 1\)

\[\mathcal{M}^2\]

Parent and daughter 56 Regge trajectories for the \(N\) and \(\Delta\) baryon families for \(\kappa = 0.5\) GeV

Oberwölz
August 31, 2010

AdS/QCD and Light-Front Holography

Stan Brodsky
SLAC
E. Klempt et al.: $\Delta^*$ resonances, quark models, chiral symmetry and AdS/QCD

Space-Like Dirac Proton Form Factor

- Consider the spin non-flip form factors

\[ F_+(Q^2) = g_+ \int d\zeta J(Q, \zeta) |\psi_+(\zeta)|^2, \]

\[ F_-(Q^2) = g_- \int d\zeta J(Q, \zeta) |\psi_-(\zeta)|^2, \]

where the effective charges \( g_+ \) and \( g_- \) are determined from the spin-flavor structure of the theory.

- Choose the struck quark to have \( S^z = +1/2 \). The two AdS solutions \( \psi_+(\zeta) \) and \( \psi_-(\zeta) \) correspond to nucleons with \( J^z = +1/2 \) and \(-1/2\).

- For \( SU(6) \) spin-flavor symmetry

\[ F_1^p(Q^2) = \int d\zeta J(Q, \zeta) |\psi_+(\zeta)|^2, \]

\[ F_1^n(Q^2) = -\frac{1}{3} \int d\zeta J(Q, \zeta) \left[ |\psi_+(\zeta)|^2 - |\psi_-(\zeta)|^2 \right], \]

where \( F_1^p(0) = 1, \ F_1^n(0) = 0 \). 
• Scaling behavior for large $Q^2$: $Q^4 F_1^P(Q^2) \to \text{constant}$  

Proton $\tau = 3$

• Scaling behavior for large $Q^2$: $Q^4 F_1^m(Q^2) \to \text{constant}$  

Neutron $\tau = 3$

Spacelike Pauli Form Factor

From overlap of $L = 1$ and $L = 0$ LFWFs

Harmonic Oscillator Confinement
Normalized to anomalous moment

$\kappa = 0.49$ GeV

G. de Teramond, sjb

$F_2^p(Q^2)$

$Q^2$(GeV$^2$)
String Theory

AdS/CFT

AdS/QCD

Semi-Classical QCD / Wave Equations

Boost Invariant 3+1 Light-Front Wave Equations

Holography

Hd l

J = 0, 1, 1/2, 3/2 plus L

Hadron Spectra, Wavefunctions, Dynamics

Goal: First Approximant to QCD
Counting rules for Hard Exclusive
Scattering
Regge Trajectories

QCD at the Amplitude Level

Mapping of Poincare' and Conformal SO
(4,2) symmetries of 3+1 space
to AdS5 space

Conformal behavior at short distances
+ Confinement at large distance

Integrable!
Consider five-dim gauge fields propagating in AdS$_5$ space in dilaton background $\varphi(z) = \kappa^2 z^2$

$$S = -\frac{1}{4} \int d^4x \, dz \, \sqrt{g} \, e^\varphi(z) \, \frac{1}{g_5^2} G^2$$

Flow equation

$$\frac{1}{g_5^2(z)} = e^{\varphi(z)} \frac{1}{g_5^2(0)} \quad \text{or} \quad g_5^2(z) = e^{-\kappa^2 z^2} g_5^2(0)$$

where the coupling $g_5(z)$ incorporates the non-conformal dynamics of confinement

YM coupling $\alpha_s(\zeta) = g_{YM}^2(\zeta)/4\pi$ is the five dim coupling up to a factor: $g_5(z) \rightarrow g_{YM}(\zeta)$

Coupling measured at momentum scale $Q$

$$\alpha^{AdS}_s(Q) \sim \int_0^\infty \zeta \, d\zeta \, J_0(\zeta Q) \, \alpha^{AdS}_s(\zeta)$$

Solution

$$\alpha^{AdS}_s(Q^2) = \alpha^{AdS}_s(0) \, e^{-Q^2/4\kappa^2}.$$

where the coupling $\alpha^{AdS}_s$ incorporates the non-conformal dynamics of confinement