Light-Front Holography: 
a New Approach to Nonperturbative QCD

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40 years of development
of Quantum Chromodynamics

Gauge/Gravity Correspondence and QCD

- Recent developments inspired by the AdS/CFT correspondence [Maldacena (1998)] between gravity in AdS space and conformal field theories in physical space-time provide physical insights into non-perturbative QCD dynamics.

- Description of strongly coupled gauge theory using a dual gravity description in a higher dimensional space (holographic).

- Isomorphism of $SO(4,2)$ group of conformal transformations with generators $P^\mu, M^{\mu\nu}, K^\mu, D$, with the group of isometries of AdS$_5$, a space of maximal symmetry, negative curvature and a four-dim boundary: Minkowski space.

- Mapping of AdS gravity theory to boundary QFT quantized at fixed light-front time gives a precise relation between holographic wave functions in AdS space and the light-front wavefunctions describing the internal structure of hadrons.

- The gauge/gravity duality leads to a simple analytical frame-independent nonperturbative semiclassical approximation to the full light-front QCD Hamiltonian: “Light-Front Holography”
• AdS\(_5\) metric: 
\[
\frac{ds^2}{L_{\text{AdS}}} = \frac{R^2}{z^2} \left( \frac{\eta_{\mu\nu}dx^\mu dx^\nu}{L_{\text{Minkowski}}} - dz^2 \right)
\]

• A distance \(L_{\text{AdS}}\) shrinks by a warp factor \(z/R\) as observed in Minkowski space \((dz = 0)\):

\[
L_{\text{Minkowski}} \sim \frac{z}{R} L_{\text{AdS}}
\]

• Since the AdS metric is invariant under a dilatation of all coordinates \(x^\mu \rightarrow \lambda x^\mu, z \rightarrow \lambda z\), the variable \(z\) acts like a scaling variable in Minkowski space

• Short distances \(x_\mu x^\mu \rightarrow 0\) maps to UV conformal AdS\(_5\) boundary \(z \rightarrow 0\)

• Large confinement dimensions \(x_\mu x^\mu \sim 1/\Lambda_{\text{QCD}}^2\) maps to large IR region of AdS\(_5\), \(z \sim 1/\Lambda_{\text{QCD}}\), thus there is a maximum separation of quarks and a maximum value of \(z\)
Forms of Relativistic Dynamics

• Different possibilities to parametrize space-time [Dirac (1949)]

• Parametrizations differ by the hypersurface on which the initial conditions are specified. Each evolve with different “times” and has its own Hamiltonian, but should give the same physical results

• Forms of Relativistic Dynamics: dynamical vs. kinematical generators [Dirac (1949)]

• *Instant form*: hypersurface defined by $t = 0$, the familiar one

\[ H, K \text{ dynamical, } L, P \text{ kinematical} \]

• *Point form*: hypersurface is an hyperboloid

\[ P^\mu \text{ dynamical, } M^{\mu\nu} \text{ kinematical} \]

• *Front form*: hypersurface is tangent to the light cone at $\tau = t + z/c = 0$

\[ P^-, L^x, L^y \text{ dynamical, } P^+, P_\perp, L^z, K \text{ kinematical} \]

\[ P^\pm = P^0 \pm P^3 \]
Light-Front Quantization of QCD

- Express the hadron four-momentum generator $P = (P^+, P^-, P_{\perp})$ in terms of dynamical fields

\[
P^- = \frac{1}{2} \int dx^- d^2 x_{\perp} \bar{\psi}_+ \gamma^+ \frac{(i\nabla_{\perp})^2 + m^2}{i\partial^+} \psi_+ + \text{(interactions)},
\]

\[
P^+ = \int dx^- d^2 x_{\perp} \bar{\psi}_+ \gamma^+ i\partial^+ \psi_+ ,
\]

\[
P_{\perp} = \frac{1}{2} \int dx^- d^2 x_{\perp} \bar{\psi}_+ \gamma^+ i\nabla_{\perp} \psi_+ ,
\]

where the integrals are over the null plane $\tau = x^+ = x^0 + x^3$

- Construct LF Lorentz invariant Hamiltonian equation for the relativistic bound state

\[
[H_{LF}\psi(P)]\equiv \mathcal{M}^2 \psi(P) = H_{LF}\psi(P)
\]

with $H_{LF} \equiv P_{\mu}P^\mu = P^- P^+ - P^2_{\perp}$
Semiclassical Approximation to QCD in the Light Front

[GdT and S. J. Brodsky, PRL 102, 081601 (2009)]

• Compute $M^2$ from hadronic matrix element

\[ \langle \psi(P')|P_\mu P^\mu|\psi(P)\rangle = M^2 \langle \psi(P')|\psi(P)\rangle \]

• State $|\psi(P)\rangle$ is expanded in multi-particle Fock states $|n\rangle$ of the free LF Hamiltonian

\[ |\psi\rangle = \sum_n \psi_n|n\rangle \]

• Relevant variable for a two-parton system $\zeta^2 = x(1-x)b_\perp^2$

• To first approximation LF dynamics depend only on the invariant variable $\zeta$, and hadronic properties are encoded in the hadronic mode $\phi(\zeta)$ from

\[ \psi(x, \zeta, \varphi) = e^{iL \varphi} X(x) \frac{\phi(\zeta)}{\sqrt{2\pi\zeta}} \]

factor angular $\varphi$, longitudinal $X(x)$ and transverse mode $\phi(\zeta)$ ($P^+, P_\perp$ and $J_z$ commute with $P^-$)
• Ultra relativistic limit $m_q \to 0$ longitudinal modes $X(x)$ decouple \hspace{1cm} \left( L = |L^z| \right)$

$$\mathcal{M}^2 = \int d\zeta \phi^*(\zeta) \sqrt{\zeta} \left( -\frac{d^2}{d\zeta^2} - \frac{1}{\zeta} \frac{d}{d\zeta} + \frac{L^2}{\zeta^2} \right) \frac{\phi(\zeta)}{\sqrt{\zeta}} + \int d\zeta \phi^*(\zeta) U(\zeta) \phi(\zeta)$$

where the confining forces from the interaction terms are summed up in the effective potential $U(\zeta)$

• LF eigenvalue equation $P_\mu P^\mu |\phi\rangle = \mathcal{M}^2 |\phi\rangle$ is a LF wave equation for $\phi$

$$\left( -\frac{d^2}{d\zeta^2} - \frac{1 - 4L^2}{4\zeta^2} \right) \frac{\phi(\zeta)}{\sqrt{\zeta}} + \left. U(\zeta) \right|_{\text{confinement}} \phi(\zeta) = \mathcal{M}^2 \phi(\zeta)$$

• Effective relativistic and frame-independent LF Schrödinger equation: $U$ is instantaneous in LF time

• Eigenmodes $\phi(\zeta)$ determine the hadronic mass spectrum and represent the probability amplitude to find $n$-massless partons at transverse impact separation $\zeta$ within the hadron at equal LF time

• Semiclassical approximation to LF QCD does not account for particle creation and absorption
Light-Front Holographic Mapping of Wave Equations

Higher Spin Modes in AdS Space

• Spin-$J$ in AdS represented by totally symmetric rank $J$ tensor field $\Phi_{M_1 \ldots M_J}$

• Action for spin-$J$ field in AdS$_{d+1}$ in presence of dilaton background $\varphi(z)$ \((x^M = (x^\mu, z))\)

\[
S = \frac{1}{2} \int d^d x \, dz \sqrt{g} e^{\varphi(z)} \left( g^{NN'} g^{M_1 M_1'} \cdots g^{M_J M_J'} D_N \Phi_{M_1 \ldots M_J} D_{N'} \Phi_{M_1' \ldots M_J'} - \mu^2 g^{M_1 M_1'} \cdots g^{M_J M_J'} \Phi_{M_1 \ldots M_J} \Phi_{M_1' \ldots M_J'} + \cdots \right)
\]

where $D_M$ is the covariant derivative which includes parallel transport

• Physical hadron has plane-wave and polarization indices along $3+1$ physical coordinates

\[
\Phi_P(x, z)_{\mu_1 \ldots \mu_J} = e^{-i P \cdot x} \Phi(z)_{\mu_1 \ldots \mu_J}, \quad \Phi_{z \mu_2 \ldots \mu_J} = \cdots = \Phi_{\mu_1 \mu_2 \ldots z} = 0
\]

with four-momentum $P_\mu$ and invariant hadronic mass $P_\mu P^\mu = M^2$

• Find AdS wave equation for spin $J$-mode $\Phi_J = \Phi_{\mu_1 \ldots \mu_J}$ and all indices along $3+1$

\[
\left[ -z^{d-1-2J} \frac{1}{e^{\varphi(z)}} \partial_z \left( \frac{e^{\varphi(z)}}{z^{d-1-2J}} \partial_z \right) + \left( \frac{\mu R}{z} \right)^2 \right] \Phi_J(z) = M^2 \Phi_J(z)
\]
Upon substitution $z \rightarrow \zeta$ and $\phi_J(\zeta) \sim \zeta^{-3/2} + J e^{\varphi(z)/2} \Phi_J(\zeta)$ in AdS WE

$$\left[ -\frac{z^{d-1-2J}}{e^{\varphi(z)}} \frac{\partial}{\partial z} \left( \frac{e^{\varphi(z)}}{z^{d-1-2J}} \frac{\partial}{\partial z} \right) + \left( \frac{\mu R}{z} \right)^2 \right] \Phi_J(z) = M^2 \Phi_J(z)$$

find LFWE ($d = 4$)

$$\left( -\frac{d^2}{d\zeta^2} - \frac{1 - 4L^2}{4\zeta^2} + U(\zeta) \right) \phi_J(\zeta) = M^2 \phi_J(\zeta)$$

with

$$U(\zeta) = \frac{1}{2} \varphi''(z) + \frac{1}{4} \varphi'(z)^2 + \frac{2J - 3}{2z} \varphi'(z)$$

and $(\mu R)^2 = -(2 - J)^2 + L^2$

- AdS Breitenlohner-Freedman bound $(\mu R)^2 \geq -4$ equivalent to LF QM stability condition $L^2 \geq 0$

- Scaling dimension $\tau$ of AdS mode $\Phi_J$ is $\tau = 2 + L$ in agreement with twist scaling dimension of a two parton bound state in QCD and determined by QM stability condition
Bosonic Modes and Meson Spectrum

Soft wall model: linear Regge trajectories [Karch, Katz, Son and Stephanov (2006)]

- Dilaton profile $\varphi(z) = +\kappa^2 z^2$
- Effective potential: $U(z) = \kappa^4 \zeta^2 + 2\kappa^2 (J - 1)$
- Normalized eigenfunctions $\langle \phi | \phi \rangle = \int d\zeta |\phi(z)|^2 = 1$

$$\phi_{n,L}(\zeta) = \kappa^{1+L} \sqrt{\frac{2n!}{(n+L)!}} \zeta^{1/2+L} e^{-\kappa^2 \zeta^2/2} L_n^L(\kappa^2 \zeta^2)$$

- Eigenvalues $\mathcal{M}_{n,L,S}^2 = 4\kappa^2 \left( n + \frac{J+L}{2} \right)$

LFWFs $\phi_{n,L}(\zeta)$ in physical space-time: L) orbital modes and R) radial modes
\( J = L + S, \ I = 1 \) meson families \( \mathcal{M}_{n, L, S}^2 = 4\kappa^2 (n + L + S/2) \)

- \( 4\kappa^2 \) for \( \Delta n = 1 \)
- \( 4\kappa^2 \) for \( \Delta L = 1 \)
- \( 2\kappa^2 \) for \( \Delta S = 1 \)

I=1 orbital and radial excitations for the \( \pi \) (\( \kappa = 0.59 \) GeV) and the \( \rho \)-meson families (\( \kappa = 0.54 \) GeV)

- Triplet splitting for the \( I = 1, \ L = 1, \ J = 0, 1, 2 \), vector meson \( a \)-states

\[ \mathcal{M}_{a_2(1320)} > \mathcal{M}_{a_1(1260)} > \mathcal{M}_{a_0(980)} \]
Multi-Partonic States and Cluster Decomposition

- Proton state $|\psi(P)\rangle$

$$|\psi\rangle = \sum_n \psi_n |n\rangle, \quad |n\rangle = \{|uud\rangle, |uudg\rangle, |uud\bar{q}q\rangle, \cdots \}$$

- Extension of LF holography to arbitrary $n$ follows from the $x$-weighted definition of the transverse impact variable of the $n-1$ spectator system

[S. J. Brodsky and GdT, PRL 96, 201601 (2006)]

$$\zeta = \sqrt{\frac{x}{1-x}} \left| \sum_{j=1}^{n-1} x_j b_{\perp j} \right|$$

where $x = x_n$ is the longitudinal momentum fraction of the active quark

- Same multiplicity of states for mesons and baryons!
Fermionic Modes and Baryon Spectrum

[LF Hard wall model: GdT and S. J. Brodsky, PRL 94, 201601 (2005)]

- Excitation spectrum of nucleon represents formidable challenge to LQCD due to enormous computational complexity beyond ground state configuration

- LF Holographic nucleon modes

\[
\psi_+ (\zeta)_{n,L} = \kappa^{2+L} \sqrt{\frac{2n!}{(n+L)!}} \zeta^{3/2+L} e^{-\kappa^2 \zeta^2/2} L_n^{L+1} (\kappa^2 \zeta^2)
\]

\[
\psi_- (\zeta)_{n,L} = \kappa^{3+L} \frac{1}{\sqrt{n+L+2}} \sqrt{\frac{2n!}{(n+L)!}} \zeta^{5/2+L} e^{-\kappa^2 \zeta^2/2} L_n^{L+2} (\kappa^2 \zeta^2)
\]

- Normalization

\[
\int d\zeta \psi_+^2 (\zeta) = \int d\zeta \psi_-^2 (\zeta) = 1
\]

- Eigenvalues

\[
\mathcal{M}^{2(+)}_{n,L,S} = 4\kappa^2 \left( n + L + S/2 + 3/4 \right)
\]

\[
\mathcal{M}^{2(-)}_{n,L,S} = 4\kappa^2 \left( n + L + S/2 + 5/4 \right)
\]
• Gap scale $4\kappa^2$ determines trajectory slope and spectrum gap between plus-parity spin-$\frac{1}{2}$ and minus-parity spin-$\frac{3}{2}$ nucleon families for the branch solutions $L + 1 = \mu R - 1/2$ and $L + 1 = \mu R + 1/2$.

Plus-minus nucleon spectrum gap for $\kappa = 0.49$ GeV.
Orbital and radial excitations for positive parity $N$ and $\Delta$ baryon families ($\kappa = 0.49 - 0.51$ GeV)

Same results for the $\Delta$ spectrum: H. Forkel, M. Beyer and T. Frederico, JHEP 0707, 077 (2007)
Light-Front Holographic Mapping of Current Matrix Elements

[S. J. Brodsky and GdT, PRL 96, 201601 (2006)]

• EM transition matrix element in QCD: local coupling to pointlike constituents

\[ \langle \psi(P') | J^\mu | \psi(P) \rangle = (P + P')^\mu F(Q^2) \]

where \( Q = P' - P \) and \( J^\mu = e_q \bar{q} \gamma^\mu q \)

• EM hadronic matrix element in AdS space from coupling of external EM field propagating in AdS with extended mode \( \Phi(x, z) \)

\[ \int d^4x \, dz \sqrt{g} \, A^M(x, z) \Phi_P^* (x, z) \overleftrightarrow{\partial}_M \Phi_P(x, z) \]

\[ \sim (2\pi)^4 \delta^4 \left( P' - P \right) \epsilon_\mu \left( P + P' \right)^\mu F(Q^2) \]

• How to recover hard pointlike scattering at large \( Q \) out of soft collision of extended objects?

[Polchinski and Strassler (2002)]

• Mapping of \( J^+ \) elements at fixed light-front time: \( \Phi_P(z) \Leftrightarrow |\psi(P)\rangle \)
• Compare with electromagnetic FF in LF QCD for arbitrary $Q$. Expressions can be matched only if LFWF is factorized

$$\psi(x, \zeta, \varphi) = e^{iM\varphi}X(x)\frac{\phi(\zeta)}{\sqrt{2\pi\zeta}}$$

• Find

$$X(x) = \sqrt{x(1-x)}, \quad \phi(\zeta) = \left(\frac{\zeta}{R}\right)^{-3/2}\Phi(\zeta), \quad z \to \zeta$$

• Form factor in soft-wall model expressed as $\tau - 1$ product of poles along vector radial trajectory (twist $\tau = N + L$) [Brodsky and GdT, Phys.Rev. D77 (2008) 056007]

$$F_{\tau}(Q^2) = \frac{1}{\left(1 + \frac{Q^2}{M_{\rho}^2}\right)\left(1 + \frac{Q^2}{M_{\rho'}^2}\right)\cdots\left(1 + \frac{Q^2}{M_{\rho^{\tau-2}}^2}\right)}$$

where $M_{\rho_{2n}}^2 \to 4\kappa^2(n + 1/2)$

• Analytical form $F(Q^2)$ incorporates correct scaling from constituents and mass gap from confinement
Nucleon Elastic Form Factors

\[
Q^4 F^p_1 (Q^2) \quad (\text{GeV}^4)
\]

\[
Q^4 F^n_1 (Q^2) \quad (\text{GeV}^4)
\]

\[
F^p_2 (Q^2)
\]

\[
F^n_2 (Q^2)
\]
Nucleon Transition Form Factors

\[ F_{1N \rightarrow N^*}^p(Q^2) = \frac{\sqrt{2}}{3} \frac{Q^2}{M_{\rho}^2} \left( 1 + \frac{Q^2}{M_{\rho}^2} \right) \left( 1 + \frac{Q^2}{M_{\rho'}^2} \right) \left( 1 + \frac{Q^2}{M_{\rho''}^2} \right). \]
Flavor Decomposition of Elastic Nucleon Form Factors


- Proton SU(6) WF:
  \[ F_{u,1}^p = \frac{5}{3} G_+ + \frac{1}{3} G_- \]
  \[ F_{d,1}^p = \frac{1}{3} G_+ + \frac{2}{3} G_- \]

- Neutron SU(6) WF:
  \[ F_{u,1}^n = \frac{1}{3} G_+ + \frac{2}{3} G_- \]
  \[ F_{d,1}^n = \frac{5}{3} G_+ + \frac{1}{3} G_- \]

\[ G_+(Q^2) = \frac{1}{\left(1 + \frac{Q^2}{M_{\rho}^2}\right)\left(1 + \frac{Q^2}{M_{\rho'}^2}\right)} \]

and

\[ G_-(Q^2) = \frac{1}{\left(1 + \frac{Q^2}{M_{\rho}^2}\right)\left(1 + \frac{Q^2}{M_{\rho'}^2}\right)\left(1 + \frac{Q^2}{M_{\rho''}^2}\right)} \]