Extended Renormalization Group and Scale Setting Using the Principle of Maximum Conformality

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SLAC-PUB-14774; arXiv:1111.6175

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The New Spring Comes With The Shining glow

旭日东升迎新春

Outline

1. Extended Renormalization Group Equation

- How to deal with the Scale- and Scheme- dependence consistently ? Extended coupling constant, E-RGE
- Solution for the Scale-equation of E-RGE
- Special case, the 't Hooft scheme $\Lambda^{'tH}_{QCD}$

2. PMC/BLM-scale setting up to NNLO / its aim?

- BLM-mechanism up to NNLO
- BLM-PMC correspondence
- Applications

A Standard Procedure

3. Summary and Outlook

Background

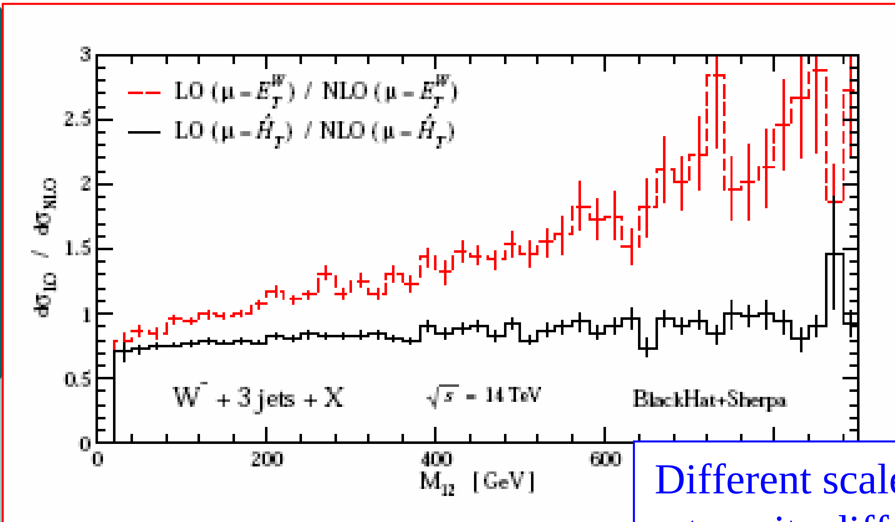
Any pQCD calculable quantity ρ

$$\rho = C_0 \left(\frac{\alpha_s(\mu)}{\pi}\right)^n + C_1(\mu) \left(\frac{\alpha_s(\mu)}{\pi}\right)^{n+1} + C_2(\mu) \left(\frac{\alpha_s(\mu)}{\pi}\right)^{n+2} + \cdots$$

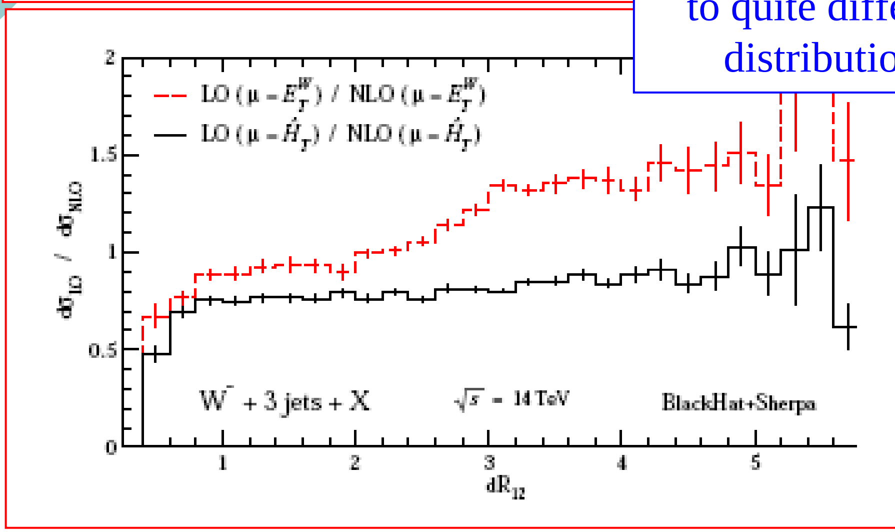
All physical predictions in QCD should in principle be invariant under any choice of renormalization scale and scheme. But at any finite order, the use of different scales and schemes may lead to quite different theoretical predictions; **which is more reliable ?**

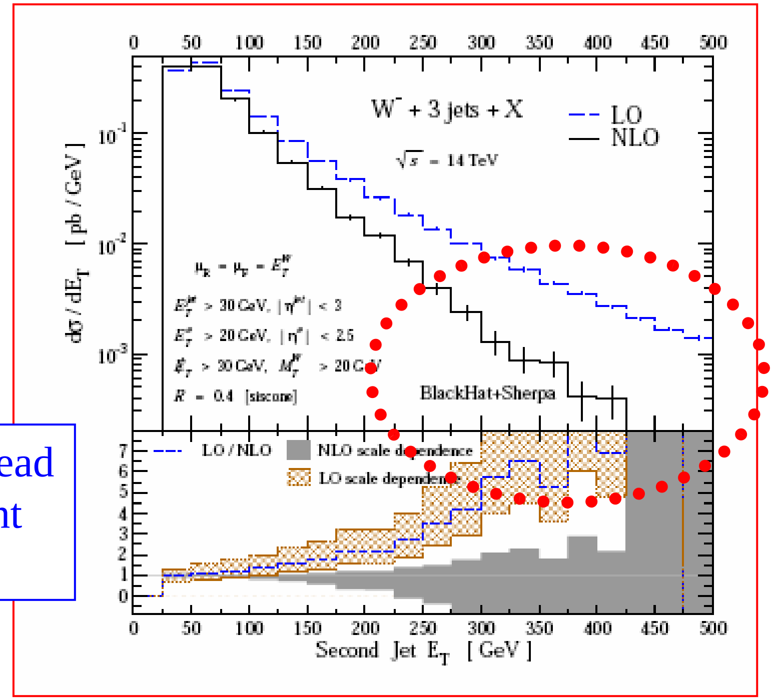
And we hope it is convergent enough, since higher-order calculation is not easy; **how to improve the convergence ?**

example



Different scales lead to quite different distributions





Because of uncancelled large logarithms

Also there are large renormalon Terms as ($n!\beta^m\alpha_s^n$), which may provide large contributions

Improper scale gives large K-factor, or even lead to negative contributions
arXiv:0907.1984

practical way to deal with the scale for MC simulation

$a \to bc$

Effective mass

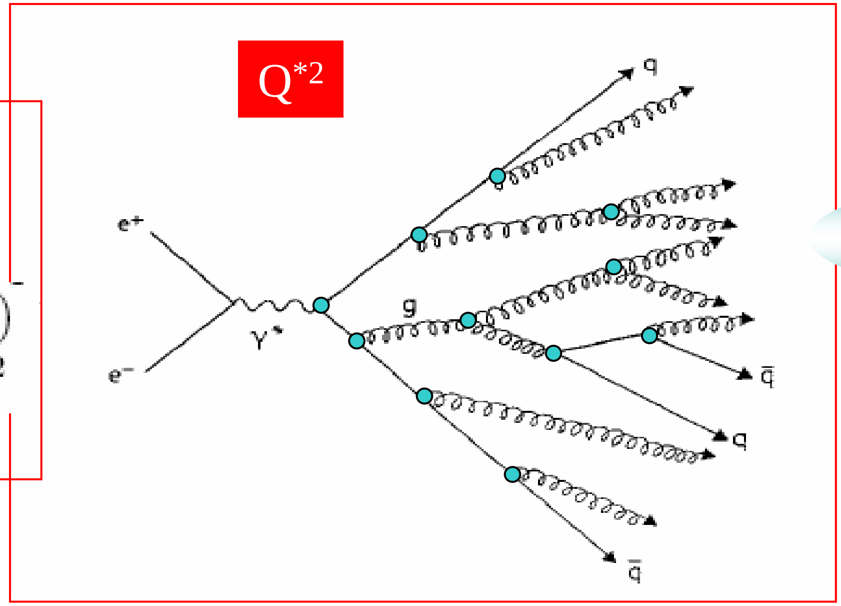
$Q^2 = m_a^2$

$Q^2 \approx m^2/(2z(1-z))$

$Q^2 = p_\perp^2 \approx z(1-z)m^2$

typical choices

PYTHIA,HERWIG



parton shower

PYTHIA 6.4
Physics and Manual
hep-ph/0603175

A typical scale Q^2 has been assigned at each interaction point

if using only **one or several effective scales** for the whole process, the scale puzzle would be more serious.

A quick review

FAC: Fastest Apparent Convergence (FAC): chooses the renormalization scale such that the NLO coefficient vanishes

Comparing: Optimized perturbation theory – minimize the higher-order contributions – **PMS**

P.M. Stevenson, Phys. Rev. D23 (1981) 2916

S.J. Brodsky, G.P. Lepage and P.B. Macke[nzie], Phys.Rev. D28, 228(1983).

Optimal procedure for obtaining precise QCD predictions is **to choose the (optimal) scale so that the result is scheme independent even at fixed order !**

How to set proper scale to achieve the goal ?

to ensure the Self-Consistence of a scale-setting method, we think it should satisfy several general properties.

First-belief → Existence of such optimal scale μ

S.J. Brodsky, SLAC-PUB-6304 (1993); S.J. Brodsky and H.J. Lu, SLAC-PUB-6000, arXiv:9211308.

A quick review

second

Reflexivity. Given a $\alpha_s(\mu)$ specified at a scale μ, we can express it in terms of itself but specified at another scale μ' **up to all orders**,

$$\alpha_s(\mu) = \alpha_s(\mu') + f_1(\mu,\mu')\alpha_s^2(\mu') + f_2(\mu,\mu')\alpha_s^3(\mu') + \cdots,$$

$$\frac{\partial \alpha_s(\mu)}{\partial \mu'} = 0$$

where $f_1(\mu,\mu') = -\frac{\beta_0}{4\pi}\ln(\mu^2/\mu'^2)$ and $\beta_0 = 11 - 2N_f/3$. If a scale-setting scheme is self-consistent, it should choose the **unique value** $\mu' = \mu$ on the right-hand side.

Note

Such an argument can be found in SJB. And a demonstration can be done with the help of E-RGE, i.e. to demonstrate the above infinite Taylor series does not depends on μ'

$$\begin{aligned} a_S(\tau_S, \{c_i^S\}) &= a(\tau_R + \bar{\tau}, \{c_i^R + \bar{c}_i\}) \\ &= a(\tau_R, \{c_i^R\}) + \left(\frac{\partial a}{\partial \tau}\right)_R \bar{\tau} + \sum_i \left(\frac{\partial a}{\partial c_i}\right)_R \bar{c}_i \\ &+ \frac{1}{2!}\left[\left(\frac{\partial^2 a}{\partial \tau^2}\right)_R \bar{\tau}^2 + 2\left(\frac{\partial^2 a}{\partial \tau \partial c_i}\right)_R \bar{\tau}\bar{c}_i + \sum_{i,j}\left(\frac{\partial^2 a}{\partial c_i \partial c_j}\right)_R \bar{c}_i \bar{c}_j\right] \\ &+ \frac{1}{3!}\left[\left(\frac{\partial^3 a}{\partial \tau^3}\right)_R \bar{\tau}^3 + \cdots\right] + \cdots, \end{aligned}$$

$$\frac{\partial a_S(\tau_S, \{c_i^S\})}{\partial \tau_R} \equiv 0$$

S.J. Brodsky, SLAC-PUB-6304 (1993); S.J. Brodsky and H.J. Lu, SLAC-PUB-6000, arXiv:9211308.

A quick review

third

Symmetry. Given two different coupling constants $\alpha_{s1}(\mu_1)$ and $\alpha_{s2}(\mu_2)$, we can express one of them in terms of the other:

$$\alpha_{s1}(\mu_1) = \alpha_{s2}(\mu_2) + r_{12}(\mu_1, \mu_2)\alpha_{s2}^2(\mu_2) + \cdots, \tag{33}$$

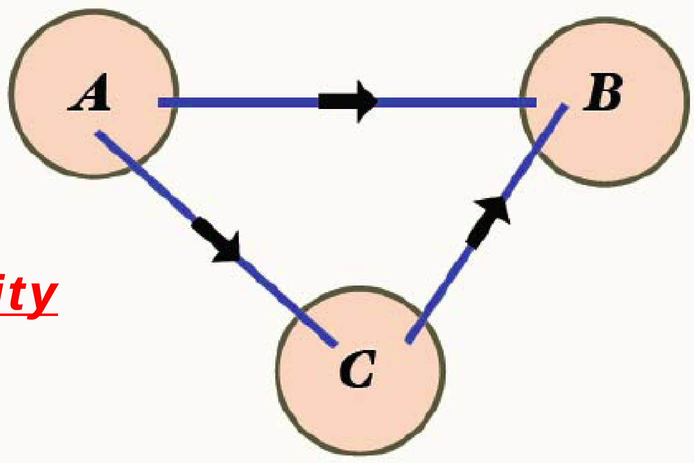
$$\alpha_{s2}(\mu_2) = \alpha_{s1}(\mu_1) + r_{21}(\mu_2, \mu_1)\alpha_{s1}^2(\mu_1) + \cdots. \tag{34}$$

If a scale-setting method gives $\mu_2 = \lambda_{21}\mu_1$ for the first series and $\mu_1 = \lambda_{12}\mu_2$ for the second series, then this scale-setting method is said to be *symmetric* if $\lambda_{12}\lambda_{21} = 1$.

fourth

Relation of observables must be independent of intermediate scheme

Transitivity



A → C   C → B   identical to   A → B

any physical observable can be used to define an effective coupling constant

S.J. Brodsky, SLAC-PUB-6304 (1993); S.J. Brodsky and H.J. Lu, SLAC-PUB-6000, arXiv:9211308.

A quick review

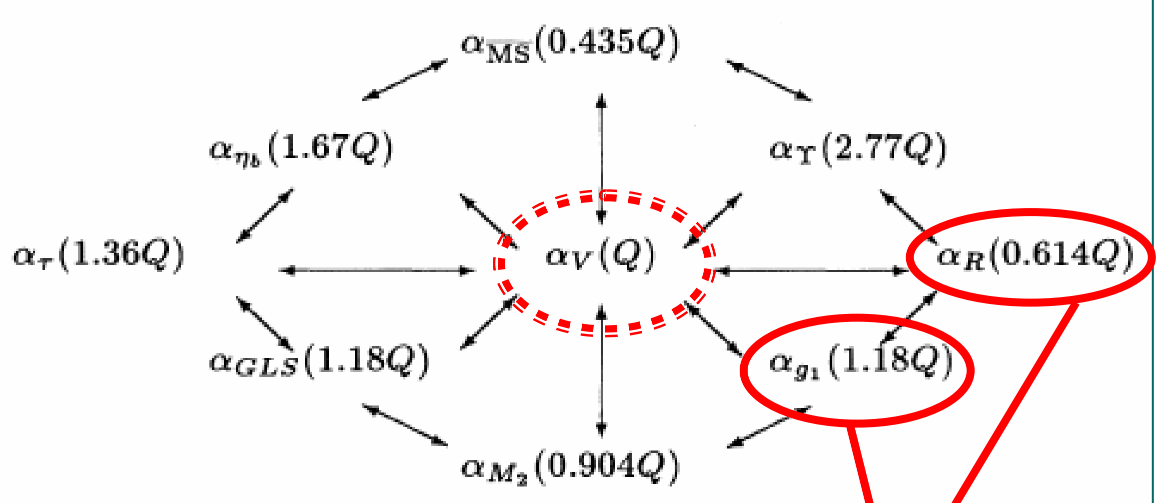
Commensurate relation among different α_s

S.J. Brodsky and H.J. Lu, Phys.Rev. D51, 3652(1995);

any physical observable can be used to define an effective coupling constant

$$R_{e^+e^-}(Q) \equiv R^0_{e^+e^-}(Q)\left[1+\frac{\alpha_s^R(Q)}{\pi}\right]$$

TABLE I. Leading order commensurate scale relations.



Transitivity

deduce

Generalized Crewther Relation

$$\left[1+\frac{\alpha_R(s^*)}{\pi}\right]\left[1-\frac{\alpha_{g_1}(q^2)}{\pi}\right]=1$$

Total hadronic e+e- annihilation

Bjorken Sum rule polarized electroproduction

$$R_{e^+e^-}(Q^2) \equiv 3\sum_{\text{flavors}} e_q^2\left[1+\frac{\alpha_R(Q)}{\pi}\right].$$

$$\int_0^1 dx\,[g_1^{ep}(x,Q^2)-g_1^{en}(x,Q^2)] \equiv \frac{1}{3}\left|\frac{g_A}{g_V}\right|\left[1-\frac{\alpha_{g_1}(Q)}{\pi}\right]$$

S.J. Brodsky and P. Huet, Phys.Lett. **B417**, 145-153 (1998).

fixed $\alpha = C_F \alpha_s$

$n_\ell = n_F / C_F$

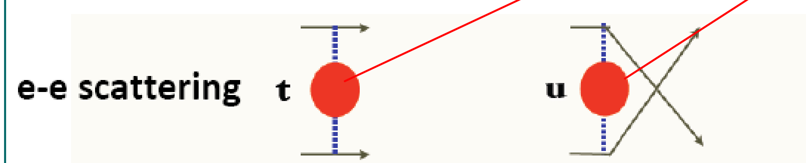
Moreover,

pQCD becomes an Abelian theory at **$N_c \to 0$**, so QCD scale-setting must agree with that of QED in this limit.

Such a comparison is useful,
since there is no renormalization scale ambiguity in QED

In the standard **Gell-Mann-Low** scheme for QED, the (optimal) renormalization scale is simply **the virtuality of virtual photon**, which naturally sums up all the vacuum polarization corrections and is unique and unambiguous

$$\mathcal{M}_{ee\to ee}(++;++) = \frac{8\pi s}{t}\,\alpha(t) + \frac{8\pi s}{u}\,\alpha(u)$$



Dressed photon propagator

$$\alpha(t) = \frac{\alpha(t_0)}{1 - \Pi(t, t_0)}$$

M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954)

S.J. Brodsky, SLAC-PUB-6304 (1993); S.J. Brodsky and H.J. Lu, SLAC-PUB-6000, arXiv:9211308.

PMS breaks Transitivity and Reflexivity

The reason (why BLM/PMC is useful) is clear

Basic features of BLM/PMC

- It satisfies all the above properties: Existence, Unitary, Transitivity, Reflexivity.
- All non-conformal and scheme-dependent β-terms in the perturbative series are summed into the running coupling. The result is conformal, and it is scheme-independent due to the proper scale-displacement in α_s.
- The active flavors n_f in β-function is correctly determined.
- Renormalons growing as ($n!\ \beta^m\ \alpha_s^n$) are avoided.
- The PMC method agrees with the standard QED results in the Nc -> 0 limit.

As a summary,

the above points show that to provide reliable QCD predictions, we need :

1) **A proper way to discuss the Scale- and Scheme-Dependence of the coupling constant (E-RGE; to get right behavior of the coupling constant)**

2) **An effective way to set scales for the coupling constant to make the prediction scheme-independent at fixed order (PMC/BLM; procedures to set scale for coupling constant)**

1. Extended Renormalization Group Equations

Naïve truncated series to know α_s at different scale --- not reliable !

$$\alpha_S(P) = \alpha_R(Q) + f(P,Q)\,\alpha_R^2(Q)$$

A way out for scale-dependence

Why it is better and useful ?

The scale is changed along the evolution trajectory with a continuous fashion, thus avoiding the presence of dissimilar scales and large expansion coefficients

Conventional RGE

$$\frac{d}{d\ln\mu^2}\left(\frac{\alpha_s^R(\mu)}{4\pi}\right) = -\sum_{i=0}^{\infty}\beta_i^R\left(\frac{\alpha_s^R(\mu)}{4\pi}\right)^{i+2}$$

universal

$\beta_0,\ \beta_1$

$$a^R = \beta_1\alpha_s^R/(4\pi\beta_0)$$

$$\tau = \frac{\beta_0^2}{\beta_1}\ln\mu^2$$

$$\frac{da^R}{d\tau} = -(a^R)^2\left[1 + a^R + c_2^R(a^R)^2 + c_3^R(a^R)^3 + \cdots\right]$$

Simpler form

Each scheme leads to different c_i^R, and vice versa

Can we discuss the uncertainty of c_i^R in a consistent way as that of the scale ?

Extended RGE !

S.J. Brodsky and H.J. Lu, Phys.Rev. D**51**, 3652(1995).
G. Grunberg, Phys.Rev. D**46**, 2228(1992).

Equivalent to usual RGE

universal coupling constant $a(\tau, \{c_i\})$

$$a^R(\tau_R) = a(\tau_R, \{c_i^R\})$$

$$\beta(a, \{c_i\}) = \frac{\partial a}{\partial \tau} = -a^2\left[1 + a + c_2 a^2 + c_3 a^3 + \cdots\right]$$

Scale equation

and

$$\beta_n(a, \{c_i\}) = \frac{\partial a}{\partial c_n} = -\beta(a, \{c_i\}) \int_0^a \frac{x^{n+2} dx}{\beta^2(x, \{c_i\})}$$

Scheme equations

Useful for a reliable error analysis on higher order

n+1 loop

New Results

Solution for the scale-equation up to the four-loop level

$$\left(\frac{\beta_0^2}{\beta_1}\ln\frac{\mu^2}{\mu_0^2}\right) = \int_{a(\tau_0,\{c_i\})}^{a(\tau,\{c_i\})} \frac{da}{\beta(a,\{c_i\})}$$

where $\tau_0 = (\beta_0^2/\beta_1)\ln\mu_0^2$ with μ_0 stands for an initial scale. Up to $\mathcal{O}(a^3)$, it leads to

Convenient way ⟹ $L = (\beta_0^2/\beta_1)\ln(\mu^2/\Lambda^2)$

$$L = C + \frac{1}{a} + \ln a + (c_2 - 1)\,a + \frac{c_3 - 2c_2 + 1}{2}a^2 + \mathcal{O}(a^3)$$

Λ the **asymptotic scale parameter**, its value is correlated with the integration parameter C.

Special case

$$\{\bar{c}_i\} \equiv \{0\}$$

'‘t Hooft scheme

$$L^{'tH} = (\beta_0^2/\beta_1)\ln(\mu^2/\Lambda^{'tH\,2})$$

$$L^{'tH} \equiv \frac{1}{a^{'tH}} + \ln\left(\frac{a^{'tH}}{1+a^{'tH}}\right)$$

‘t Hooft scale is the pole of this equation

‘t Hooft scale associated with *R*-scheme $\Lambda_R^{'tH}$

$$a^R(\mu) = a(2\beta_0^2/\beta_1\ln(\mu/\Lambda_R^{'tH}),\{c_i^R\}) \qquad a^{'tH-R}(\mu) = a(2\beta_0^2/\beta_1\ln(\mu/\Lambda_R^{'tH}),\{0\})$$

$$\Lambda_R^{'tH} = \exp\left(\frac{\beta_1}{2\beta_0^2}\mathcal{C}_R\right)\Lambda_R.$$

by choosing $\mathcal{C}_{\overline{MS}} = \ln\beta_0^2/\beta_1$,

$$\Lambda_{\overline{MS}}^{'tH} = \left(\frac{\beta_1}{\beta_0^2}\right)^{-\beta_1/2\beta_0^2}\Lambda_{\overline{MS}},$$

W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys.Rev. D**18**, 3998(1978). ‘$\overline{MS}$’
W. Furmanski and R. Petronzio, Z.Phys. C**11**, 293(1982).

‘t Hooft solution $\quad a_H \sim -\dfrac{\beta_0^2}{\beta_1(1+W_{-1}(z_w(\mathrm{L})))}$

$$L = \mathcal{C} + \frac{1}{a} + \ln a + (c_2 - 1)\, a + \frac{c_3 - 2c_2 + 1}{2} a^2 + \mathcal{O}(a^3)$$

Scale-equation to be solved iteratively

- Setting $a = \frac{1}{L}$ to cancel the L^1-term. And we can find the coefficient L^0
- Setting $a = \frac{1}{L} + \frac{c_2}{L^2}$ to cancel the L^0-term. And we can find the coefficient for L^{-1}
- Setting $a = \frac{1}{L} + \frac{c_2}{L^2} + \frac{c_3}{L^3}$ to cancel the L^{-1}-term. And we can find the coefficient for L^{-2}
- Setting $a = \frac{1}{L} + \frac{c_2}{L^2} + \frac{c_3}{L^3} + \frac{c_4}{L^4}$ to cancel the L^{-2}-term. And the final renormalization equation is of accuracy $\mathcal{O}(1/L^3)$, which is rightly our present required accuracy.

Final four-loop formulae

$$a = \frac{1}{L} + \frac{1}{L^2}(\mathcal{C} - \ln L) + \frac{1}{L^3}\left[\mathcal{C}^2 + \mathcal{C} + c_2 - (2\mathcal{C} - \ln L + 1)\ln L - 1\right] + \frac{1}{L^4}\left\{\mathcal{C}\left(\mathcal{C}^2 + \frac{5}{2}\mathcal{C} + 3c_2 - 2\right) - \frac{1 - c_3}{2} - \left[3\mathcal{C}^2 + 5\mathcal{C} + 3c_2 - 2 - \left(3\mathcal{C} - \ln L + \frac{5}{2}\right)\ln L\right]\ln L\right\} + \mathcal{O}\left(\frac{1}{L^5}\right)$$

Only gives four-loop formulae for a particular MSbar $\mathcal{C}$ ← K.G. Chetyrkin, B.A. Kniehl and M. Steinhauser, Phys.Rev.Lett. **79**, 2184(1997).

2. BLM/PMC-scale setting up to NNLO

- It satisfies all the above properties: Existence, Unitary, Transitivity, Reflexivity.
- All non-conformal and scheme-dependent β-terms in the perturbative series are summed into the running coupling. The result is conformal, and then scheme-independent.
- The active flavors n_f in β-function is correctly determined.
- Renormalons growing as $(n!\ \beta^m\ \alpha_s^n)$ are avoided.
- The PMC method agrees with the standard QED results in the Nc -> 0 limit.

LO BLM/PMC scale setting can be found in SJB

S.J. Brodsky, G.P. Lepage and P.B. Mackenzie, Phys.Rev. D**28**, 228(1983).

S.J. Brodsky and L.D. Giustino, arXiv: 1107.0338.

Main points for setting BLM scales

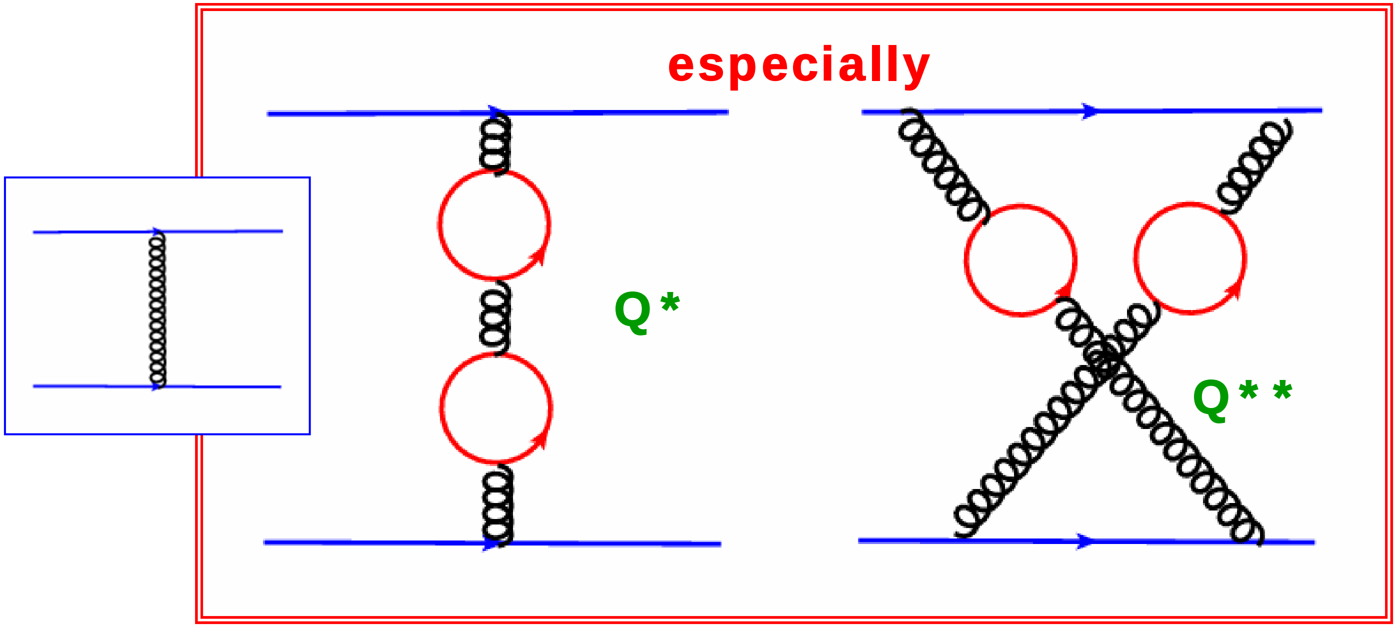
scales can be set up in a general scheme-independent way

- We shift the initial renormalization scale Q into **effective ones** **until we fully absorb those higher-order terms with n_f-dependence into the running coupling**.

"We need to set at least one effective scale at each order"

"Different terms at the same perturbative order may contribute to different scales Q*, Q, and etc., which depends on how these n_f-terms come from."**

- Order-by-order, we set **different renormalization scales for the perturbative series**. The remaining terms are free of β and are identical to that of **a conformal theory.**



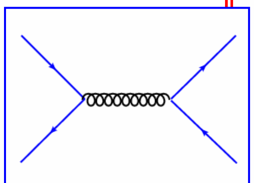
Another way: **a unified effective scale Q*** is used for all orders

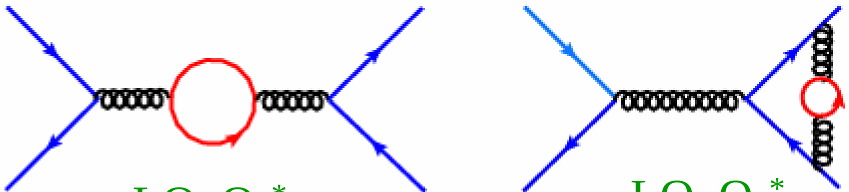
G. Grunberg and A.L. Kataev, Phys.Lett. B**279**, 352(1992).

S.V. Mikhailov, JHEP **0706**, 009(2007).

there is no compelling reason why we should set it in such a way

Especially in the threshold region





LO Q_1^* LO Q_2^*

$$\frac{x}{1-e^{-x}}$$

$$\left(1+\frac{\alpha(s\beta^2)\,\pi}{4\beta}\right)$$

$$\frac{\pi}{\beta}-\text{Coulomb terms}$$

Instantaneous coulomb potential

Sommerfeld's rescattering formula

Hard transverse gluon exchange

Another way: **a unified effective scale Q*** is used for all orders

G. Grunberg and A.L. Kataev, Phys.Lett. B**279**, 352(1992).

S.V. Mikhailov, JHEP **0706**, 009(2007).

there is no compelling reason why we should set it in such a way

To set the BLM scales up to NNLO, the starting point

$$\rho = r_0\Big[a_s^n(Q) + (A_1 + A_2 n_f)a_s^{n+1}(Q) + (B_1 + B_2 n_f + B_3 n_f^2)a_s^{n+2}(Q) + (C_1 + C_2 n_f + C_3 n_f^2 + C_4 n_f^3)a_s^{n+3}(Q) + \cdots\Big]$$

free of $a_s = \left(\frac{\alpha_s}{\pi}\right)$

first

to set the effective scale Q^* at LO

$$\rho = r_0\Big[a_s^n(Q^*) + \widetilde{A}_1 a_s^{n+1}(Q^*) + (\widetilde{B}_1 + \widetilde{B}_2 n_f)a_s^{n+2}(Q^*) + (\widetilde{C}_1 + \widetilde{C}_2 n_f + \widetilde{C}_3 n_f^2)a_s^{n+3}(Q^*) + \cdots\Big]. \quad (11)$$

second

The second step is to set the effective scale Q^{**} at NLO

$$\rho = r_0\Big[a_s^n(Q^*) + \widetilde{A}_1 a_s^{n+1}(Q^{**}) + \widetilde{\widetilde{B}}_1 a_s^{n+2}(Q^{**}) + (\widetilde{\widetilde{C}}_1 + \widetilde{\widetilde{C}}_2 n_f)a_s^{n+3}(Q^{**}) + \cdots\Big], \quad (12)$$

standard procedures

LO NLO NNLO

final

and the final step is to set the effective scale Q^{***} at NNLO

$$\rho = r_0 \left[a_s^n(Q^*) + \widetilde{A}_1 a_s^{n+1}(Q^{**}) + \widetilde{\widetilde{B}}_1 a_s^{n+2}(Q^{***}) + \widetilde{\widetilde{\widetilde{C}}}_1 a_s^{n+3}(Q^{***}) + \cdots \right] . \quad (13)$$

At least three-effective scales

$$\tilde{A}_1 = A_1 + \frac{33}{2} A_2 \, , \; \tilde{\tilde{B}}_1 = \tilde{B}_1 + \frac{33}{2} \tilde{B}_2 \, , \; \tilde{\tilde{\tilde{C}}}_1 = \tilde{\tilde{C}}_1 + \frac{33}{2} \tilde{\tilde{C}}_2$$

$$\tilde{B}_1 = \frac{1}{4n} \left[1089(n+1) A_2^2 + 153 n A_2 + 66(n+1) A_1 A_2 + (4B_1 - 1089 B_3) n \right]$$

$$\tilde{B}_2 = \frac{-1}{4n} \left[66(n+1) A_2^2 + 19 n A_2 + 4(n+1) A_1 A_2 - 4n(B_2 + 33 B_3) \right]$$

$$\tilde{C}_1 = \frac{1}{64 A_2 n^2} \Big[-40392 C_4 n^3 + 143748 A_2{}^4 (3 + 5n + 2n^2) + 8 A_2 n^2 (8C_1 + 35937 C_4 + 5049 B_3 n) - 13464 A_2{}^3 n (n^2 - 3n - 7) + 72 A_1 A_2 (1+n)(34 A_2 n - 242 B_3 n + 121 A_2{}^2 (3 + 2n)) + 3 A_2{}^2 n (2857 n + 352 B_1 (2+n) - 95832 B_3 (3 + 2n)) \Big]$$

$$\tilde{C}_2 = \frac{1}{192 A_2 n^2} \Big[22392 C_4 n^3 - 52272 A_2{}^4 (3 + 5n + 2n^2)(3 + 2n) - 24 A_2 n^2 (-8C_2 + 6534 C_4 + 933 B_3 n) - 48 A_1 A_2 (1+n)(19 A_2 n - 132 B_3 n + 66 A_2{}^2 (3 + 2n)) + A_2{}^2 n (-5033 n - 192 B_1 (2+n) + 3168 B_2 (2+n) + 52272 B_3 (8 + 5n)) + 24 A_2{}^3 n (-1871 + n(-627 + 311 n)) \Big]$$

$$\tilde{C}_3 = \frac{1}{576 A_2 n^2} \Big[-2736 C_4 n^3 + 4752 A_2{}^4 (3 + 5n + 2n^2) + 144 A_2 n^2 (4C_3 + 198 C_4 + 19 B_3 n) - 912 A_2{}^3 (n^3 - 4n) + 288 A_1 A_2 (1+n)(-2 B_3 n + A_2{}^2 (3 + 2n)) - A_2{}^2 n (-325 n + 576 B_2 (2+n) + 9504 B_3 (5 + 3n)) \Big]$$

$$\tilde{\tilde{C}}_1 = \frac{1}{4(n+1)\tilde{A}_1} \left[33(n+2) \tilde{B}_2 (2\tilde{B}_1 + 33 \tilde{B}_2) + (n+1)(153 \tilde{B}_2 + 4\tilde{C}_1 - 1089 \tilde{C}_3) \tilde{A}_1 \right]$$

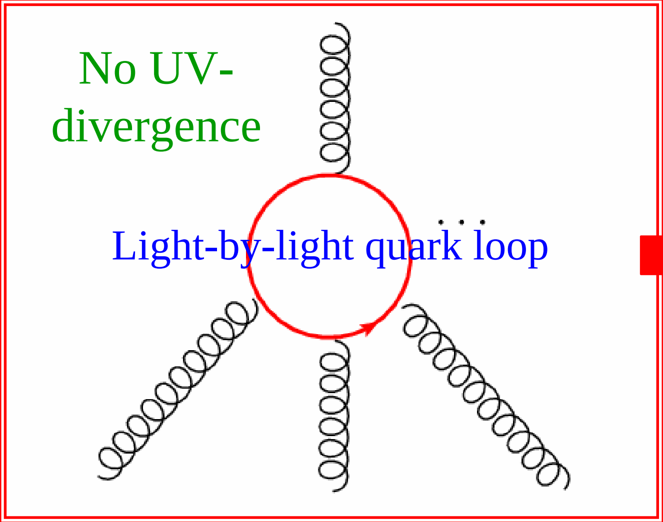
$$\tilde{\tilde{C}}_2 = \frac{-1}{4(n+1)\tilde{A}_1} \left[2(n+2) \tilde{B}_2 (2\tilde{B}_1 + 33 \tilde{B}_2) + (n+1)(19 \tilde{B}_2 - 4(\tilde{C}_2 + 33 \tilde{C}_3)) \tilde{A}_1 \right]$$

Coefficients

standard procedures

First point

When performing the scale shifts $Q \to Q^*$, $Q^* \to Q^{**}$ and $Q^{**} \to Q^{***}$, we eliminate the n_f-terms associated with the $\{\beta_i\}$-terms completely,



Note (one subtle point)

Those n_f-terms, which are irrelevant to the ultra-violet cutoff and have no relation to the β-terms, should be identified and kept separately after the BLM scale setting

$$\ln\frac{Q^{*2}}{Q^2} = \ln\frac{Q_0^{*2}}{Q^2} + \frac{x\beta_0}{4}\ln\frac{Q_0^{*2}}{Q^2}a_s(Q) + \frac{y}{16}\left(\beta_0^2\ln^2\frac{Q_0^{*2}}{Q^2} - \beta_1\ln\frac{Q_0^{*2}}{Q^2}\right)a_s^2(Q) \quad (14)$$

$$\ln\frac{Q^{**2}}{Q^{*2}} = \ln\frac{Q_0^{**2}}{Q^{*2}} + \frac{z\beta_0}{4}\ln\frac{Q_0^{**2}}{Q^{*2}}a_s(Q^*) \quad (15)$$

$$\ln\frac{Q^{***2}}{Q^{**2}} = \ln\frac{Q_0^{***2}}{Q^{**2}} \quad (16)$$

where the effective scales $Q_0^{*,**,***}$ are determined so as to eliminate $A_2 n_f$, $\widetilde{B}_2 n_f$ and $\widetilde{\widetilde{C}}_2 n_f$-terms completely, the parameters x and z are used to eliminate the $B_3 n_f^2$ and the $\widetilde{C}_3 n_f^2$ terms respectively, and the parameter y is used to eliminate the $C_4 n_f^3$-term. It is found that

$$\ln\frac{Q_0^{*2}}{Q^2} = \frac{6A_2}{n} \quad (17)$$

$$\ln\frac{Q_0^{**2}}{Q^{*2}} = \frac{6\widetilde{B}_2}{(n+1)\widetilde{A}_1} \quad (18)$$

$$\ln\frac{Q_0^{***2}}{Q^{**2}} = \frac{6\widetilde{\widetilde{C}}_2}{(n+2)\widetilde{\widetilde{B}}_1} \quad (19)$$

and

$$x = \frac{3(n+1)A_2^2 - 6nB_3}{nA_2} \quad (20)$$

$$y = \frac{(n+1)(2n+1)A_2^3 - 6n(n+1)A_2B_3 + 6n^2C_4}{nA_2^2} \quad (21)$$

$$z = \frac{3(n+2)\widetilde{B}_2^2 - 6(n+1)\widetilde{A}_1\widetilde{C}_3}{(n+1)\widetilde{A}_1\widetilde{B}_2} \quad (22)$$

Second point

The **effective scales should be a perturbative series of α_s** so as to absorb all n_f-dependent terms properly

The effective scales depends on the scheme.

Relations between different scales give **scale displacements** among different schemes

$$\alpha_s^{\overline{MS}}(e^{-5/3}Q^2) = \alpha_s^{GM-L}(Q^2)$$

well-known one-loop relation

The BLM – PMC correspondence

PMC, **dealing with the β-series**, provides the principle underlying BLM scale setting.

However to find what's the β-expansion series like ?

1) There are few cases, people have calculated the β-terms directly, since it is more convenient to calculate the n_f-terms (light-quark loops). So usually, we need to transform the n_f-terms into β-terms for PMC.

2) The relation between β and n_f is not in a simple way, i.e. β_2 include the 2-quark -loop, 1-quark-loop and 0-quark-loop contributions. So to get the same n_f-series, the combination of β-term is not unique, which is more adaptable ?

$$\beta_0 = 11 - \frac{2}{3}n_f$$

$$\beta_1 = 102 - \frac{38}{3}n_f$$

$$\beta_2^{\overline{MS}} = \frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2$$

$$\beta_3^{\overline{MS}} \simeq 29243.0 - 6964.30 n_f + 405.089 n_f^2 + 1.49931 n_f^3$$

O.V. Tarasov, A.A. Vladimirov and A. Yu Zharkov, Phys.Lett. B**93**, 429(1980); T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, Phys.Lett. B**400**, 379(1997); M. Czakon, Nucl.Phys. B**710**, 485(2005).

S.V. Mikhailov, JHEP **0706**, 009(2007).

A.L. Kataev and S.V. Mikhailov, Teor.Mat.Fiz. **170**, 174-186 (2012)

An Naïve way: all β-terms are constructed by all possible combinations of the n_f-terms at the same order.

Questioning: too much β-terms shall be introduced and the coefficients of them cannot be solitarily determined.

More reliable way:

1) **The purpose of the running coupling in any gauge theory is to sum up all the terms involving the β-functions;**

2) **PMC is to absorb all β-terms into running coupling;**

So, inversely,

all the needed β-terms at any concerned order should be the same as those emerged in the expansion of the running coupling.

$$a_s(Q^*) = a_s(Q) - \frac{1}{4}\beta_0 \ln\frac{Q^{*2}}{Q^2} a^2{}_s(Q) + \frac{1}{4^2}\left(\beta^2{}_0 \ln^2\frac{Q^{*2}}{Q^2} - \beta_1 \ln\frac{Q^{*2}}{Q^2}\right) a^3{}_s(Q) + \cdots$$

S.V. Mikhailov, JHEP **0706**, 009(2007).

A.L. Kataev and S.V. Mikhailov, Teor.Mat.Fiz. **170**, 174-186 (2012)

$const., \beta_0, \beta_1, \beta_0^2$

$const., \beta_0, \beta_1, (\beta_0\beta_1), \beta_2, \beta_0^2, \beta_0^3$

Up to NNLO / PMC expansion

More explicitly, up to NNLO, the physical observable can be expanded in the $\{\beta_i\}$-series as,

$$\rho = r_0 \Big[a_s^n(Q) + (A_1^0 + A_2^0\beta_0) a_s^{n+1}(Q) + (B_1^0 + B_2^0\beta_1 + B_3^0\beta_0^2) a_s^{n+2}(Q) + (C_1^0 + C_2^0\beta_2 + C_3^0\beta_0\beta_1 + C_4^0\beta_0^3) a_s^{n+3}(Q) \Big]. \quad (33)$$

We call it `The BLM – PMC correspondence`

$$A_1 = A_1^0 + 11A_2^0$$
$$A_2 = -\frac{2}{3}A_2^0$$
$$B_1 = B_1^0 + 102B_2^0 + 121B_3^0$$
$$B_2 = -\frac{2}{3}(19B_2^0 + 22B_3^0)$$
$$B_3 = \frac{4}{9}B_3^0$$
$$C_1 = C_1^0 + \frac{2857}{2}C_2^0 + 1122C_3^0 + 1331C_4^0$$
$$C_2 = -\frac{1}{18}(5033C_2^0 - 3732C_3^0 - 4356C_4^0)$$
$$C_3 = \frac{1}{54}(325C_2^0 + 456C_3^0 + 792C_4^0)$$
$$C_4 = -\frac{8}{27}C_4^0$$

Under such correspondence,

BLM and PMC are related with each other exactly

One-to-One

Application A: Scale setting in 3-jets events at LO

R. K. Ellis *et Al*, Nucl. Phys. B178, 421-456 (1981)

S.J. Brodsky and L.D. Giustino, arXiv: 1107.0338.

$e^+e^- \to q + \bar{q} + g$ up to NLO

$$\frac{1}{\overline{\sigma}_0}\frac{d\sigma^{(s)} + d\sigma^3}{dy} = \int_y^{1-2y} dz \int_y^{1-y-z} dx\, T[1-x-z, x, z]\alpha_s(s) \left[1 - \frac{\alpha_s(s)}{\pi}\left(\frac{\beta_0}{4}\left(\ln[x] + \ln[z] - \frac{5}{3}\right) + \cdots\right)\right]$$

y: the maximum virtuality of the jet

$$= \alpha_s(s)\left[T(y) - \frac{\alpha_s(s)}{\pi}\left(\left(C(y) - \frac{5}{3}T(y)\right)\frac{\beta_0}{4} + \cdots\right)\right]$$

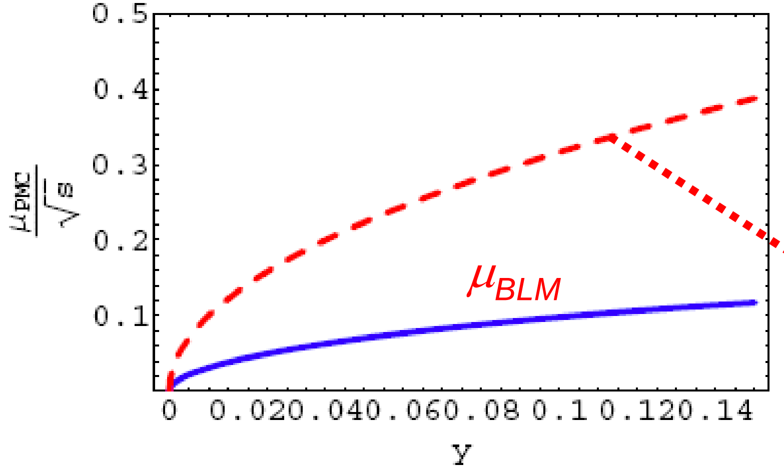
$$= T(y)\alpha_s(s)\left[1 - \frac{\alpha_s(s)}{\pi}\left(\frac{1}{4}\left(\frac{C(y)}{T(y)} - \frac{5}{3}\right)\beta_0 + \cdots\right)\right]$$

$$= T(y)\alpha_s(\mu^2_{BLM}) + \cdots$$

$$\mu^2_{BLM} = s \times \exp\left(-\frac{5}{3} + \frac{C(y)}{T(y)}\right)$$

LO-BLM/PMC scale

Smaller PMC-scale improve the perturbative expansion

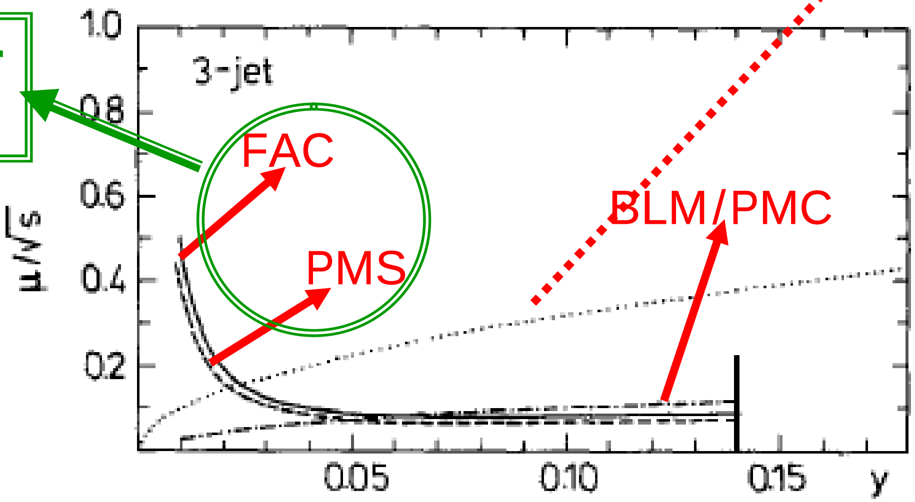


GM-L scale

$\mu/\sqrt{s} = \sqrt{y}$

gluon jet

Wrong behavior At small *y*



G. Kramer and B. Lampe, Z. Phys. C 39, 101 (1988).

$r_1(\mu) = 0.$ **FAC**

$\frac{dR_N}{d\mu} = 0.$ **PMS**

Application B: Scale setting for heavy quark pair production at threshold at NLO – QED corrections

Physics Letters B 359 (1995) 355–361

Need two light-quark loop

Dirac Pauli

$$\bar{u}\Lambda_\mu v = ieQ_f\bar{u}\left[\gamma_\mu F_1(q^2) + \frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2(q^2)\right]v,$$

$$\frac{d\sigma(e^+e^-\to f\bar{f})}{d\Omega} = \frac{N_c\alpha^2 Q_f^2\beta}{4s}\left[(1-\beta^2)|G_e|^2\sin^2\theta + |G_m|^2(1+\cos^2\theta)\right]$$

$$G_e = F_1 + \tfrac{s}{4m^2}F_2 \text{ and } G_F = F_1 + F_2.$$

Anisotropy

$$A = \frac{|G_m|^2 - (1-\beta^2)|G_e|^2}{|G_m|^2 + (1-\beta^2)|G_e|^2} = \frac{\tilde{A}}{1-\tilde{A}}$$

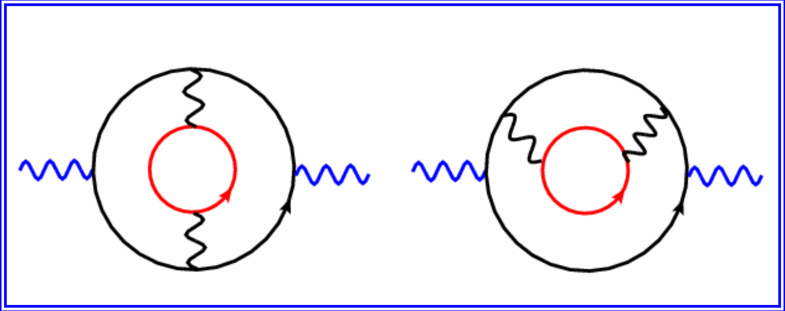
$$\tilde{A} = \frac{\beta^2}{2}\frac{|F_1|^2(1-\beta^2) - |F_2|^2}{|F_1+F_2|^2(1-\beta^2)}$$

$$\frac{dN}{d\cos\theta} \propto 1 + A\cos^2\theta$$

application

Near the threshold region, NNLO with light-quark pair

Take F_1 as an example



$$F_1 = 1 + \frac{\alpha\pi}{4\beta}\left[1 + \frac{\alpha}{\pi}\sum_{i=1}^{n_f}\frac{1}{3}\left(\ln\frac{s\beta^2}{m_i^2} - \frac{8}{3}\right)\right] - \frac{3\alpha}{2\pi}\left[1 + \frac{\alpha}{\pi}\sum_{i=1}^{n_f}\frac{1}{3}\left(\ln\frac{s}{4m_i^2} - \frac{1}{2}\right)\right]$$

$$= 1 + \frac{\alpha(s\beta^2/e)\pi}{4\beta} - \frac{3\alpha(e^{7/6}s/4)}{2\pi} + \mathcal{O}(\beta, \alpha^2)$$

$$\simeq \left(1 + \frac{\alpha(s\beta^2/e)\pi}{4\beta}\right)\left(1 - \frac{3\alpha(e^{7/6}s/4)}{2\pi}\right)$$

Real part is suppressed by β^3.

Coulomb potential

Hard photon exchange

Two effective scales

the square of the relative momentum.

$$\frac{\alpha(Q^2)}{\pi} = \frac{\alpha}{\pi}\left[1 + \left(\frac{\alpha}{\pi}\right)\sum_{i=1}^{n_f}\frac{1}{3}\left(\ln\frac{Q^2}{m_i^2} - \frac{5}{3}\right)\right]$$

S.J. Brodsky, A.H. Hoang, J.H. Kuhn and T. Teubner, Phys.Lett. **B359**, 355(1995)

Application C: Scale setting for R(Q) at NNLO

$$R_{e^+e^-}(Q) = 3\sum_q e_q^2\Big[1+\left(a^{\overline{MS}}(Q)\right)+(1.9857-0.1152n_f)\left(a^{\overline{MS}}(Q)\right)^2$$
$$+\left(-6.63694-1.20013n_f-0.00518n_f^2-1.240\frac{(\sum_q e_q)^2}{3\sum_q e_q^2}\right)\left(a^{\overline{MS}}(Q)\right)^3$$
$$+\left(-156.61+18.77n_f-0.7974n_f^2+0.0215n_f^3+C\frac{(\sum_q e_q)^2}{3\sum_q e_q^2}\right)\left(a^{\overline{MS}}(Q)\right)^4\Big]$$

$$\frac{\sigma(e^+e^-\to \mathrm{hadrons}, Q)}{\sigma(e^+e^-\to \mu^+\mu^-, Q)} \equiv R(Q)$$

C is for singlet contribution and is small

As usual, we set $C=0$

P.A. Baikov, K.G. Chetyrkin and J.H. Kuhn, Phys.Rev.Lett.**101**, 012002(2008); arXiv:0906.2987[hep-ph]; K. Nakamura et al. (Particle Data Group), J.Phys. G**37**, 075021 (2010).

$$R_{e^+e^-}(Q) = 3\sum_q e_q^2\Big[1+\left(a_s^{\overline{MS}}(Q^*)\right)+\widetilde{A}\left(a_s^{\overline{MS}}(Q^{**})\right)^2$$
$$+\widetilde{\widetilde{B}}\left(a_s^{\overline{MS}}(Q^{***})\right)^3+\widetilde{\widetilde{\widetilde{C}}}\left(a_s^{\overline{MS}}(Q^{***})\right)^4\Big], (44)$$

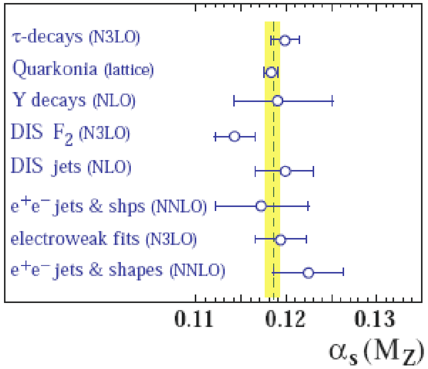
application

Discussions

• **If taking the experimental results for R(Q)**

From the experimental value, $r_{e^+e^-}(31.6GeV) = \frac{3}{11}R_{e^+e^-}(31.6GeV) = 1.0527 \pm 0.0050$ [26], we obtain

$$\Lambda_{\overline{MS}}^{'tH} = 412^{+206}_{-161}\text{MeV}$$
$$\Lambda_{\overline{MS}} = 359^{+181}_{-140}\text{MeV}$$



$$\alpha_s^{\overline{MS}}(M_Z) = 0.129^{+0.009}_{-0.010}$$

is consistent with those obtained from e^+e^- colliders, i.e. $\alpha_s^{\overline{MS}}(M_Z) = 0.13 \pm 0.005 \pm 0.03$ by the CLEO Collaboration [28] and $\alpha_s^{\overline{MS}}(M_Z) = 0.1224 \pm 0.0039$ from the jet shape analysis

S. Bethke, Eur.Phys.J. C**64**, 689 (2009)

- **Inversely, if taking the value of $\alpha_s(M_z)$**

$$\alpha_s^{\overline{MS}}(M_Z) = 0.1184 \pm 0.0007$$

$$\Lambda^{'tH}_{\overline{MS}}|_{n_f=5} = 245^{+9}_{-10}\ \mathrm{MeV}\ \text{and}\ \Lambda_{\overline{MS}}|_{n_f=5} = 213^{+19}_{-8}\ \mathrm{MeV}$$

Four-loop

Eur.Phys.J. C**64**, 689 (2009)
$215 \pm 9MeV$
wrong 4-loop coefficient

- **Discuss the four-loop uncertainty caused by *C***
  (C-> c_3) => using scheme-equation

Even take *C* to have the value comparable with other terms at the same order

$|c_3^R| = 16.1239$

Several percent around **2 %** of experimental error

$$a_s^R[31.6GeV] = 0.0665 \pm 0.0063$$

R. Marshall, Z.Phys. C**43**, 595 (1989).

	$a^R = 0.07276$	$a^R = 0.06645$	$a^R = 0.06016$
$c_3^R = -16.1239$	10.3512 (726)	11.6325 (477)	13.1768 (287)
$c_3^R = 0$	10.3932 (717)	11.6673 (471)	13.2054 (284)
$c_3^R = +16.1239$	10.4348 (707)	11.7020 (466)	13.2338 (282)

After PMC/BLM scale-setting, all higher-order β-terms have been absorbed into the coupling-constant, i.e. in some sense parts of the higher order corrections have been absorbed into the lower order, so the convergence of the perturbative series will be improved in principle.

$$Q^* = (0.757 \pm 0.008)\, Q$$

$$a_s^{\overline{MS}}(Q^*)/a_s^{\overline{MS}}(Q) = 1.060 \pm 0.004$$

Application D: Scale setting for top-pair production at NLO - needs NNLO n_f-terms

(Rough results; To Be Continued)

$$p+p(\bar{p}) \rightarrow Q+\bar{Q}+X$$
$$q+\bar{q} \rightarrow Q+\bar{Q}+X$$
$$g+g \rightarrow Q+\bar{Q}+X$$
$$g+q \rightarrow Q+\bar{Q}+X$$
$$\sigma(S,m^2) = \sum_{ij}\int dx_1\, dx_2\, \hat{\sigma}_{ij}(s,m^2,\mu^2)\, f_i(x_1,\mu^2)\, f_j(x_2,\mu^2)$$

Main Point: The expansion coefficients also include factorization scale-dependence, heavy-quark mass dependence, how to improve the PMC/BLM procedure?

application

known — known — partly known

$$\hat{\sigma}_{ij} = \frac{1}{m_t^2}\left\{ f^0_{ij}(\rho,Q)a_s^2(Q) + f^1_{ij}(\rho,Q)a_s^3(Q) + f^2_{ij}(\rho,Q)a_s^4(Q)\right\}$$

$$f^1_{ij}(Q) = \left[\left(A^{10}_{ij} + A^{11}_{ij}\ln\frac{Q^2}{m_t^2}\right) + \left(B^{10}_{ij} + B^{11}_{ij}\ln\frac{Q^2}{m_t^2}\right)n_f\right] + \left[D^{10}_{ij} + D^{11}_{ij}\ln\frac{Q^2}{m_t^2}\right]\left(\frac{\pi}{\beta}\right) \quad (8)$$

$$f^2_{ij}(Q) = \left[\left(A^{20}_{ij} + A^{21}_{ij}\ln\frac{Q^2}{m_t^2} + A^{22}_{ij}\ln^2\frac{Q^2}{m_t^2}\right) + \left(B^{20}_{ij} + B^{21}_{ij}\ln\frac{Q^2}{m_t^2} + B^{22}_{ij}\ln^2\frac{Q^2}{m_t^2}\right)n_f + \left(C^{20}_{ij} + C^{21}_{ij}\ln\frac{Q^2}{m_t^2} + C^{22}_{ij}\ln^2\frac{Q^2}{m_t^2}\right)n_f^2\right] + \left[\left(D^{20}_{ij} + D^{21}_{ij}\ln\frac{Q^2}{m_t^2} + D^{22}_{ij}\ln^2\frac{Q^2}{m_t^2}\right) + \left(E^{20}_{ij} + E^{21}_{ij}\ln\frac{Q^2}{m_t^2} + E^{22}_{ij}\ln^2\frac{Q^2}{m_t^2}\right)n_f\right]\left(\frac{\pi}{\beta}\right) + \left(F^{20}_{ij} + F^{21}_{ij}\ln\frac{Q^2}{m_t^2} + F^{22}_{ij}\ln^2\frac{Q^2}{m_t^2}\right)\left(\frac{\pi}{\beta}\right)^2 \quad (9)$$

initial

Find coefficients For all n_f-terms

Final total CS =

$$f^0_{ij}(\rho,Q)a_s^2(Q_1^*) + \left[\tilde{A}^{10}_{ij} + \tilde{A}^{11}_{ij}\ln\frac{Q_1^{*2}}{m_t^2}\right]a_s^3(Q_1^{**}) +$$
$$\left[\tilde{\tilde{A}}^{20}_{ij} + \tilde{\tilde{A}}^{21}_{ij}\ln\frac{Q_1^{**2}}{m_t^2} + \tilde{\tilde{A}}^{22}_{ij}\ln^2\frac{Q_1^{**2}}{m_t^2}\right]a_s^4(Q_1^{**}) +$$
$$\left[D^{10}_{ij} + D^{11}_{ij}\ln\frac{Q^2}{m_t^2}\right]\left(\frac{\pi}{\beta}\right)a_s^3(Q_2^*) +$$
$$\left[\tilde{D}^{20}_{ij} + \tilde{D}^{21}_{ij}\ln\frac{Q_2^{*2}}{m_t^2} + \tilde{D}^{22}_{ij}\ln^2\frac{Q_2^{*2}}{m_t^2}\right]\left(\frac{\pi}{\beta}\right)a_s^4(Q_2^*) +$$
$$\left[\tilde{F}^{20}_{ij} + \tilde{F}^{21}_{ij}\ln\frac{Q_2^{*2}}{m_t^2} + \tilde{F}^{22}_{ij}\ln^2\frac{Q_2^{*2}}{m_t^2}\right]\left(\frac{\pi}{\beta}\right)^2 a_s^4(Q_2^*)$$

Coulomb contributions should be resummed

Sommerfeld's rescattering formula

To be programmed

3. Summary and Outlook

Fitzpatrick
JHEP 1111 (2011) 095

I. **BLM / PMC** provide self-consistent way to set the effective scales, which leads to **scheme-independent result**. QCD is not confromal, however one can use the PMC to convert a PQCD series to the corresponding conformal QCD series.

II. After PMC/BLM, one can use the **Mellin-space analysis** to compute the conformal QCD correlators and amplitudes. To be considered.

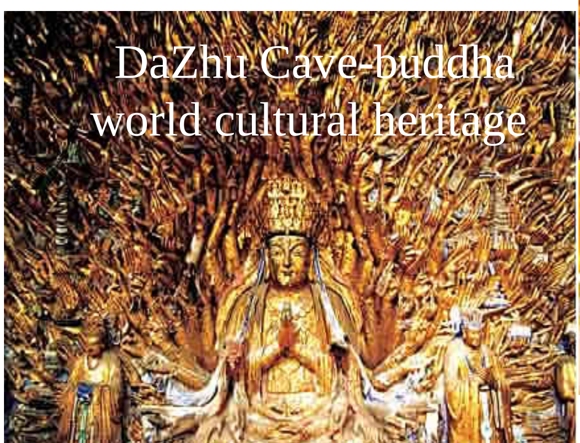
II. **With the extended RGE**, the α_s scale-dependence can be solved and the four-loop results are given; and the α_s scheme-dependence can be reliably estimated by it scheme equations, e.g. an estimation of higher-order correction.

III. A combination of BLM/PMC to E-RGE can be used to derive a **precise QCD estimation**.

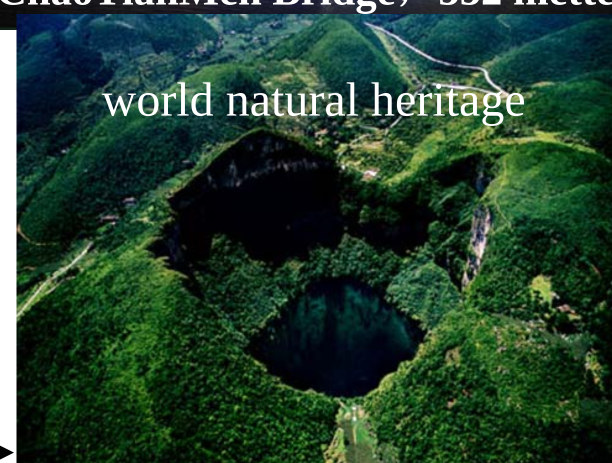
IV. More applications will appear.

Different style
but also beautiful city

Welcome to Chongqing,
when you have time



Karst physiognomy→



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Thanks