Bunched Beam Diagnostics

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H. Signals From Longitudinal Motion

A. Introductory Remarks

It is natural to consider beam diagnostics and beam dynamics in the frequency domain. Particles in a storage ring can be thought of as oscillating in potential wells formed by the magnetic lattice in the transverse and the accelerating fields in the longitudinal. An extremely useful first approximation is that the oscillation frequencies are independent of amplitude, and these frequencies, the tunes, are among the most basic accelerator parameters. Departures from this first approximation are also well described in the frequency domain. Cubic and higher order contributions to the potentials show up as tune shifts, and collective phenomena can be analyzed as perturbations of the basic motion. For these reasons this discussion concentrates on frequency domain analysis and measurements.

Fourier transformations are used for changing between time domain and frequency domain functions. If \( f(t) \) is a time domain function, then the convention used in this paper for the Fourier transform, \( \mathcal{F}(\omega) \), is

\[
\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt
\]

where \( j = \sqrt{-1} \). The inverse transformation is

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{i\omega t} d\omega
\]

Angular frequencies, \( \omega \), are the appropriate mathematical variables. However, spectrum analyzers read out frequencies, \( \nu = \omega/2\pi \), and when discussing measurements, these are more convenient quantities. In this paper the word "frequency" depends on context, but formulæ keep the distinction between \( \omega \) and \( \nu \) when it appears, it is always an angular frequency. Uniquely subscripted different physical frequencies, and the subscript are the same whether referring to frequency or angular frequency. For example, \( \nu_1 \) and \( \omega_1 = 2\pi \nu_1 \), both refer to the rotation frequency.

B. Single Particles

1. Constant Revolution Frequency: Consider a single, unit-charge particle with a constant revolution frequency \( \omega_0 \) (Figure 1a). At some azimuth of the machine there is a detector that measures the beam current which is

\[
i(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)
\]

where \( T = 2\pi/\omega_0 \) is the revolution period. Taking the Fourier transform

\[
i(\omega) = \sum_{n=-\infty}^{\infty} \delta(-n\pi) e^{-i\omega t} = \sum_{n=-\infty}^{\infty} e^{-in\omega T} = \sum_{n=-\infty}^{\infty} \exp(-i2\pi n\omega_0) \]

The last sum can be thought of as an infinite sum of phasors of unit magnitude that will add up to zero unless all the phasors are in phase, i.e., unless \( n = 0 \), it is equal to an integer.

Formally this follows from the Poisson Sum Rule which states that if \( \mathcal{F}(\omega) \) is the Fourier transform of \( i(t) \), then

\[
\sum_{n=-\infty}^{\infty} \mathcal{F}(n\omega_0) = \frac{1}{\Delta} \sum_{n=-\infty}^{\infty} \mathcal{F}(2\pi n/\Delta) \]

Taking \( \Delta = 2\pi/\omega_0 \) and \( \omega = 2\pi n/\Delta \), gives \( \mathcal{F}(2\pi n/\Delta) = 2\pi \delta(\omega - n\omega_0) \) and eq. (5) becomes

\[
I(\omega) = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \]

\[
(6)
\]

\( \delta(\omega) \) is a series of \( \delta \)-functions at \( n\omega_0 \), \( n \) where \( n \) is an integer; this is in agreement with the qualitative argument above. The spectrum is a comb of lines at the revolution harmonics - frequencies that are integer multiples of the revolution frequency (Figure 1b).

![Figure 1: a) A single particle with constant revolution frequency passing a detector once per turn, b) The spectrum of the current measured in the detector.](image)

2. Bunched Beams: In a machine where the beam is bunched by the accelerating fields of an RF cavity the expression for the beam current must include the effects of synchrotron oscillations. In this case the current, \( i(t) \), is given by

\[
i(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\piT) \cos(\omega_nTt + \phi_n) \]

In this equation \( T \) is the revolution period of a synchronous particle, \( \omega_n \) the amplitude of the synchrotron oscillation, \( \omega_0 \) is the synchrotron oscillation frequency, and \( \phi_n \) is the phase of the oscillation. Taking the Fourier transform

\[
i(\omega) = \sum_{n=-\infty}^{\infty} \exp(-j\omega_0 t) \cos(\omega_n T t + \phi_n) \]

\[
(8)
\]

Before proceeding formally, a central feature of this expression, the correlation between the amplitudes envelope and the frequencies of the spectral lines, can be seen by Taylor expanding eq. (8)

\[
i(\omega) = \sum_{n=-\infty}^{\infty} \exp(-j\omega_0 t) \left( 1 - \frac{j\omega_0}{2} \cos(\omega_n T t + \phi_n) \right) \]

\[
= \sum_{n=-\infty}^{\infty} \exp(-j\omega_0 t) \left( 1 - \frac{(\omega_0)^2}{4} \cos(\omega_n T t + \phi_n) \right)
\]

\[
= \sum_{n=-\infty}^{\infty} \exp(-j\omega_0 t) \frac{(\omega_0)^2}{4} \cos(2\omega_n T t + \phi_n) \]

\[
(9)
\]

When \( \omega_n \ll \omega_0 \) this reduces to eq. (6); the spectrum is a comb of lines at the revolution harmonics. As \( \omega_n \) increases the lines term becomes important, and \( I(\omega) \) equals

\[
I(\omega) = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \]

\[
(6)
\]
\[
1(\omega) = \sum_{n=-\infty}^{\infty} \exp(-j\alpha n \tau) - \frac{1}{j2\pi} \sum_{n=-\infty}^{\infty} \frac{\exp\left(-j\left((\omega - \alpha n)\tau + \phi\right)\right)}{n}
\]

The new terms are combs of lines offset from the rotation harmonics at \(\omega = \alpha n + \omega_0\) and \(\omega = \alpha n - \omega_0\). These lines have an amplitude envelope that is linear in \(\alpha n\) and, therefore, didn't appear when \(\alpha n \ll 1\). For \(\alpha n\) large enough that the quadratic terms must be considered, there are two more combs at \(\omega = \alpha n + 2\omega_0\) and \(\omega = \alpha n - 2\omega_0\). In addition, the comb at the rotation harmonics decreases in amplitude.

Formatively, eq. (6) can be rewritten so that it can be interpreted in the same way as eq. (5) and figure 1b by using a Bessel function sum:

\[
j \cos am = \sum_{k=-\infty}^{\infty} j^{-k} J_k(a) e^{-jka}\tan\theta.
\]

Substituting this into eq. (6) and using the Poisson sum rule:

\[
1(\omega) = \alpha \sum_{k=-\infty}^{\infty} (-1)^k J_k(\alpha n) e^{jka} \sum_{n=-\infty}^{\infty} \delta(\omega - k\alpha - n\omega_0).
\]

The spectrum is a double-infinity of lines at \(\omega = k\alpha - n\omega_0\), where \(n = \ldots, -1, 0, 1, \ldots\) and \(k = \ldots, -1, 0, 1, \ldots\).

As expected from the discussion above the value of the integer \(k\) has consequences for the lococones and amplitudes of spectral lines. The lines are sidebands of the rotation harmonics; they are spaced by \(\alpha\), and off sets from the rotation harmonics by \(\pm \omega_0\). The factor \(J_k(\alpha n)\) determines the envelope of the spectral lines. The first maxima of the Bessel function \(J_k(x)\) are approximately at \(x = \pm k\). When the magnitude of the argument, \(\alpha n\), is sufficiently less than \(\pm k\), the envelope never is small and the sidebands with that value of \(k\) have small amplitude. Alternatively, a sideband of a given \(k\) can be identified by the offset from a rotation harmonic, and, if such a sideband is prominent, the synchronous oscillation amplitude must satisfy

\[
t = \frac{1}{k\omega_0} \frac{1}{\omega_0}+1
\]

where \(\omega_0\) is the frequency of observation. Sample spectra for \(k = 0\) and \(k = 1\) are shown in figure 2.

The sum in (12) contains positive and negative frequencies. How does a spectrum analyzer measure negative frequencies? Consider the two phasors shown on the right: one has frequency \(\omega_0 > 0\) and the other has frequency \(\omega_0 - \omega_0\). The physical quantities represented by these phasors are the projections onto the real axis, and these projections are the same for the two phasors. A spectrum analyzer cannot distinguish positive and negative frequencies.

This has several consequences that are illustrated in figure 2. On the left-hand side of this figure are the 'mathematical' spectra given by the \(\delta\) functions in eq. (12), and on the right-hand side are the 'physical' spectra that would be measured. For \(k = 0\) the measured lines are at \(\omega = \pm \omega_0\), where \(\omega = \pm 0.12\); the \(n = \omega 0\) line is one-half the amplitude of the other lines. When the quantity being measured is the beam current, the \(n = 0\) line is 2n times the DC current and all the other lines have an amplitude equal to twice the \(n = 0\) line (4n times the DC currents). When \(n = 1\), the physical spectrum has sidebands spaced \(\pm \omega_0\) from the rotation lines, and \(k = 1\) and \(k = -1\) cannot be distinguished. Both \(k = 1\) and \(k = -1\) sidebands appear at the same frequency, and, from eq. (12), the real parts of the currents are in phase. Therefore, the measured current is twice the current from a single sideband. Of course, these remarks hold for all \(n > 0\).

The envelope in eq. (12) has a factor \(\exp(-j(k\phi + \phi))\) that gives a phase, but this phase has no meaning for a single particle. For multiple particles the phase takes on statistical significance.

C. Multiple Particles: Introduction

Consider longitudinal phase space with Cartesian coordinates \(t\), the deviation from the synchronous time, and

\[
e = \frac{\alpha}{\omega_0} \frac{\Delta E}{E_0}
\]

where \(\Delta E/E_0\) is the fractional deviation from the nominal energy and \(\alpha\) is the momentum compaction. The quantities \(t\) and \(e\) satisfy

\[
t = \frac{1}{2} \cos(\omega_0 t + \phi)\]

and

\[
e = \frac{1}{2} \sin(\omega_0 t + \phi).
\]

---


The charge density at $t=0$ is written as a function of the polar coordinates, $r$ and $\theta$, where
$$p(t, r, \theta) = \text{charge in phase space area } t \, dt \, dr \, d\theta.$$ \hspace{1cm} (16)

To determine the spectrum of the beam, the single particle spectrum, eq. (12), is multiplied by $p$ and integrated over phase space. For $k=0$, the rotation harmonic is
$$I_0(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \omega \, d\omega \int_{0}^{2\pi} p(t, r, \theta) \, \frac{d\omega}{n\nu} \sum_{n=1}^{\infty} \delta(\omega - n\nu).$$ \hspace{1cm} (17)

The comb of lines at integer multiples of the rotation frequency remains, and there is an envelope given by the convolution of the charge density and $I_0$. The measured current is given by combining the positive and negative frequency components
$$I_0^{\pm}(\omega) = \frac{Q \delta(\omega - k_0 \nu)}{\pi} \int_{0}^{2\pi} p(t, r, \theta) \, \frac{d\omega}{n\nu} \sum_{n=1}^{\infty} \delta(\omega - n\nu) + \sum_{n=1}^{\infty} \delta(\omega - n\nu).$$ \hspace{1cm} (18)

where $Q$ is the total charge of the bunch.

For an example, assume that the distribution is uniform in $\theta$ and Gaussian in $t$ with an rms bunch length $\sigma_t$.
$$p(t, r, \theta) = \frac{Q}{\pi \sigma_t^2} \exp\left(-\frac{t^2}{2\sigma_t^2}\right).$$ \hspace{1cm} (19)

The resulting spectrum is
$$I_0^{\pm}(\omega) = Q \delta(\omega - k_0 \nu) \pm 2Q
\nu \exp\left(-\frac{(\omega \pm k_0 \nu)^2}{2\nu^2}\right) \sum_{n=1}^{\infty} \delta(\omega - n\nu).$$ \hspace{1cm} (20)

It is a comb of lines at multiples of the rotation frequency $\omega$ with a Gaussian envelope with an rms width equal to $\nu$. Measurements of rotation harmonics over a range of frequencies can be used to determine the bunch length; this requires a detector with a bandwidth from a fraction of $10^5 \nu$ to several times $10^5 \nu$.

There are two different physical situations that lead to different interpretations of the spectra for $k \neq 0$. These are:

CASE I: Particles move independently of each other.

CASE II: The motion of each individual particle depends on the other particles in the beam. This situation arises when beam induced electromagnetic fields, wakefields, are important.

In CASE I measurements of the spectra for $k \neq 0$ are measurements of Schottky noise, and in CASE II they are measurements of collective effects.

D. CASE I: Schottky Noise

1. Approximate Picture: A beam can be injected into a storage ring in a way such that the charge density has dependence on both $t$ and $\theta$. When the beam particles move independently, the $\theta$ dependence vanishes due to filamentation and/or randomization. Assume that at $t=0$ this function $p(t)$ only
$$p(t, \theta) = \frac{1}{2\pi} p(t).$$ \hspace{1cm} (21)

For a given $k$ the spectrum is
$$I_k(\omega) = \frac{(2\pi)^{\frac{1}{2}}}{2\pi} \sum_{n=1}^{\infty} \delta(\omega - k_0 \nu + n\nu) \int_{0}^{2\pi} p(t, \theta) \exp\left(\frac{\pi}{k_0 \nu} \int_{0}^{2\pi} p(t, \theta) \, d\theta \right) \frac{\partial I_0(\omega)}{\partial \omega}.$$ \hspace{1cm} (22)

The integral over $\theta$ gives this result independent of the form of $p(t)$. Why? Individual particles have current at $\omega = n\nu - k_0 \nu$, but as the phase of the current depends on $\theta$ when $k_0 > 0$ (see eq. (21)). Therefore, when the charge density has no azimuthal structure, the currents of the individual particles cannot lead to the result in eq. (21).

However, if the situation is as stated, the cancellation cannot be complete. The beam is made up of a finite number of particles, and if these particles are independent as assumed, they cannot be positioned such that the cancellation is perfect. If they were, they wouldn't be independent! The calculation of the beam spectrum is the same as other statistical calculations. The particles in the beam are drawn from a parent distribution that satisfies eq. (21). However, they are a finite sample, and the actual beam distribution has fluctuations from the parent distribution. The beam spectrum shows these fluctuations. Since this is a statistical problem, we expect the deviation from eq. (22) to depend on the square root of number of particles. This residual, statistical current is the Schottky Noise of the beam.

The charge density for $N$ particles is
$$p(t, \theta) = \sum_{p=1}^{N} \delta(t - t_{ap}) \delta(\theta - \theta_p).$$ \hspace{1cm} (23)

where q is the charge of an individual particle, and \( T_{mp} \) and \( \omega_p \) are the synchrotron oscillation amplitude and phase of particle p. The current is given by substituting this expression for the charge density into eq. (22) and performing the \( \phi \) and \( \Omega \) integrals. The result is

\[
I_{nk}(\omega) = \sum_{p=1}^{P} \sum_{s=1}^{S} F_{nk}(\omega k_{ps} - \omega_p s)
\]

where

\[
F_{nk}(\omega k_{ps} - \omega_p s) = q \omega_p \sum_{p=1}^{P} \sum_{s=1}^{S} \int_{0}^{1} j_{nk}(\omega k_{ps} - \omega_p s) \exp(-j \Omega k_{ps} \phi_p) \, d\Omega / (2\pi)
\]

(24)

For each set of P particles the \( F_{nk}(\omega) \)'s are different; they depend on the phase space coordinates of the particles. It follows that at a given time the current is unknown because it depends on the particular set of particles in the accelerator. However, over time, the particle distribution changes due to filamentation and randomization, and the mean and rms currents are the meaningful quantities. For stationary random variables time averaging and ensemble averaging are equivalent, \(^5\) and the mean and rms values of \( I_{nk}(\omega) \) can be determined by averaging over all samples of P particles. This averaging is denoted by \( \langle \cdot \rangle \).

The mean value of the current is

\[
\langle I_{nk}(\omega) \rangle = q \omega_p \sum_{p=1}^{P} \sum_{s=1}^{S} \int_{0}^{1} j_{nk}(\omega k_{ps} - \omega_p s) \exp(-j \Omega k_{ps} \phi_p) \, d\Omega / (2\pi)
\]

(25)

because the phase factor averages to zero. This is the same result as eq. (22). The square of the rms current is

\[
\langle I_{nk}^2(\omega) \rangle = \frac{1}{2} \langle I_{nk}(\omega) \rangle^* \langle I_{nk}(\omega) \rangle
\]

\[
= \frac{q^2 \omega_p^2}{2} \sum_{p=1}^{P} \sum_{s=1}^{S} \int_{0}^{1} j_{nk}(\omega k_{ps} - \omega_p s) \exp(j \Omega k_{ps} \phi_p) \, d\Omega / (2\pi)
\]

(26)

When averaged, the phase factor is equal to zero for \( p \neq s \) and equal to one for \( p = s \). This removes one of the sums in the eq. (26), and

\[
\langle I_{nk}^2(\omega) \rangle = \frac{q^2 \omega_p^2}{2} \sum_{p=1}^{P} \int_{0}^{1} j_{nk}(\omega k_{ps} - \omega_p s) \, d\Omega / (2\pi)
\]

(27)

As expected, the rms current is proportional to the square root of the number of particles. Rewriting eq. (27) in terms of \( p(t_j) \) from eq. (23)

\[
\langle I_{nk}^2(\omega) \rangle = \frac{q^2 \omega_p^2}{2} \int_{0}^{1} \left( \frac{1}{2} \int_{0}^{1} \int_{0}^{1} p(t_j) \, dt \right) \left( \frac{1}{2} \int_{0}^{1} \int_{0}^{1} p(t_j) \, dt \right) \, d\Omega / (2\pi)
\]

(28)

The measured rms current has contributions from positive and negative frequencies. Applying the same reasoning used in section B for a single particle, at any measurement frequency these contributions are in phase for any set of particles. Therefore, the rms measured Schottky current is twice that given by eq. (28)

\[
\langle I_{nk}^2(\omega) \rangle = \frac{q^2 \omega_p^2}{2} \sum_{p=1}^{P} \int_{0}^{1} j_{nk}(\omega k_{ps} - \omega_p s) \, d\Omega / (2\pi)
\]

(29)

where

\[
\Gamma_{nk}(\omega) = \left( \langle I_{nk}^2(\omega) \rangle \right)^{1/2}
\]

and \( \omega = \omega_p - \omega_{ps} \), \( p \neq s \).

---


A measurement of the ratio of the currents at the different sidebands can be used to determine the bunch length. This has the advantage of working at a small frequency range. A resonant, narrow-band detector with high sensitivity can be used, and there is no need to correct frequency-dependent effects like cable attenuation.

2. Synchrotron Frequency Spread: Up to this point in the discussion, synchrotron frequency spreads have been treated inconsistently to simplify the development of some concepts. Starting at eq. (20) up to eq. (30) it is assumed implicitly that all particles have the same synchrotron frequency, but irradiation was invoked at several points to argue that phase space structure evolves in time. To have each irradiation there must be a spread in synchrotron frequency covered by nonlinearities in the RF waveform. The effect of this incoherence treatment depends on a comparison of the synchrotron frequency spread, $\Delta \nu_s$, and the bandwidth of the spectrum analyzer used for measurements. $\Delta \nu_s$. For a Gaussian density distribution and sideband $k$, the rms current is given by eq. (30) when $\Delta \nu_s > 8 \Delta \nu_0$.

However, when $\Delta \nu_s \approx \Delta \nu_0$, the sidebands have a width $8 \Delta \nu_0$ rather than a unique frequency, and the signal is reduced in amplitude because the number of particles within the bandwidth must be used instead of $P$. If $\nu_0 \Delta \nu_0$ is the fraction of particles with synchrotron frequency between $\nu_0$ and $\nu_0 + \Delta \nu_0$, the rms current is

$$I_{\text{rms}}(\phi) = I_0 \left( 2P \int_{0, \nu_0}^{\nu_0 + \Delta \nu_0} |w(\omega)|^2 \exp(-\frac{\omega^2}{2\nu_0^2}) \right)^{1/2}$$  \hspace{1cm} (31)

where $w(\omega)$ is the fraction of the particles in the bandwidth. It is given by

$$w(\omega) = \frac{1}{\Delta \nu_0} P\int_{\nu_0}^{\nu_0 + \Delta \nu_0} \exp(-\frac{\omega^2}{2\nu_0^2}) \, d\omega.$$  \hspace{1cm} (32)

The noise spread can be calculated approximately using the method of Kryloff and Bogoliubov.\textsuperscript{7} For a sinusoidal RF voltage the equation of motion is

$$\frac{d^2 \phi}{dt^2} + \omega_0^2 \phi(t) = \sum_{n \neq 0}^{\infty} \sin(n \phi) \cdot \chi_{nm} \cdot \sin(\omega_n t) \cdot \sin(\omega_0 t).$$  \hspace{1cm} (33)

where $\omega_0$ is the RF frequency. The first term contains the deviation from a linear restoring force. In the approximate solution the amplitude is unchanged by the non-linearity but the frequency is dependent on the amplitude

$$\tau = \frac{1}{2} \cos(\phi \cos(\Delta \phi/t + \phi)).$$  \hspace{1cm} (34)

The quantity $\Delta \phi$ is given by

$$\Delta \phi = \frac{\omega_0^2}{\omega_n^2 - \omega_0^2} \int_0^{2\pi} \sin \theta \cos \theta \sin(\omega_0 t) \cos(\omega_n t) \, d\theta.$$  \hspace{1cm} (35)

For a quantitative example consider a single bunch in the Tevatron Collider with parameters given in Table 1. The numerical values in figure 4 apply to this example.

From eq. (20) the current in a resonance harmonic near $\phi = 0$ is $J_0 = 2.9 mA$.

---

**Table 1: Longitudinal Motion in the Tevatron Collider, 1987**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>RF Voltage</td>
<td>$V_{RF} = 1.1 \times 10^7$ V</td>
</tr>
<tr>
<td>Beam Energy</td>
<td>$E = 900$ GeV</td>
</tr>
<tr>
<td>Revolution Frequency</td>
<td>$v_R = 477.7$ kHz</td>
</tr>
<tr>
<td>Harmonic Number</td>
<td>$h = 1113$</td>
</tr>
<tr>
<td>RF Frequency</td>
<td>$V_{RF} = 4 \pi h / 3$ M</td>
</tr>
<tr>
<td>Momentum Conservation</td>
<td>$\alpha = 0.0038$</td>
</tr>
<tr>
<td>Synchrotron Oscillation Frequency</td>
<td>$\nu_s = 37$ Hz</td>
</tr>
<tr>
<td>RMS Longitudinal Emittance</td>
<td>$\sigma_{\nu_s} = 0.16$ eV-cm</td>
</tr>
<tr>
<td>RMS Bunch Length</td>
<td>$\nu_0 = 1.4$ eV-cm</td>
</tr>
<tr>
<td>RMS Fractional Energy Spread</td>
<td>$\sigma_{\nu_s}/\nu_0 = 1.3 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

---

Figure 4 shows the comparison of eqs. (20) and (30); the current at the resonance harmonics is greater than the rms Schottky current out to about 600 MHz where $\omega_n = 5$.

4. For $\Delta \nu_s = 5$ Hz and small $\nu_0$, all the sidebands are well separated from each other and eq. (30) is valid. However, caution is advised; when $\omega_n = 5$ the resonance harmonics are so large compared to the Schottky sidebands that it may not be possible to resolve the sidebands.

5. For $\nu_0 > 10$ the sidebands overlap. Figure 5a shows the results from eqs. (31) and (32) for $\Delta \nu_s = 1$ Hz and $\omega_n = 6$.

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Figure 5: Synchrotron frequency spread in the Tevatron. a) Synchrotron frequency as a function of amplitude. b) Synchrotron frequency distribution. c) Sideband overlap for large $k$, $\omega_n = 6$.
E. CASE II: Collective Effects

In the case of Schottky Noise, particles are independent, and signals are due to statistical fluctuations from the average. Particles do not always move independently; they can influence one another through beam generated electromagnetic fields. At high energies the dominant fields, called wakefields, are generated as the beam passes changes to the vacuum chamber profile. When the wakefields are strong, the particles are no longer independent. They act collectively, and the beam develops stable structure in phase space called coherent modes.

A full discussion of collective effects is outside the scope of the present paper; a number of reviews are available. Most theoretical studies of collective effects are done in the frequency domain and use the Fourier transforms of the wakefields which are called impedances. There are an extensive literature devoted to wakefields and impedances, and there are a number of review articles from past accelerator schools with detailed discussions and references to original works.

1. Approximate Picture: For a first approximation assume that there is some stable structure in phase space that rotates with frequency $\omega_b$. The density $\rho$ at time $t$ can be written as a Fourier expansion in $\phi$:

$$p(t, \phi, t) = \exp(j \omega_b t) \sum_{m=-\infty}^{\infty} \rho_m \exp(j m \phi).$$  \hspace{1cm} (36)

The quantity $\rho_m$ is the component with azimuthal periodicity $m$, and assuming that the structure is stable means that the $\rho_m$'s change slowly compared to measurement time scales. Usually $\rho_0$ is the dominant term and one or more of the other $\rho_m$'s is non-zero.

The beam spectrum is given by multiplying the single particle spectrum eq. (12b) by the charge density at $t=0$ (eq. (30) with $t=0$), and integrating over phase space. For a given rotation harmonic:

$$I(\omega) = \frac{2 \pi}{k_{\omega}} \sum_{m=-\infty}^{\infty} \rho_m \int_0^{2\pi} \rho_0 \rho_0 \sin(\omega t) \exp(-j \omega t) \sin(\omega \phi) \exp(-j \omega \phi) \ ds \sin(\omega \phi)$$ \hspace{1cm} (37)

The $\omega$ integral is equal to zero unless $k = m$, and, therefore, phase space with azimuthal structure gives spectral lines at $\omega = \omega_0 - m \omega_b$, and signals at $\omega = \omega_0 + m \omega_b$. Conversely, a signal at $\omega = \omega_0 - m \omega_b$ with amplitude greater than the Schottky signal indicates a coherent mode with azimuthal phase space structure $\omega_0 \omega_b$ or $\omega \omega_b$. Since there may be more than one structure in phase space, there can be more than one sideband present.

In contrast to the Schottky case, the signal:

$$I(\omega) = \frac{2 \pi}{k_{\omega}} \sum_{m=-\infty}^{\infty} \rho_m \int_0^{2\pi} \rho_0 \rho_0 \sin(\omega t) \exp(-j \omega t) \sin(\omega \phi) \exp(-j \omega \phi) \ ds \sin(\omega \phi)$$ \hspace{1cm} (38)

has a non-zero amplitude because of the proportionality to $\rho_m$. For a coherent mode with a dominant structure at amplitude $\rho_m$, the signal will be stronger in the frequency range $\omega = m \omega_b$ because of the coincidence with $\omega_0$. I.e. the signal of a structure of high azimuthal periodicity (large $m$) will be strong at high frequencies.

2. Coherent Tune Shifts: In writing eq. (36) frequency shifts due to wakefields were neglected. This simplification leads to the relationships between signal frequency, amplitude, and phase space structure (from the bare results), but must be modified to account for coherent frequency shifts and instabilities. A more complete expression for the density at time $t$ is:

$$p(t, \phi, t) = \exp(j \omega_b t) \sum_{m=-\infty}^{\infty} \rho_m \exp(j m \phi) \left( \frac{\sum_{n=-\infty}^{\infty} \rho_n / n_{mp} \exp(j n \Delta \omega_p) \right)$$ \hspace{1cm} (39)

As in eq. (36), the charge density is expanded in a Fourier series in $\phi$. In addition, the $\rho_m$'s are expanded using an appropriate basis; this is symbolized by the summation over $p$. The resulting modes are characterized by two integers describing the azimuthal and radial $(\ell, m)$ structure, and $n$ and $\rho$, and have phase space rotation frequencies $\omega_0 - \omega_b = \Delta \omega_p$.

Calculation of the mode structure $(\ell, m, n)$ and frequency shifts $(\Delta \omega_p)$ is the subject of papers on collective effects. Much of the underlying physics in these papers is contained in the discussion above. A mode has an associated current given by a generalization of eq. (37): this current induces a voltage equal to the product of the current and the accelerator impedance summed over all spectral lines. This voltage modifies the beam density distribution, and the problem becomes an eigenvalue problem finding the phase space distribution consistent with the induced voltage.

A summary of the important results of detailed calculations is:

1. The coherent modes and frequency shifts are solutions of the Vlasov equation. A perturbation approach leads to an integral equation that involves a convolution over frequency of the current and the longitudinal impedance.

2. Approximate solutions for the $\Delta \omega_p$'s are proportional to beam current and longitudinal impedance weighted by a frequency dependence of the mode. As a consequence, the frequency shifts are different for different modes.

3. The $\Delta \omega_p$'s are complex. The real part, $\Delta \omega_p = \Re \Delta \omega_p$, changes the phase space rotation frequency from $\omega_0$ to $\omega_0 + \Delta \omega_p$. The imaginary part, $\Delta \omega_p = \Im \Delta \omega_p$, can lead to mode growth or damping. At low beam current values such as Landau damping or radiation damping can dominate and prevent instabilities.

4. $\Delta \omega_p$ depends upon the imaginary part of the impedance which gives the component of the beam induced voltage in phase with the current, and, therefore, can add or remove energy from a mode.

5. $\Delta \omega_p$ depends on the real part of the impedance. This component of the beam induced voltage is in phase with the current and, therefore, can add or remove energy from a mode.

6. When $\Delta \omega_p < 0$, modes can couple leading to an instability (the micro-wave instability) with large growth rates.

When a coherent mode is stable the measurement time is long compared to the instability growth time, the signal can be calculated in a way analogous to that leading to eq. (38). The appropriate single particle spectrum is given by eq. (32) with the appropriate frequency shifted by $\Delta \omega_p$:

$$I(\omega) = \frac{2 \pi}{k_{\omega}} \sum_{m=-\infty}^{\infty} \rho_m \int_0^{2\pi} \rho_0 \rho_0 \sin(\omega t) \exp(-j \omega t) \sin(\omega \phi) \exp(-j \omega \phi) \ ds \sin(\omega \phi)$$ \hspace{1cm} (40)

Multiplying by the charge density of the mode at $t=0$, and integrating over $\phi$ gives the signal

$$I(\omega) = \frac{2 \pi}{k_{\omega}} \sum_{m=-\infty}^{\infty} \rho_m \int_0^{2\pi} \rho_0 \rho_0 \sin(\omega t) \exp(-j \omega t) \sin(\omega \phi) \exp(-j \omega \phi) \ ds \sin(\omega \phi)$$ \hspace{1cm} (41)

This result is essentially the same as eq. (38); the difference is:

1. The need for a third "quantum number" $p$, to describe the $\omega_0$ dependence of the phase space structure of a mode; the $\phi$ dependence is not a sufficient description.


2. The current of a mode is not necessarily a constant. If the mode is stable it is constant, but it grows with rate $\Gamma_{max}$ if the mode is unstable.

3. The oscillation frequency can shift with beam current. Data such as shown in figure 6 for CESR\textsuperscript{10} can give information about the acceleration impedance, but that information is qualitative at best (see the CESR reference for an example). For transverse motion frequency shift measurements are easier to interpret quantitatively, and they are often performed and the results quoted.

Figure 6: Synchrontron tune shift with beam current in CESR. A, B, & C are for different accelerator conditions.

F. Multiple Particles: Summary

This is a summary of the previous three sections.

Rotation Harmonics

<table>
<thead>
<tr>
<th>Spectrum</th>
<th>A comb of lines at integer multiples of the rotation frequency, $\omega = n \omega_r$, $n = 0,1,2,\ldots$ (eq. (18))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Envelope</td>
<td>1. DC ($\omega = 0$) current = $Q_{nc}$ (eq. (18))</td>
</tr>
<tr>
<td></td>
<td>2. For a general charge density the envelope is the convolution of the charge density and $I_{nc}$ (eq. (18))</td>
</tr>
<tr>
<td></td>
<td>3. For a Gaussian bunch the envelope is a Gaussian with rms width equal to $1/\theta_0$ (eq. (25))</td>
</tr>
<tr>
<td>Measurement</td>
<td>A measurement of the current over a range of frequencies can be used to determine the rms bunch length.</td>
</tr>
</tbody>
</table>

Schottky Noise

<table>
<thead>
<tr>
<th>Spectrum</th>
<th>A comb of lines at integer multiples of the synchronous frequency, $\omega = m \omega_s + k \omega_r$, $k = -\infty,-1,1,\ldots,\infty$ (eq. (39))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Envelopes</td>
<td>1. The overall normalization is known; it is proportional to the square root of the number of particles. (eq. (35))</td>
</tr>
<tr>
<td></td>
<td>2. If $\eta$ is known, the frequency dependence of the rms current is known. There are no relative or absolute uncertainties. (eq. (28))</td>
</tr>
</tbody>
</table>

1. For a general charge density the envelope is a convolution of charge density with $\hat{I}_c^2$. (eq. (28))

4. For a Gaussian bunch the envelopes rise from 0 at $\omega = 0$ to a maximum value at $\omega = \omega_r$, and are approximately independent of frequency above that value. (eq. (30) and figure 6)

Measurement

1. The sideband frequencies give the synchronous frequency $\omega_r$.

2. The ratio of the amplitudes of the sidebands in a narrow frequency range give the bunch lengths.

Collective Effects

Spectrum

A collective mode with azimuthal dependence $e^{im\phi}$ produces sidebands of the rotation harmonics at $n \omega_r$. In addition, there is a frequency shift $\Delta \omega$ that depends on the mode, $\omega = n \omega_r + m(\omega_0 + \Delta \omega)$. (eq. (49))

Envelopes

1. The maximum detectable signal is that due to Schottky noise.

2. The overall normalization is not known. (eq. (48))

3. The frequency dependence is determined by the $\theta_0$ dependence of the coherent mode. Signals from coherent modes with high phase space periodicity tend to be strong at high frequencies. (eq. (30))

Measurement

1. A signal above the Schottky noise level is an indication of a coherent mode. The displacement of the sideband from the rotation harmonic gives the azimuthal structure of the mode.

2. Frequency shift as a function of current gives qualitative information about the acceleration impedance.

III. Signals From Transverse Motion

A. Introductory Remarks

The signals generated by the transverse motion of particles in a beam can be analyzed with the same techniques. Many of the results are similar, but there are important differences that arise from the dependence of the transverse signal on the longitudinal motion.

The beam current is the quantity that is measured in the longitudinal; in the transverse it is the dipole moment:

\[ d(t) = i(t) \times (i(t)) \]  

where \( i(t) \) is the transverse displacement (from the center of the detector). It has contributions from closed orbit errors, betatron oscillations, and synchrotron oscillations. Begin by considering single particles.

B. Single Particles

1. Constant Revolution Frequency: The dipole moment of an ion charge particle is

\[ d(t) = \left[ X' + \frac{\Delta E}{E_0} \right] \sum_{n} \left( \frac{1}{n} \right) \delta(t - nT) \text{.} \]  

The first term is from the closed orbit error, \( X' \), and the second is from the deviation from the nominal energy, \( \Delta E \), and the revolution at the detector, and \( T \). The revolution period, \( T \), depends on \( \Delta E/E_0 \).

The third term is due to a betatron oscillation with amplitude \( A_\beta \) and frequency

\[ \omega_\beta = \frac{q}{n} \text{,} \]  

\[ \Omega_\beta = n \frac{\Delta E}{E_0} \text{,} \]  

\[ \delta_t(t) = \sum_{n} \left( \frac{1}{n} \right) \delta(t - nT) \text{.} \]  

\[ Q_\beta \] is the betatron tune, and \( n_\beta \) and \( q_\beta \) are the integer and fractional parts of the tune, respectively. Figure 3a shows \( \delta_t(t) \); the dark lines are the dipole moment on successive turns. The average value is

\[ X' + \frac{\Delta E}{E_0} \text{,} \]  

and the variations from the average have amplitude \( A_\beta \).

The Fourier transform is

\[ \mathcal{D} \delta t(t) = X' + \frac{\Delta E}{E_0} \sum_{n} \delta(t - nT) \text{.} \]  

\[ \text{The closed orbit error and energy deviation lead to lines at the revolution harmonics. This signal is the same as for the longitudinal.} \]  

\[ \text{The betatron oscillations are offset from the revolution harmonics by } \omega_\beta \text{. Only the fractional part of } Q_\beta \text{ can be determined; nothing depends on } n_\beta \text{ because it can be absorbed in index of summation.} \]  

Figure 1b illustrates the spectrum. The signals from the closed orbit error and energy deviation are separated from those due to betatron oscillations. The measured spectrum is obtained by combining the positive and negative frequency signals (figure 1c).

There are two betatron lines between each pair of revolution harmonics: one of these is the positive frequency line and the other the negative frequency line. There is no way to distinguish them without changing the tune, and \( q_\beta \) derived from the spectrum has an ambiguity between \( q_\beta \) and -1 - \( q_\beta \). However, the change in signal frequency with a small change in \( Q_\beta \) can resolve the ambiguity.

2. Bunched Beams: Synchrony oscillations affect the dipole moment in three ways: 1) the arrival time is modulated, 2) the energy deviation is not constant, and 3) \( Q_\beta \) can be energy dependent.

The dipole moments is
is a series of lines offset from the rotation harmonics by multiples of $\omega_q$. The offsets and spectral envelopes are related as they are in the longitudinal.

The signal associated with the energy deviation is

$$
d(t) = \frac{\Delta E}{n_0} \sum_{n=-\infty}^{\infty} \delta(t-nT \tau_c \cos(\omega_q nT \phi))
$$

$$
= -\frac{n_0}{\Delta E} \sin(\omega_q t + \phi) \sum_{n=-\infty}^{\infty} \delta(t-nT \tau_c \cos(\omega_q nT \phi)).
$$

Equations (II.14) and (II.15) have been used. Taking the Fourier transform and using the Bessel function sum (eq. (II.11)) and the Poisson sum rule (eq. (II.9)) gives

$$
D(\omega) = \sum_{n=-\infty}^{\infty} j^{n} \frac{\omega}{2a} e^{-j\omega \tau_c \cos(\omega_q \tau_c)}
$$

Again, the signal is offset from the rotation harmonics by multiples of $\omega_q$, but the relationship between the offsets and envelopes is different than for a closed orbit error.

In particular, synchronous sidebands appear at low frequencies, $\omega_0 < 1$, where the dominant contribution is from $k = 0$. For $\omega_0 \gg \omega \gg \omega_0$, the Bessel functions are approximately equal, and

$$
D_0(\omega) = \sum_{n=-\infty}^{\infty} j^{n} \frac{\omega}{2a} e^{-j\omega \tau_c \cos(\omega_q \tau_c)}
$$

This spectrum is a comb of lines offset from the rotation harmonics by $\omega_q$. It is illustrated in figure 2 together with the spectrum from a closed orbit error. At high frequencies there is a similar contrast between the closed orbit and energy oscillation contributions to the signal.

The momentum deviation of the betatron wave produces qualitatively new features in the spectrum from betatron oscillations. The chromaticity, $\chi$, is defined by

$$
\delta Q_\chi = \frac{\Delta E}{n_0} \delta Q_\chi
$$

where $Q_\chi$ is the tune of a nominal energy particle. Let $w_\chi(t)$ denote the phase of the betatron oscillation at time $t$. The time rate of change of $w_\chi$ is

$$
\frac{dw_\chi}{dt} = \omega_\chi \frac{\Delta E}{n_0} \omega_\chi \frac{\Delta E}{n_0} (1 + \delta Q_\chi) = \omega_\chi \frac{\Delta E}{n_0} \frac{\Delta E}{n_0} dt.
$$

Equations (II.14) and (II.15) have been used, $\omega_\chi = Q_\chi \omega_q$, and $\omega_0 = \omega_q \Delta E$. Integrating

$$
w_\chi(t) = \omega_\chi t + \omega_0 t
$$

is the phase of the synchronous oscillation at $t = 0$, and $\omega$ is a constant of integration related to the phase of the betatron oscillation at $t = 0$

$$
\omega = w_\chi(0) = \omega_\chi t + \omega_0 t.
$$

The dipole moment is

$$
d(t) = A_\beta e^{j\omega \tau_c \cos(\omega_q \tau_c)}
$$

and the Fourier transform is

$$
D(\omega) = \sum_{n=-\infty}^{\infty} e^{j\omega \tau_c \cos(\omega_q \tau_c)}
$$

Both amplitude modulation and the dependence of $w_\chi(t)$ contribute to the argument of the second exponential. By analogy to the discussion for longitudinal motion, both can be expressed in terms of the spectral envelopes. Using the Bessel function sum (eq. (II.11)) and the Poisson sum rule (eq. (II.9))

$$
D(\omega) = \sum_{n=-\infty}^{\infty} e^{j\omega \tau_c \cos(\omega_q \tau_c)}
$$

The index $k$ enters into the spectral frequencies and the envelopes. For a fixed $k$, the lines are offset from the rotation harmonics by $\omega_q = k \omega_q$, and this is independent of $\chi$. The envelopes have most of the properties discussed for the longitudinal. They are Bessel functions centered at $\omega = \omega_q$ and $\chi = 0$, rather than $\omega = 0$. Chromaticity shifts the envelopes away from $\omega = 0$. This is illustrated for $k = 0, 1$ in figure 3a. The envelope shifts to positive or negative frequency depending on $\chi$. The envelope width is determined by $\tau_c$. $\omega_\beta(\chi)$ has a central maximum at $\chi = 0$ and the first minimum at $\chi = \pm 2\chi_0$. In terms of frequencies, the maximum is at $\omega_\chi$, and the first minima are at $\omega = \omega_q + \omega_0 \pm \chi_0 \omega_q \Delta E$. Figure 3c shows this for two values of $\chi_0$. When $\chi_0 \neq 0$, the first maximum of the envelope occurs at

$$
\omega = \omega_q \omega_\chi \chi_0 \omega_q \Delta E
$$

Figure 3c shows this for $k = 1$, $\omega_0 = 1$ and two values of $\omega_q$, the two envelopes are centered at $\omega = -1$ but the envelopes have different scales.
Figure 3: Spectral envelopes. a) k = 0 for different chromaticities. b) k = 0 for different synchrotron oscillation amplitudes. c) k = 1 for different synchrotron oscillation amplitudes.

Consequences of these chromaticity effects are that synchrotron sidebands of the betatron frequency can appear near ω = 0, and positive and negative frequency lines need not have equal amplitude. In figure 4a ω = 0; the only lines near ω = 0 are betatron lines (k = 0). At ω = 2kπ, the betatron lines are small (γ/ω相比于) and the synchrotron sidebands for k = ±1, ±2, and ±3 can be seen. The figure was drawn for Q = 0.38; consequently the right-hand set of lines are the positive frequency lines and the left-hand set the negative frequency ones. They have equal amplitudes because k = 0.

In figure 4b ω = 2, and the spectrum is strikingly different: 1) the betatron lines at low frequency have smaller amplitudes. 2) there are synchrotron sidebands at low frequency. 3) there are betatron lines at ω = 2kπ, and 4) the positive and negative frequency lines have equal amplitudes at ω = 0, but at ω = 2.4 they have different amplitudes. The last of these follows because the positive and negative frequency lines come from ω_0 = ω_0 ω_0 + ω_0 kω_0 (ω_0 + kω_0) and ω_0 = n, a_0 = 0k_0 (ω_0 - kω_0), respectively. For ω = 0, k = 0, n = 1, and the Bessel functions do not change significantly for this small a frequency difference. At ω = 2.4 the arguments of the Bessel functions are different (ω_0 - ω_0 = kω_0 - 4.4kω_0), and as a result, the amplitudes of the lines are unequal.

Figure 4: Spectral lines near ω = 0 and ω = 2 kπ for k = 0. A) k = 0. B) k = ±1, ±2, ±3.

C. Multiple Particles: Introduction

The discussion will concentrate on betatron oscillations. In analogy with eq. (21.16) the beam is described by a 6-dimensional charge density, ρ_0, defined by

\[ \rho_0(\tau, \phi, \lambda, 0) = \int \phi \, d\tau \, d\lambda \, d\psi \, d\omega \, d\varepsilon \, d\gamma \]

where ρ_0 = the synchrotron oscillation amplitude,
\[ \phi = \text{the synchrotron oscillation phase at t = 0} \]
\[ \lambda = \text{the betatron oscillation phase at t = 0} \]
\[ \psi = \text{the phase of the beam oscillation at t = 0 in terms of } \psi \]

The beam signal is given by multiplying the single particle signal, eq. (17), by ρ_0 andintegrating over phase space. 1 Schoonmaker noise and collective effects are considered in detail.

D. Schoonmaker Noise

This treatment parallels section II.D. The particles move independently, and, as a consequence, the average charge density doesn't depend on n and w. In addition, the betatron and synchrotron amplitudes of a particle are assumed to be independent of each other. The average charge density is

\[ \rho_0(\tau_0, \phi, \lambda, 0) = \int \phi \, d\tau_0 \, d\lambda_0 \, d\psi \, d\omega_0 \, d\varepsilon_0 \, d\gamma_0 \]

1. Note that ρ_0 could have been written as a function of ω_0t = 0 rather than ω_0, but the above choice is more natural because ω_0 enters directly into the phase of eq. (17). Using ω_0t would introduce another Bessel function term.
\[ \bar{\rho}_k = \frac{1}{\sigma_k} k \rho_k (\tau_k) \rho_P (\lambda_P). \] (20)

When eq. (17) is multiplied by this and the integral over \( v \) performed, the result is zero. The signals from individual particles cancel on average because of their different betatron phases.

As in the longitudinal fluctuations from the average lead to Schockley noise. For \( P \) particles the charge density is

\[ \bar{\rho}_k = \frac{1}{P} \sum_{p=1}^{P} \delta(\tau_k - \bar{\tau}_p) \delta(\phi - \phi_p) \delta(\lambda_P - \lambda_{P,P}) \delta(\psi - \psi_p). \] (21)

where \( \tau_{p,P} \) are the coordinates of particle \( p \). The dipole moment at \( \omega = \omega_P + \sigma_P \cdot k \lambda_P \) is

\[ D_{nk}(\omega) = q n \sum_{p=1}^{P} A_{np} \exp[j(\omega_n - \omega_P) \frac{n}{2}] \sum_{p=1}^{P} \exp[j(\phi_{np} - \phi_p)]. \] (22)

The mean value, given by averaging over all samples of \( P \) particles, is zero. This holds even for \( k = 0 \) because the phase depends on the betatron oscillation phase.

The square of the rms dipole moment is

\[ \langle D_{nk}(\omega) \rangle = \frac{1}{2} \int D_{nk}(\omega) \overline{D_{nk}(\omega)} \, d\omega \]

\[ = \frac{1}{2} \sum_{p=1}^{P} A_{np}^2 \exp[j(\omega_n - \omega_P) \frac{n}{2}] \sum_{p=1}^{P} \exp[j(\phi_{np} - \phi_p)]. \] (23)

The average phase factor equals zero unless \( p = c \); this removes one of the terms.

\[ \langle D_{nk}(\omega) \rangle = \frac{1}{2} \sum_{p=1}^{P} A_{np}^2 \exp[j(\omega_n - \omega_P) \frac{n}{2}] \sum_{p=1}^{P} \exp[j(\phi_{np} - \phi_p)]. \] (24)

Equation (23) shows that the rms fluctuations is proportional to the typical betatron amplitude and the square root of the number of particles.

Following section III D

\[ \langle D_{nk}(\omega) \rangle^{1/2} = |\frac{1}{2} A_{nk}| \exp[-\tau_k / \gamma + \frac{1}{2} \lambda_P^2 / \gamma_P^2]. \] (25)

For a Gaussian beam with bunch length \( \sigma_B \) and width \( \sigma_P \),

\[ \bar{\rho}_k = \frac{\bar{\rho}_P}{\frac{1}{\sigma_k^2} \frac{1}{\sigma_P^2}} \exp[-\tau_k^2 / 2 \sigma_P^2 - \lambda_P^2 / 2 \gamma_P^2]. \] (26)

the integral over \( A_n \) equals \( 2 \gamma_P^2 \). Performing the integral over \( \tau_k^2 \)

\[ \langle D_{nk}(\omega) \rangle^{1/2} = |\frac{1}{2} A_{nk}| \exp[-\tau_k^2 / 2 \sigma_P^2 - \lambda_P^2 / 2 \gamma_P^2]. \] (27)

The envelopes are centered at \( \omega = \omega_P \) and have broad maxima at \( \omega = \omega_P \pm k \) (figure 5).

Most of this result is just what would be expected from a simple statistical argument. The mean position of the beam is being measured with \( P \) independent particles, and, as with any measurement, the error in the mean is \( 1 / \sqrt{P} \) times the rms spread of a single measurement - \( \sigma_{\rho_P} \). Therefore, the expected signal should be equal to that from a charge \( P \delta P \) with a displacement \( \sigma_{\rho_P} \). This argument doesn’t give the dependence on frequency, bunch length, and chromaticity.

Typically \( P \) is measured with a beam current monitor, and \( \sigma_P \) is measured with a longitudinal detector. Given these, the rms dipole moment can be used to measure transverse properties of the beam including the fractional part of the tune, the beam width, and the chromaticity.

The rms signal voltage is

\[ V^{(1/2)}_{\text{rms}}(\omega) = \frac{1}{\sqrt{2}} \langle D_{nk}(\omega) \rangle^{1/2} \] (28)

where \( S_{\rho} \) is the detector sensitivity. The signal levels are low even with a high sensitivity detector.

In addition, there can be large, unaugmented signals from closed orbit errors. These have different frequencies, but they can be overloaded electronics, and they must be removed by filtering. From these considerations, the solution is to analyze the rate of a detector electronics system with a relatively narrow frequency range (less than \( \sigma_{\rho_P} \) centered at an appropriately chosen frequency.

In the following discussion examples are based on experience at the Tevatron where the parameters were approximately those in Table I. Some of the points above are illustrated in figure 6. Figure 6a shows a 1 MHz span centered at the peak response of the detector. The dominant signals are the rotation harmonics spaced at 47.7 kHz; they don’t have equal amplitudes because there were three bunches in the machine. Figure 6b shows a 100 kHz span centered at the same frequency; the beam position can be seen but only because the section is being simulated by external excitation. Schockley signals would be 20 to 40 dB lower in amplitude.

First consider betatron tune measurement (\( k = 0 \)). Just as for a single particle, the signal frequency gives the fractional part of the tune, \( \omega_P = \frac{\omega}{\gamma} - \omega_0 \), where \( \omega_0 \) is the frequency in the range \( n \lambda_P \). The ambiguity can be resolved by changing the phase of a known direction. The rms voltage is given by eqs. (37) and (28) with \( k = 0 \). This is plotted for the Tevatron parameters in figure 7. The signal voltage is 0.25 \( \mu V \) near \( \omega_0 = 0 \). This corresponds to an rms motion of the mean position, \( \sigma_{\omega_0} = 3.4 \text{ mm} \).

For comparison the Johnson Noise voltage

\[ V^{(1/2)}_{\text{noise}}(\omega) = \frac{2 \gamma P_{\text{nuke}}}{2 \pi k T} (\omega_0 / 2 \pi)^{1/2} \] (30)

\[ \text{References}
\[ 1. \text{L.S. Ginzburg and I.M. Rybik, Table of Integrals, Series, and Products (New York: Academic Press, 1963), eq. 6.633.2.}
Table 1: Transverse Motion in the Tevatron Collider, 1987

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam Tunes</td>
<td>H: 19.407, V: 19.404</td>
</tr>
<tr>
<td>Normalized Emittance</td>
<td>7 x 10^{-3} m</td>
</tr>
<tr>
<td>Beam Energy</td>
<td>E: 900 GeV</td>
</tr>
<tr>
<td>Emittance</td>
<td>7 x 10^{-9} m</td>
</tr>
<tr>
<td>β Functions at Detector Locations</td>
<td>0.84 mm</td>
</tr>
<tr>
<td>Beam Sizes at Detector Locations</td>
<td>27.7 kHz</td>
</tr>
<tr>
<td>Revolution Frequency</td>
<td>ν₀ = 47.7 kHz</td>
</tr>
<tr>
<td>Synchronous Oscillation Frequency</td>
<td>ν = 37 kHz</td>
</tr>
<tr>
<td>RMS Beam Length</td>
<td>5 x 10^{-6} m</td>
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<tr>
<td>Momentum Compression</td>
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<tr>
<td>Number of Particles in a Single Bunch</td>
<td>5 x 10^{11}</td>
</tr>
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</table>

Detector/Electronics Properties

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Center Frequency</td>
<td>21.4 MHz</td>
</tr>
<tr>
<td>Frequency Range</td>
<td>15 kHz</td>
</tr>
<tr>
<td>Sensitivity</td>
<td>S = 27.3 pm</td>
</tr>
</tbody>
</table>

* Related longitudinal parameters are given in Chapter II, Table 1.

It is Boltzmann's constant, T is the absolute temperature. K is the size parameter of the electronics, and δv is the measurement bandwidth. Choosing δv = 10 Hz, the approximate spread of beam frequencies, and R = 50 Ω gives a noise voltage of 20 nV, which is 10 times less than the maximum expected signal. It is difficult to reach the Johnson noise level for several reasons: (a) the signal processing electronics introduces noise (although amplifiers with noise figures of only a few dB are commercially available), (b) there is a high load on the power connecting cables, and (c) the combination of closed orbit signals and nonlinearities in the filters can combine to generate modulation products. The noise from these sources can be minimal, but it requires careful work.

There is a spread of beam frequencies caused by effects such as nonlinearities leading to amplitude dependence of tune and the beam-beam interaction. The tune noise can be measured by reducing the measurement bandwidth, δv, below the beam frequency spread. The rms bunch mode is given by eq. (27) with the P replaced by the number of particles in the δv, change in particle beam. The noise sensitivity of the measurement, δv, is given by eq. (33) and (34). Near the peak of the bunch distribution, the noise sensitivity is 

\[ \text{Noise Sensitivity} = \sqrt{10(KTBR)} \]

where K is the radiation constant and T is the noise temperature, and the signal-to-noise ratio affects the measurement time needed to see a meaningful signal.

The beam width follows directly from the signal amplitude through eq. (27). The principal uncertainty is whether that equation is the correct interpretation of the signal. It isn't if there are collective effects (section III.E) or if there is an external excitation of the beam. The latter was important in the initial running of the Tevatron collider. The beam size determined by eq. (27) was consistent with the size measured with flying wires. This led to the conclusion that the noise of the counter-of-charge was substantially larger than δv/W and that there must be external excitations of the beam. A rapidly fluctuating 1 Gauss-mm magnetic field could explain the data. Ultimately, the RF system, abort kickers, and interaction region quadrupoles were found to be sources of excitation.

4. K = 8.626 x 10^{-5} V^2 Hz GHz kK = 293 K.

Figure 6: Tevatron beam signals: the detector was tuned to 26.2 MHz for these pictures.
It should be noted that such external excitation does not affect tune measurements. In fact, low level noise excitation of the beam is often used to “shake” the beam for rapid tune measurement.

The most direct way to measure the chromaticity is to vary the beam energy while keeping the magnetic fields fixed by changing the RF frequency. The change of tune with energy gives \( \xi \) (eq. (11)).

\[
\frac{\Delta \nu}{\nu} = \frac{\Delta \nu}{\nu} \frac{\partial \omega}{\partial \nu} \frac{\nu}{\partial \nu} \tag{31}
\]

An alternative method is to measure the relative amplitudes of the synchrotron subbands of the beam. Eq. (27) gives the rms signal at different subband frequencies when the signal is due to Schottky noise. The relative amplitudes are correct also if the beam is excited by an external signal, but the equation is incorrect if there are collective effects. Assume this is the case. The amplitudes for the Tevatron are plotted in Figure 7. Near \( \xi = 0 \) the only signals are at \( \nu = \nu_0 + \omega_0 (0 \pm k) \) and \( \nu = \nu_0 + \omega_0 \kappa \kappa \). As \( \xi \) increases more subbands appear.

Figure 7: Tevatron signals at 21.4 MHz.

Figure 8 shows some data from the Tevatron taken when the beam was externally excited and both the horizontal and vertical beams were present. The vertical \( k = 2 \) are not useful below the \( \kappa = 0 \) line, and the \( \kappa = 2 \) lines are barely visible; from Figure 7 the chromaticity is \( \xi \approx -0.4 \) or \( \xi \approx -0.3 \). The beam would be unstable if the chromaticity were negative (i.e., the beam is unstable). For both subbands, the horizontal \( k = 0, 2, 4 \) and \( \kappa = 0 \) are clearly seen, and from the ratio of amplitudes the chromaticity is measured to be \( \xi_0 \approx 0.7 \).

\[ \Delta \nu = 6.3 \text{ Hz} \]

\[ \Delta \nu_b = 0.0524 \]

\[ \Delta \nu_b = 2.5 \text{ kHz} \]

\[ \Delta \nu_b = 0.4152 \]

\[ \Delta \nu_b = 4.15 \text{ dB} \]

\[ \Delta \nu_b = 21.395 \text{ MHz} \]

\[ \Delta \nu_b = 0.4 \text{ MHz} \]

\[ \Delta \nu_b = 1350189 \text{ MHz} \]

\[ \Delta \nu_b = 0.092 \text{ MHz} \]

\[ \Delta \nu_b = 0.084 \text{ MHz} \]

\[ \Delta \nu_b = 0.072 \text{ MHz} \]

\[ \Delta \nu_b = 0.061 \text{ MHz} \]

\[ \Delta \nu_b = 0.050 \text{ MHz} \]

\[ \Delta \nu_b = 0.040 \text{ MHz} \]

\[ \Delta \nu_b = 0.031 \text{ MHz} \]

\[ \Delta \nu_b = 0.022 \text{ MHz} \]

\[ \Delta \nu_b = 0.013 \text{ MHz} \]

\[ \Delta \nu_b = 0.004 \text{ MHz} \]

\[ \Delta \nu_b = 0 \text{ MHz} \]

E. Collective Effects

In this case particles do not move independently, and the beam develops space structure by interacting with the surrounding environment. The mechanism is closely related to that discussed in section 4.1: 1) the beam generates a dipole signal (the transverse impedance of the accelerating) and produces deflecting wakefields. 2) these act back on the beam to modify the phase space distribution. 3) this changes the beam generated signal and the wakefields. The coherent modes are the solution of this eigenvalue problem.

The charge density at \( \omega = 0 \) is a periodic function of \( \varphi \) and \( \psi \), and it can be expanded in a Fourier series in these angles

\[ \rho_0 \left( \varphi, \psi, \varphi, \psi \right) = \sum_{r=-\infty}^{\infty} \sum_{\varphi} \rho_{r,\varphi} e^{i \varphi \varphi} \tag{32} \]

The expansion coefficients are functions of \( \varphi \) and \( \psi \). Multiplying eq. (17) by \( \rho_0 \) to get the dipole signal and performing the \( \psi \) integration gives

\[ \Delta \psi \left( \varphi \right) = \sum_{r=-\infty}^{\infty} \bar{\rho}_{r} e^{i \varphi \varphi} \tag{33} \]

This integral equals zero unless \( r = -1 \), and, therefore, only that term has a dipole signal.

The transverse coherent modes have transverse amplitudes that are the same for all particles with the same synchrotron amplitude. Using this and the result from eq. (33), \( \rho_0 \) is

\[ \rho_0 \left( \varphi, \psi, \varphi, \psi \right) = \left( \sum_{r=-\infty}^{\infty} \bar{\rho}_{r} e^{i \varphi \varphi} \right) \left( \sum_{r=-\infty}^{\infty} \bar{\rho}_{r} e^{i \varphi \varphi} \right) \]

This is the charge density for calculating the signals of coherent modes.

1. Approximate Picture: Begin with the approximation that the phase space structure is evolving slowly, the tune is constant, and the beam is slowly turning. i.e., at a later time the expression similar to eq. (34) could be written, but the \( \rho_0 \)'s would differ only by a phase factor, \( e^{i \varphi \varphi} \), which accounts for the phase space motion of particles. The signal is given by multiplying eq. (17) by \( \rho_0 \) and integrating over phase space

\[ \Delta \psi \left( \varphi \right) = \sum_{r=-\infty}^{\infty} \bar{\rho}_{r} e^{i \varphi \varphi} \tag{35} \]

2. The integral equals zero unless \( q = -k \). The \( \varphi \) periodicity determines the signal frequency; if \( \rho_0 = e^{i \varphi \varphi} \), the signal frequency is \( \omega = \omega_0 + \varphi^k + \omega_0 \).

The general result is

\[ \Delta \psi \left( \varphi \right) = \sum_{r=-\infty}^{\infty} \bar{\rho}_{r} e^{i \varphi \varphi} \left( \Delta \psi \left( \varphi \right) \right) \]

\[ \Delta \psi \left( \varphi \right) = \sum_{r=-\infty}^{\infty} \bar{\rho}_{r} e^{i \varphi \varphi} \left( \Delta \psi \left( \varphi \right) \right) \]

where

\[ \sum_{r=-\infty}^{\infty} \bar{\rho}_{r} e^{i \varphi \varphi} \left( \Delta \psi \left( \varphi \right) \right) \]

3. The \( \rho_0 \)'s depend on the phase space structure, and that needs to be known to calculate the signal. This is in contrast to the Schottky case where there are no unknowns. The Bessel function in the integrand has the usual effect of correlating the periodicity and the frequency range of the signal. As \( \kappa \) increases the signal shifts to higher values of \( \varphi \).


To show the qualitative features of this result assume i) the coherent eigenmodes are given by Hermite polynomials, and ii) $A_0$ is independent of $t$. Begin with $k = 0$

$$
\rho_0 = \exp\left(-\frac{t^2}{2\sigma_e^2}\right).
$$

Substituting into eq. (36) and performing the integrals,

$$
D_0(\omega) = \exp\left(-\frac{(\omega_0^2 - \omega^2)}{2}\sigma_e^2\right).
$$

$D_0(\omega)$ is a Gaussian with an rms width of $1/\sqrt{2\pi}$ centered at $\omega = \omega_0$. This is similar to point being illustrated in figure 3b for a single particle. In both cases, $\xi$ determines the center frequency and the longitudinal amplitude determines the width. However, in the present case the distribution of synchronous amplitudes washes out the multiple zeros in figure 3b.

For $k = 1$,

$$
D_1(\omega) = \exp\left(-\frac{(\omega_0^2 - \omega^2)}{2}\sigma_e^2\right).
$$

(39)

and

$$
D_0(\omega) = \exp\left(-\frac{(\omega_0^2 - \omega^2)}{2}\sigma_e^2\right).
$$

(40)

This envelope has two maxima: one is at $\omega \equiv 1/\sqrt{2} \sigma_e$ and the other at $\omega \equiv -1/\sqrt{2} \sigma_e$. Eqs. (38) and (40) illustrate the second way that the longitudinal phase space structure affects the signal. In addition to giving the frequencies of the spectral lines ($\omega_0, \omega_0 + \omega_0, \omega_0^2 - \omega_0^2$), it determines the spectral envelope. As $k$ increases, the peaks in the envelope move to higher values of $\omega$ and $\xi$.

Whether this is higher or lower heat depends on $\omega_0$; for example, for $\omega_0 = 10\sigma_e$, the $k = 0$ signal is centered at $\omega = 10\sigma_e$, while one peak of the $k = 1$ envelope is at a lower frequency, $\omega = 0$ (the other one is at $-20\sigma_e$).

The result form eq. (33), that only the $-i1$ term in eq. (32) has a dipole signal, leads to the chromaticity effects that have been shown formally. They can be understood semi-quantitatively by considering the dipole moment (eq. (1)). Using eq. (33) the transverse displacement of a bunch at $t = 0$ is

$$
x(t) = A_0 \exp\left(\frac{j}{2} \int_0^t g\left(\omega_0, \omega_0^2 - \omega^2\right)\right).
$$

(41)

The dipole moment is given by multiplying $x(t)$ by $p_d$ and integrating over phase space. The goal is to calculate $d(t)$, and this is easiest if eq. (3.15) is used to switch from polar to Cartesian coordinates. Doing this

$$
d(t) = \frac{1}{k} \int_0^t dt' p_d(t') A_0(t') \exp\left(\frac{j}{2} \int_0^t g\left(\omega_0, \omega_0^2 - \omega^2\right)\right).
$$

(42)

where $\omega = \omega_0^2$ and $t'^2 = t^2 + k^2$. The dipole moment depends on $t$ through the longitudinal structure of the bunch and the phase shift of the transverse displacement along the bunch. The latter is proportional to $\omega$. The dipole signal depends on the invariance between these two terms.

9. These arise naturally for a bunch with a Gaussian distribution.


12. $d(t)$ on turn $N$ is given by multiplying eq. (42) by $(2\pi)^N$.
2. Coherent Tune Shifts: Growth, damping, and frequency shifts due to wakefields can be accounted for with a treatment similar to that in section II.2.2. The charge density at time is expanded in a Fourier series in $q$ and an appropriate basis in the radial coordinate $r_0$:

$$\rho_q(t) = \sum_q \rho_q(t_0) \delta[q - q_0] \exp[i((\eta_q + \sigma_q) r_0 + \phi_q) + \omega_q t]$$

The dipole signal can be calculated for each term in eq. (43), so that signal multiplied by the transverse impedance gives the deflecting fields that act back on the beam. The frequency shifts and modes are the eigenvalues and eigenvectors, respectively, of the resultant eigenvalue problem.

The detailed formulation starting with the Vlasov equation is contained in the references reviewed in section II.2. A summary of results in:

1. A perturbative approach to the Vlasov equation leads to an integral equation involving a convolution (over frequency) of the dipole signal and the transverse impedance.

2. The $\Delta \omega_{\text{cap}}$'s are proportional to the beam current and the transverse impedance weighted by the frequency spectrum of the mode. The mode spectrum depends on the $\phi$ periodicity, the radial $(r_0)$ structure, and chromaticity; therefore, the frequency shifts are different for each mode and depend on $r_0$.

3. The $\Delta \omega_{\text{cap}}$'s are complex. The real part, $\Delta \omega_{\text{cap}} = \text{Re}(\Delta \omega_{\text{cap}})$, changes the betatron frequency from $\omega_{\text{cap}}$ to $\omega_{\text{cap}} + \Delta \omega_{\text{cap}}$. The imaginary part, $\Delta \omega_{\text{cap}} = \text{Im}(\Delta \omega_{\text{cap}})$, can lead to growth or damping.

4. Landau damping and the radiation damping can prevent instabilities.

5. The imaginary part of the transverse impedance is the component of the deflecting field in phase with the displacement. This component shifts the betatron frequency and determines $\Delta \omega_{\text{cap}}$.

6. At low currents signal frequencies depend on $q$, the longitudinal phase space periodicity, as $\Delta \omega_{\text{cap}} = \omega_{\text{cap}} + \Delta \omega_{\text{cap}}$. As the beam current increases the frequencies shift away from these values. When $\Delta \omega_{\text{cap}} - \omega_{\text{cap}}$, modes with different periodicity can couple; this leads to the "fast head-tail instability" which has a large growth rate.

Some of these points are illustrated with data from CESR in figure 10. The data in figure 10 show that the tune shifts for modes with $q = 0, 1, 2$. These measurements were made at low frequency, $\omega_0 = \omega_{\text{cap}}$, and positive chromaticity. The beam was driven with a "chirp" magnet (an external field); the drive frequency matched the measurement frequency. The azimuthal periodicity, $q$, was determined by the signal frequency, and different radial modes could not be distinguished. These data illustrate point 2 above. The signal spectra is different for these modes, and, therefore, the tune shifts are unequal because they are determined by the transverse impedance in different frequency ranges.

The data in figure 10b show coherent damping as a function of chromaticity. These data were taken by giving the beam an impulse and measuring the decay time constant of the coherent signal with a tuned receiver centered at $\omega = \omega_{\text{cap}}$ (the $q = 0$ frequency). The spectrum shifts as $\omega_0$ is changed, and this affects the frequency range over which the impedance is sampled. If the lines in figure 10b are extrapolated to $\omega_0 = 0$ we would be positive and there would be an instability. This is the head-tail instability that was discovered by Pinczon and Grassie used in later experiments by Pelligrini and Sands.

F. Multiple Particles: Summary

This is a summary of the previous two sections.

Schottky Noise

Spectrum A comb of sidebands at integer multiples of the synchronous frequency offset from the motion harmonic by $\omega_q = \omega_{\text{cap}} + \omega_{q_{\text{cap}}}$. $k = 1, 2, 3, \ldots$.

Envelopes 1. The overall normalization is known; it is proportional to the square root of the number of particles and the typical betatron amplitude $(\omega_0, \omega)$.

2. $\rho(x_0, \omega_0)$ and $p(x_0, \omega_0)$ are known, the frequency dependence of the rms dipole signal is known. There are no relative or absolute uncertainties. $(\omega_0, \omega)$.

3. For a general charge density $\omega_{\text{cap}}$ parameter $\omega_{\text{cap}}$ is given by the convolution of charge density with $\lambda^2$. The envelope is centered at $\omega_0 = \omega_{\text{cap}}$. $(\omega_0, \omega)$.

4. For a Gaussian bunch the envelopes are centered at $\omega_0 = \omega_{\text{cap}}$. There are broad maxima at $\omega_0 = \omega_{\text{cap}} + k \lambda_{\text{cap}}$. $(\omega_0, \omega)$.

Measurements 1. The beam's frequency fits to the fractional part of the tune $\omega_{\text{cap}}$. $(\omega_0, \omega)$.

2. The signal amplitude $\delta \rho$ given by the beam size and $\omega_0$ can be used to detect excitations of the beam.

3. The shift of tune with RF frequency gives the chromaticity $(\omega_0, \omega)$.

4. The relative amplitudes of sidebands measures the chromaticity. $(\omega_0, \omega)$.

Collective Effects

Spectrum A collective mode with periodicity in longitudinal phase space given by $\exp(\omega q)$.

Envelopes 1. The maximum detectable signal is due to Schottky noise.

2. The overall normalization is not known $(\omega_0, \omega)$.

3. Frequency dependence is determined by the $\omega_{\text{cap}}$ dependence of the mode. Signatures from coherent modes with high periodicity in longitudinal phase space tend to be strong at high frequencies $(\omega_0, \omega)$.

Measurements 1. A signal above the Schottky noise level is an indication of a coherent mode. The displacements of the sidebands from the betatron frequency gives the azimuthal structure of the mode.

2. Frequency shift at a fraction of current gives qualitative information about the accelerator impedance.

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IV. Beam Detectors

A. Introduction

For a relativistic beam, the line density of the image current flowing on the vacuum chamber wall equals the line density of the beam (with an overall minus sign because it's an image current)

\[ i_t(x) = -i_s(x) \]  
\[ i_t(y) = -i_s(y) \]  
\[ (1) \]

This follows from the Lorentz contraction of the electromagnetic fields of a relativistic particle. From a circuit point-of-view the image current is as close as you can get to an ideal current source - the image current will flow through any impedance placed in its path. A beam detector (or any other object) with impedance \( Z(0) \) can be analyzed as a circuit with a current source driving the impedance (figure 1). If \( Z \) is a beam detector which is sensitive to longitudinal motion and voltage \( V_{out} \) is produced across load \( R \), the detector's longitudinal sensitivity is defined as

\[ S_L(0) = \frac{V_{out}(0)}{i_s(0)} \]  
\[ (2) \]

Figure 1: a) The beam and wall currents. b) The equivalent circuit.

The distribution of image current around the beam pipe circumference depends on the beam pipe geometry and the position of the beam. Figure 2 shows an example of a detector that measures the transverse displacement of the beam. The impedance, \( Z \), and load resistor, \( R \), are connected between the two sides of the detector. The transverse sensitivity of the detector in terms of the voltage \( V_{out} \) across \( R \)

\[ S_T(0) = \frac{V_{out}(0)}{i_s(0)} \]  
\[ (3) \]

\( D(\alpha) \) is the dipole moment which was discussed extensively in chapter III.

Two specific detectors are discussed in the following sections.


B. The SPS Wideband Wall Current Pick-Up

The detector is a gap in the beam pipe (figure 1a). There is a cladded metallic chamber around the gap which provides shielding, giving a DC current path, and, for this particular detector, is the vacuum enclosure. The latter could be avoided by using a ceramic gap, but this was decided against for reasons discussed below. The wall current flows through a parallel combination of the gap capacitance, \( C \), the measuring resistor, \( R \), and the impedance of the shielding box, \( Z \). The voltage developed across the gap is

\[ V(\alpha) = \frac{1}{Z} \left[ \frac{1}{RC + Z} \right] \]  
\[ (4) \]

The shielding box is inductive at low frequencies; the current is \( V_{in} = RL \). The gap capacitance determines the high frequency cutoff; it would equal \( V_{in} \) if \( C \) were purely inductive. The beam pipe-shielding box structure can have resonances, and these will modify the high frequency behavior.

Some comments about the design criteria are: 1) it pays to increase the impedance to reduce the low frequency cutoff, 2) the gap capacitance should be decreased to raise the high frequency cutoff, 3) attention must be paid to resonances and other aspects of the high frequency behavior of the structure, 4) \( R \) and \( Z \) should be distributed around the circumference of the beam pipe to avoid beam position sensitivity. Usually there are other factors such as available space and cost which strongly influence the design. These are impossible to include in a general way, so the choice was made to describe a particular detector, the SPS wall current pick-up. Figure 3 is a schematic.

The inductance can be increased by inserting ferrite between the beam pipe and shielding box, but ferrite becomes lossy at high frequencies and gives a poorly characterized high frequency performance. Instead, the shielding box (figure 4) and beam pipe (figure 5) were configured like the conductors of a coaxial line with a characteristic impedance \( Z_0 = 50 \Omega \). Inductive material (FERRITE) placed between the conductors produces a matched load. The line is terminated above \(-10 \, \text{MHz}\), and the reflection coefficient is less than 0.13 for frequencies from 100 MHz to about 4 GHz. The low frequency inductance is about 0.25 \( \mu \text{H} \). The coaxial line-inductor design involved bench measurements of different absorber materials and configurations; the original references discuss these in detail.

The gap was chosen to be 2 mm long (detail drawings att bottom of figure 2) which was a compromise between low capacitance and mechanical constraints. With this gap and an 83 mm diameter, 1.5 mm thick beam pipe, the gap capacitance is 7 pF. The design specification was for \( V_{in} = 4 \, \text{GHz} \), so the parallel combination of \( R \) and \( Z_0 \) must be less than 5.75 \( \Omega \). A vacuum rather than ceramic gap was selected because ceramic would have increased the capacitance and needed a thin metallic coming to prevent charge build-up. A factor in this decision was that the RF absorber could be designed to have good vacuum properties without affecting RF properties.

The signal was coupled out at eight points equally spaced around the circumference: this removed sensitivity to beam position. Stripline (50 \( \Omega \), figure 3) detail A) was used between the gap and coax connectors; without this the series inductance of the laser determined the high frequency response. The output powers were added with a 50 \( \Omega \) hybrid, and with the hybrid output terminated in 50 \( \Omega \), each of the eight signal points has a 50 \( \Omega \) resistance.

Figure 4: The equivalent circuit of the SPS detector.

The equivalent circuit for this detector is shown in Figure 4. At intermediate frequencies where the capacitance and inductance can be ignored, $i_2$ flows through $25\Omega$ and $i_3$ flows through each of the $50\Omega$ resistors. The power generated in each resistor is

$$P = i^2R = (1/10)^2 \times 50\Omega = 2.5\Omega.$$  

(5)

The hybrid adds power: there is a loss, $K = 1.32$, in the hybrid. The output power is

$$P_{out} = \frac{P}{K} = \frac{2.5\Omega}{1.32} = 1.9\Omega.$$  

(6)

and the output voltage and sensitivity are

$$V_{out} = 1/\sqrt{(50 \times 3.565)} \times \frac{1}{2} \times 0.5\Omega; \quad S = 13.4\Omega.$$  

(7)

The frequency response of the detector was measured under several conditions. In all cases $V_{low} = 4.2$ to 4.4 MHz. Looking directly at one of the eight outputs with all the others terminated $V_{high} = 4.4$ GHz. When the eight signals are combined in the hybrid, $V_{out} = 3.4$ GHz; the hybrid was limiting the frequency response. When the signal is run through two of high quality cable to beam diagnostic area, $V_{high} = 2.0$ GHz, this is consistent with specified cable attenuation. This loss can be avoided by placing the analyzing instrument, a spectrum analyzer or sampling head, close to the detector. Of course, the instrument must be under remote control and have remote readouts, but with the IEEE bus this has become considerably easier than it used to be.

This wideband detector is used at the SPS for pulse measurements. How would it is do with Schottky signal? Using $i_2$ and $i_3$ (figure 4A) and $S = 13.4\Omega$, the signal is 24 eV; this is a factor of ten above the Johnson noise (eq. (11.30)) for $\Delta f v = 1$ Hz. The detector could be used for Schottky measurements.

C. The Tevatron Transverse Beam Detector

The goals were to design and build a high sensitivity detector that could be used for measurements with Schottky noise. The design concept, shown in figure 5, is based on earlier work by line and Sparkes at the SPS. It is a set of parallel plates with impedance $Z$ connected

between them. $Z$ has several elements: the main one is a loosely coupled transformer. Figure 5 is a detailed schematic of the detector.

1. Background: Let $i_2(u)$ and $i_3(u)$ denote the wall currents on the left- and right-hand planes at the upstream and downstream end of the detector. They depend on the transverse displacement of the beam and the fraction of the total wall current that flows on the plates $L$.

$$i_2(u) = L = \frac{1}{Z} = \frac{(2\pi - \tan^{-1}(1)) \times \frac{2\pi}{8} \times \frac{1}{1 + \frac{1}{Z}}}{3}.$$  

(8)

In this equation $g$ is the plate separation, $w$ is the wall height, $x$ is the displacement towards the right-hand plane, and the $+\tan^{-1}$ sign applies for $i_2(u)$. When $g \gg 1$, $Q(1 + \frac{1}{Z}) = 1/2$. Only the difference current will develop a voltage across $Z$: at the upstream end it is

$$i_{out} = i_2(u) - i_3(u) = Z \times \frac{\frac{1}{g}}{1 + \frac{1}{Z}}.$$  

(9)

At the downstream end of the detector the current equals

$$i_{out} = -\exp(-g/s) \times \frac{1}{g} \times \frac{1}{s} \times i_{out}.$$  

(10)

There is an overall minus sign because current flows off the plates, and the phase shift accounts for the one it takes the beam to travel the length of the plate ($v_0$ is the velocity of the beam and $s$ is the length of the plates).

The plates together with the surrounding enclosure form a transmission line with characteristic impedance $Z_0$. Assume, for the purpose of discussion, that $Z$ has been connected at the upstream end of the detector as shown above. The upstream current source seen in $Z$ is parallel with the impedance of the transmission line terminated in a short circuit. The voltage from the upstream current source is

$$V = \frac{i}{V_0}; \quad V = \frac{1}{Z} = \frac{\tan^{-1} \frac{V_0}{v_0}}{Z_0}.$$  

(11)

where $v_0$ is the signal phase velocity. From now on take $v_0 = v = c$, and fix $\theta = 0 = \phi = 0$ is the length of the plate in units of phase. In terms of $\theta$

$$V = \frac{Z_0}{Z} \times \frac{1}{\tan^{-1} \frac{V_0}{v_0}}.$$  

(12)

---

4. The SPS has a narrowband Schottky detector with $S = 385 \Omega$ (CERN SPS/ATF7/PS-17).

5. D. Martin et al., to be presented at the 1989 IEEE Particle Accelerator Conference.


The voltage across Z from Iₜ can be calculated by transforming Z to downstream end, calculating the voltage there, and transforming back to the upstream end; the result is

\[
V_0 = \frac{1}{2} \frac{Z}{Z_0} \left( \frac{Z}{Z_0} \right)^{1/2} \text{......(13)}
\]

Adding them together

\[
V = V_a + V_b = \frac{1}{2} \frac{Z}{Z_0} \left( \frac{Z}{Z_0} \right)^{1/2} \text{......(14)}
\]

Where \( \theta < 1 \), eq. (14) becomes

\[
V = \frac{1}{2} \frac{Z}{Z_0} \left( \frac{Z}{Z_0} \right)^{1/2} \text{......(15)}
\]

The transmission line capacitance per unit length, \( C \), and characteristic impedance are related as \( Z_0 = 1/\sqrt{C} \). Using this eq. (15) becomes

\[
V = \frac{1}{2} \frac{Z}{Z_0} \left( \frac{Z}{Z_0} \right)^{1/2} \text{......(16)}
\]

The circuit is a current source \( I_\theta \) driving the parallel combination of \( Z \) and capacitor \( C \).

The complete circuit is in figure 6a. It is a resonant circuit with the signal transformer coupled to the load, \( R \). The principal element of \( Z \) is the transformer which is modelled by the parallel combination of shunt resistor, \( R_a \), ideal inductance, \( L \), and an ideal n-turn transformer. \( Z \) has two parallel capacitances also. \( C \) is the stray capacitance between leads (ideally \( C = 0 \)), and \( C_a \) is a variable capacitor that is adjusted to keep the resonant frequency fixed when the gap is changed.

The resonant frequency is \( f_{res} = \frac{1}{2\pi\sqrt{L \text{ C} \text{ R \text{ a}}}} \). The impedance of \( L \) and \( C \) cancel at resonance, and when \( L \) is transformed to the primary the circuit is that in figure 6b. \( R_a \) and \( C \) can be expressed in terms of the \( Q \)'s of the circuit which are measurable quantities. Without the load the circuit \( Q \) would be the "unloaded" \( Q \), \( Q_0 \), and with the load it is the "loaded" \( Q \), \( Q \).

\[
Q_0 = \frac{R_a}{\text{res} \cdot C} \quad \text{and} \quad Q = \frac{1}{Q_0} + \frac{R}{\text{res} \cdot C} \text{......(17)}
\]

Combining these two equations the transformer ratio is

\[
\eta^2 = \frac{1}{Q_0} \left( 1 + \frac{Q}{Q_0} \right) \text{......(18)}
\]

The final circuit is in figure 6c; the current source \( I_\theta \) is driving a resistor \( Q \)\text{res} \cdot C \). The output voltage is \( V_{\text{out}} = V \). Using eq. (3) and (9) the sensitivity is

\[
S_v = \frac{41\pi}{24} \left( \frac{Q_0 \text{res} \cdot C}{Q_0 (1 + \frac{Q}{Q_0})} \right)^{1/2} \text{......(19)}
\]

![Figure 6: The equivalent circuit for the Tevatron detector. The circuit at resonance b) in terms of \( R_a \) and \( R \), and c) in terms of \( Q_0 \).](image)

Make some approximations. When \( \phi \leq 1 \) and \( C_a \gg C \), the coefficient equals \( 2\pi(f \\text{ C} \text{ g}) \). If the plates are far from the surrounding enclosure and fringe fields ignored, \( C = k \phi/\pi \). \( S_v \) is a maximum when \( Q/Q_0 = 1/2 \); then

\[
S_v = \left( \frac{Q_0 \text{res} \cdot C}{Q_0} \right)^{1/2} \text{......(20)}
\]

where \( Q_0 = 377 \Omega \) is the characteristic impedance of free space. \( Q_0 \) should be as large as possible consistent with the range of tube to be measured, \( Q_0 \leq \text{seal} \cdot \text{res} \cdot \text{C} \), the gap and plate height should be small, and the plates should be long.

2. The Tevatron Detector: With this background the Tevatron detector, shown in figure 5, is described. The detector frequency was chosen by the availability of components. A filter is needed to attenuate resonant harmonics (figure 3b), and relatively inexpensive crystal filters can be purchased "off-the-shelf" at 10.7 and 21.4 MHz. The latter frequency equals \( 448.51 \text{ Vp} \), and with a 21.4 MHz filter that has a 15 kHz bandwidth a rate range 0.35 < \( g_q < 0.65 \) can be measured. This covers the Tevatron operating point, at \( 10^4 \times 21.4 \text{ MHz} \) was chosen.

The gap and plate height should be small (with \( \phi \leq 1 \)) for high sensitivity. However, the gap must be large at injection time and during energy ramps so the aperture won't be restricted. The solution is to make the plate movable. The translation mechanism should keep the plates parallel to avoid unnecessary loss of aperture; \( l_p = 1 \text{ m} \) was chosen as a compromise between increasing the sensitivity (\( S_v = \text{d} \phi \)) and minimizing mechanical problems.

The transformer primary is an air-core inductor made of several turns of transformer tubing. In early models of the detector it was four turns of 1/2" tubing wound with a 1/2" pitch on a 1 3/4"
diameter form: this gave L = 0.65 μH and Q₀ = 410. The transformer secondary is a loop that intersects some of the flux of the primary. The "turns-ratio" and inductance were adjusted empirically. First, the gap was reduced to the minimum desired value, \( g = 10 \) mm, and \( \text{Q} \) was set equal to zero. \( L \) was adjusted by making small changes in the inductor length until \( V_{in} = 21.4 \) MHz. Second, the turn-to-turn ratio was adjusted by removing load \( R \) and measuring the input impedance at what would normally be the signal output using a vector impedance meter. \( S \) transformed to the secondary equals

\[
S_{R} = \left( \frac{Q_2}{Q_1} \right) R.
\]

The area and/or position of the secondary were adjusted until the impedance at 21.4 MHz equaled \( R \); then \( Q_2 = Q_0/2 \).

The detector parameters are given in Table 1. The spectra in figures 111.6 and 11.8 were taken with (unintentional) external excitation of the beam and high sensitivity wasn't required. The gap was \( g = 22 \) mm, and \( S = 27 \) Ωmm. With \( g \) gap reduced to the design value, \( g = 10 \) mm, the sensitivity would be \( S_0 = 59 \) Ωmm.

### Table 1: Tevatron Transverse Beam Detector, 1987

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resonant frequency</td>
<td>( 21.4 ) MHz</td>
</tr>
<tr>
<td>Phase length</td>
<td>( 1.0 ) m</td>
</tr>
<tr>
<td>Phase height</td>
<td>( 2.54 ) cm</td>
</tr>
<tr>
<td>Capacitance</td>
<td>( C = 85 \mu F )</td>
</tr>
<tr>
<td>Inductance</td>
<td>( L = 0.5 ) mH</td>
</tr>
<tr>
<td>Unloaded Q</td>
<td>( Q_0 = 410 )</td>
</tr>
<tr>
<td>Loaded Q</td>
<td>( Q_0 = 205 )</td>
</tr>
<tr>
<td>Load resistance</td>
<td>( R = 50 ) Ω</td>
</tr>
<tr>
<td>g</td>
<td>( 22 ) mm</td>
</tr>
<tr>
<td>S</td>
<td>( 27 ) Ωmm</td>
</tr>
</tbody>
</table>

The sensitivity can be doubled and the signals from counterrotating beams separated by adding signals from two detectors that are one-quarter wavelength, \( 3/4 \), apart. Let detector \#1 be \( f \) from detector \#2 and let \( P_1 \) be upstream of \( f \) for protons and downstream of \( f \) for anti-protons. Assuming equal detector sensitivities \( V_0(a) = V_0(b) \) for protons, and \( V_0(a) = V_1(a) \exp(-j/2) \) for anti-protons. If the signal from detector \#1 is added to the signal from detector \#2 after being delayed by a cable with a \( 90^\circ \) phase shift, the proton and anti-proton signals are

\[
\begin{align*}
V_p(a) &= V_1(a) \exp(j/2) \\
V_p(b) &= V_0(a) \exp(-j/2) \\
V_\bar{p}(a) &= V_1(a) \exp(-j/2) \\
V_\bar{p}(b) &= V_0(a) \exp(j/2)
\end{align*}
\]

The anti-proton signals have interacted destructively, and the proton signals have interacted constructively giving a signal which is twice that of a single detector. If \( V_0 \) is delayed rather than \( V_1 \), the \( p \) signal doubles while the \( \bar{p} \) signal cancels.

10. It is necessary to compensate for the self inductance of the secondary loop by working at the "denuded" position. See B. L. Ginman, Microwave Measurements (New York: McGraw-Hill Book Company, 1957), sect. 9.5.

11. In practice the sensitivities aren't equal, but attenuators can compensate for this.

### 3. Signal Processing: For some purposes it is adequate to filter the signal to remove the rotation harmonics, amplify it, and measure the resultant spectrum directly. However, the resolution bandwidths of spectrum analyzers useful at 21.4 MHz are typically \( \Delta f \geq 10 \) Hz, and this limits measurements of true distributions.

It is possible to get much higher resolution by mixing the signal down to a frequency range where an audio spectrum analyzer can be used. This is illustrated schematically for the Tevatron detector in figure 7. The detector signal is filtered, amplified, and mixed with the signal from a local oscillator running at \( 448 v \). If the signal frequency is \( v \) \( (v \geq 21.4 \) MHz), the mixer output frequency are \( v + 448 v_0 \) (-43 MHz) and \( v - 448 v_0 \). The latter is well within the frequency range of Fast Fourier Transform signal analyzers. These have resolution bandwidths which are a percentage of the frequency span, 0.25% is typical. This signal processing arrangement is simple and has some flexibility: for example, the filter and local oscillator frequencies can be changed to study coupled bunch effects. This does require a custom made filter, and delivery can be long and costs high.

It is easier to use a frequency synthesizer as the local oscillator, but this has the disadvantage of not tracking the revolution frequency, \( v \), changes by about 1 Hz \((\Delta v/v \approx 2 \times 10^{-5})\) as the Tevatron ranges from injection, 135 GeV, to collisions, 900 GeV. If the local oscillator frequency was fixed and \( q_0 \) was constant during the test ramp, the output frequency of the circuit in figure 7 would change by 430 Hz. This large a frequency change makes it difficult to measure true changes during the ramp.

This problem can be avoided if the local oscillator frequency is derived from the RF as shown in figure 8. Phase-locked loops are used to derive local oscillator signals at 371 GeV \((v = 17.7 MHz)\) and 771 GeV \((v = 3.67 MHz)\). The output from the first stage of mixing is \( v + 371 v_0 \) and \( v - 371 v_0 \), where \( v \) is the signal frequency. The higher frequency signal is removed by the 3.7 MHz filter, and the lower frequency signal is mixed giving outputs \( v - 794 v_0 \) and \( v - 448 v_0 \). The former is filtered out and the latter is sent to the audio spectrum analyzer. With this solution changes in the signal frequency are changes in tone, and it is easy for the operator to measure true changes during the ramp.

![Figure 7: Frequency translation by mixing.](image)

![Figure 8: The Tevatron tune receiver with local oscillators locked to the RF.](image)