Electric Field of a Uniformly Charged Ellipse

This note contains a calculation of the electric field of a uniformly charged ellipse with size *a* in the x-direction, *b* in the y-direction, and extending infinitely in the z-direction with charge *Q* per unit length. The calculation is an application of the method employed by Sam Kheifets to calculate the potential of a three dimensional Gaussian,[1] and uses the Green's function for the diffusion equation and takes the limit as the diffusion strength approaches zero.
The diffusion equation is

$$\nabla^2\psi - A^2\frac{\partial\psi}{\partial t} = -4\pi\rho \tag{1}$$

The two-dimensional Green's function is[2]

$$G(x,y,t) = \frac{1}{\tau}\exp\left(-\frac{A^2R^2}{4\tau}\right) \tag{2}$$

where (x, y, t) are the coordinates and time of the point where the field is measured and (x_0, y_0, t_0) are the coordinates and time of the source point. In terms of these

$$R = \left|\vec{r}-\vec{r}_0\right|,\ R^2 = (x-x_0)^2+(y-y_0)^2,\ \tau = t-t_0 \tag{3}$$

The solution is

$$\begin{aligned}\psi(x,y,t) &= \int_0^t dt_0\int_{-\infty}^{\infty}dx_0\int_{-\infty}^{\infty}dy_0\,\frac{1}{\tau}\exp\left(-\frac{A^2R^2}{4\tau}\right)\rho(x_0,y_0,t_0)\\ &= \int_0^t\frac{d\tau}{\tau}\int_{-\infty}^{\infty}dx_0\int_{-\infty}^{\infty}dy_0\,\exp\left(-\frac{A^2R^2}{4\tau}\right)\rho(x_0,y_0,\tau)\end{aligned} \tag{4}$$

Define $q = 4\tau/A^2$, and the solution becomes

$$\psi(x,y,t) = \int_0^{4t/A^2}\frac{dq}{q}\int_{-\infty}^{\infty}dx_0\int_{-\infty}^{\infty}dy_0\,\frac{1}{\tau}\exp\left(-\frac{R^2}{q}\right)\rho \tag{5}$$

Apply this result to the uniformly charged ellipse described above. The charge density is

$$\rho = \frac{Q}{\pi ab}\quad\text{within the ellipse} \qquad\qquad 0\quad\text{outside} \tag{6}$$

and

$$\psi(x,y,t) = \frac{Q}{\pi ab}\int_0^{4t/A^2}\frac{dq}{q}\int_{-a}^{a}dx_0\int_{-b}^{b}dy_0\,\exp\left(-\frac{R^2}{q}\right) \tag{7}$$

Taking the limit $A\to 0$ gives the potential

$$V(x,y) = \frac{Q}{\pi ab}\int_0^{\infty}\frac{dq}{q}\int_{-a}^{a}\exp\left(-\frac{(x-x_0)^2}{q}\right)dx_0\int_{-b}^{b}\exp\left(-\frac{(y-y_0)^2}{q}\right)dy_0 \tag{8}$$

The x and y integrals can be performed. For example

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[1] Sam Kheifets, "Potential of a three-dimensional Gaussian bunch", PETRA Kurtzmitteilung 119, 1/10/1976.

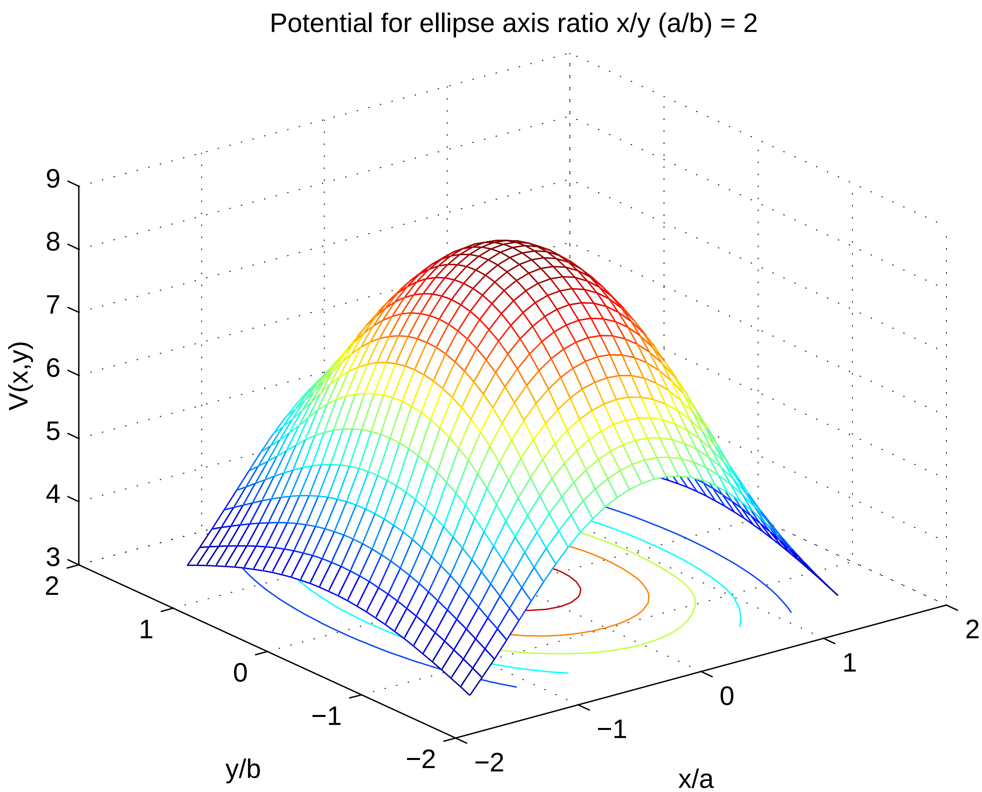
[2] P. M. Morse and H. Feshbach, "Methods of Theoreticcal Physics", pg. 861 (7.4.10)

$$\int_{-a}^{a} \exp\left(-\frac{(x-x_0)^2}{q}\right)dx_0 = \frac{\sqrt{\pi q}}{2}\left(erf\left(\frac{x+a}{\sqrt{q}}\right)+erf\left(\frac{x-a}{\sqrt{q}}\right)\right) \qquad (9)$$

Making a final change of variables $p = q/ab$ gives

$$V(x,y) = \frac{Q}{2}\int_0^\infty dp\left(erf\left(\sqrt{\frac{a}{bp}}(1+x/a)\right)+erf\left(\sqrt{\frac{a}{bp}}(1-x/a)\right)\right)\times \left(erf\left(\sqrt{\frac{b}{ap}}(1+y/b)\right)+erf\left(\sqrt{\frac{b}{ap}}(1-y/b)\right)\right) \qquad (10)$$

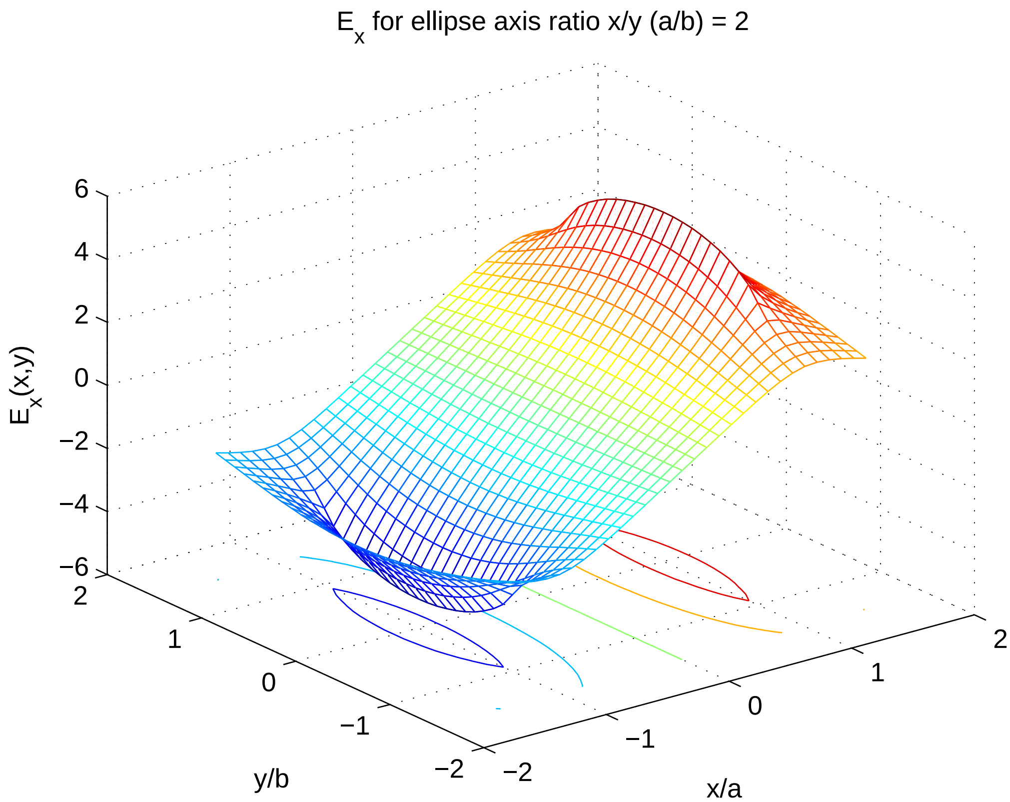
The potential is plotted below

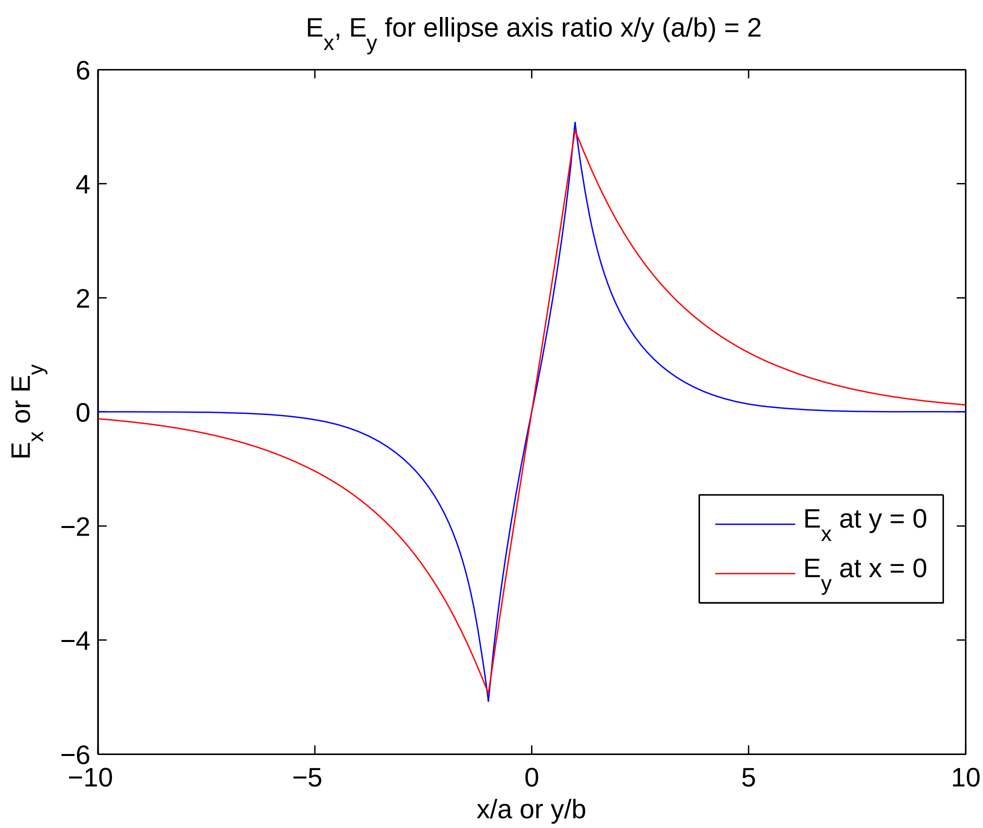


The electric field can be derived from the potential. For example, in the x-direction

$$E_x = -\frac{\partial V}{\partial x} = -\frac{Q}{2}\frac{1}{\sqrt{ab}}\int_0^\infty \frac{dp}{\sqrt{p}}\left(\exp\left(-\frac{a}{bp}(1+x/a)^2\right)-\exp\left(-\frac{a}{bp}(1-x/a)^2\right)\right)\times \left(erf\left(\sqrt{\frac{b}{ap}}(1+y/b)\right)+erf\left(\sqrt{\frac{b}{ap}}(1-y/b)\right)\right) \qquad (11)$$

Plots of E_x are below


E_x for ellipse axis ratio x/y (a/b) = 2
E_x(x,y)
6
4
2
0
−2
−4
−6
y/b
2
1
0
−1
−2
x/a
−2
−1
0
1
2


E_x, E_y for ellipse axis ratio x/y (a/b) = 2
E_x or E_y
6
4
2
0
−2
−4
−6
−10
−5
0
5
10
x/a or y/b
E_x at y = 0
E_y at x = 0

E_x can be written

$$E_x = -\frac{Q}{2}\frac{1}{\sqrt{ab}}\int_0^\infty \frac{dp}{\sqrt{p}} F(x,p)G(y,p)$$
$$F(x,p) = \exp\left(-\frac{a}{bp}(1+x/a)^2\right) - \exp\left(-\frac{a}{bp}(1-x/a)^2\right) \tag{12}$$
$$G(y,p) = erf\left(\sqrt{\frac{b}{ap}}(1+y/b)\right) + erf\left(\sqrt{\frac{b}{ap}}(1-y/b)\right)$$

Taylor expanding within the ellipse

$$E_x = E_x(x=y=0) + x\left.\frac{\partial E_x}{\partial x}\right|_0 + y\left.\frac{\partial E_x}{\partial y}\right|_0 + \frac{1}{2}x^2\left.\frac{\partial^2 E_x}{\partial x^2}\right|_0 + \frac{1}{2}y^2\left.\frac{\partial^2 E_x}{\partial y^2}\right|_0 + xy\left.\frac{\partial^2 E_x}{\partial y \partial x}\right|_0 + \ldots \tag{13}$$

Look at the terms in the series

$$E_x(x=y=0) = -\frac{Q}{2}\frac{1}{\sqrt{ab}}\int_0^\infty \frac{dp}{\sqrt{p}} F(0,p)G(0,p) = 0 \tag{14}$$

because

$$F(x=0,p) = 0 \tag{15}$$

It also follows from eq. (15) that

$$\left.\frac{\partial E_x}{\partial y}\right|_0 = \left.\frac{\partial^2 E_x}{\partial y^2}\right|_0 = 0 \tag{16}$$

The first x-derivative is

$$\left.\frac{\partial E_x}{\partial x}\right|_0 = -\frac{Q}{2}\frac{1}{\sqrt{ab}}\int_0^\infty \frac{dp}{\sqrt{p}} \left.\frac{\partial F(x,p)}{\partial x}\right|_0 G(y,p) \tag{17}$$

Taking the first partial of F

$$\frac{\partial F}{\partial x} = -\frac{2}{bp}\left((1+x/a)\exp\left(-\frac{a(1+x/a)}{bp}\right) + (1-x/a)\exp\left(-\frac{a(1-x/a)}{bp}\right)\right)$$
$$\left.\frac{\partial F}{\partial x}\right|_0 = -\frac{4}{bp}\exp\left(-\frac{a}{bp}\right) \tag{18}$$

and

$$G(y=0,p) = 2erf\left(\sqrt{\frac{b}{ap}}\right) \tag{19}$$

to give

$$\left.\frac{\partial E_x}{\partial x}\right|_0 = 4Q\sqrt{\frac{1}{ab^3}}\int_0^\infty \frac{dp}{p^{3/2}}\exp\left(-\frac{a}{bp}\right) erf\left(\sqrt{\frac{b}{ap}}\right) \tag{20}$$

Taking the second partial of F gives

$$\frac{\partial^2 F}{\partial x^2} = \frac{2}{bp}\left(\exp\left(-\frac{a\left(1+x/a\right)^2}{bp} \right) - \exp\left(-\frac{a\left(1-x/a\right)^2}{bp} \right) \right)$$
$$+\frac{4}{b^2 p^2}\left(\left(1-x/a\right)^2 \exp\left(-\frac{a\left(1-x/a\right)^2}{bp} \right) - \left(1+x/a\right)^2 \exp\left(-\frac{a\left(1+x/a\right)^2}{bp} \right) \right) \quad (21)$$

So

$$\left.\frac{\partial^2 F}{\partial x^2}\right|_0 = 0 \text{ and } \left.\frac{\partial^2 E_x}{\partial x^2}\right|_0 = 0 \quad (22)$$

Taking the partial derivative of G with respect to y

$$\frac{\partial G}{\partial y} = \sqrt{\frac{1}{abp}}\left(\exp\left(-\frac{b\left(1+y/b\right)^2}{ap} \right) - \exp\left(-\frac{b\left(1-y/b\right)^2}{ap} \right) \right) \quad (23)$$

and

$$\left.\frac{\partial G}{\partial y}\right|_0 = 0 \quad (24)$$

It follows from this that

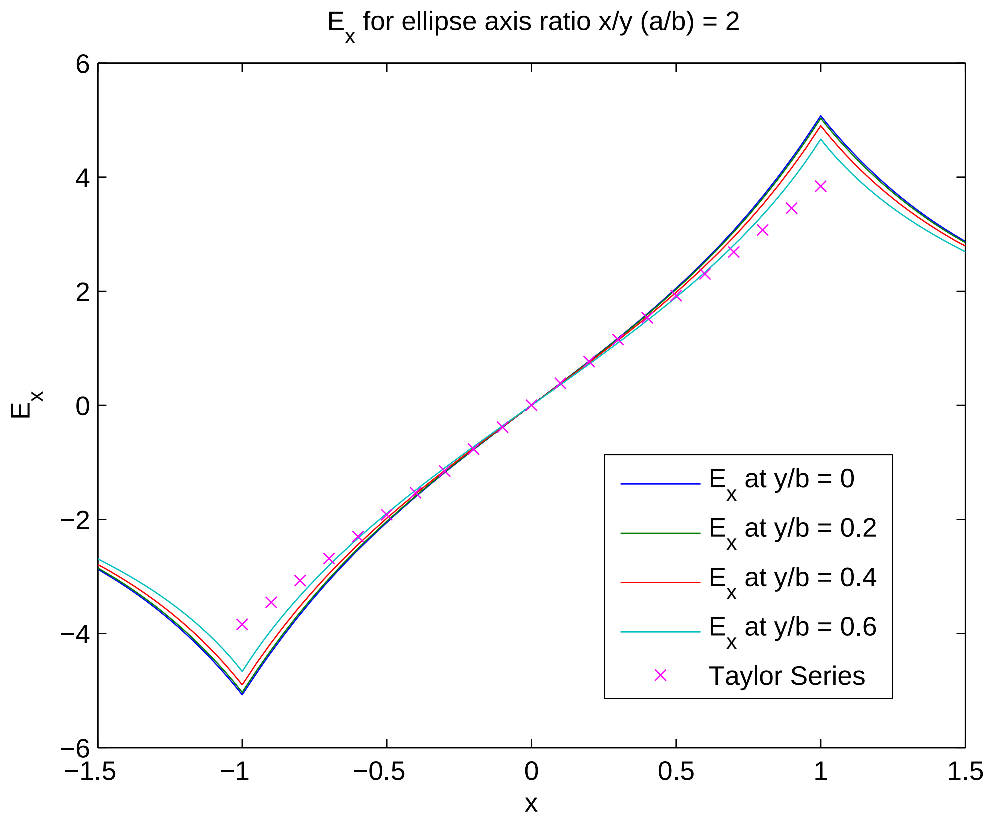
$$\left.\frac{\partial E_x}{\partial y}\right|_0 = \left.\frac{\partial^2 E_x}{\partial x \partial y}\right|_0 = 0 \quad (25)$$

Combining equations (13), (15), (16), (20), (22) and (25) gives

$$E_x(x,y) = 4xQ\sqrt{\frac{1}{ab^3}}\int_0^\infty \frac{dp}{p^{3/2}} \exp\left(-\frac{a}{bp} \right) erf\left(\sqrt{\frac{b}{ap}} \right) + O(x^3, y^3, x^2 y, ...) \quad (26)$$

A comparison of this Taylor series approximation and the exact calculation is shown below.

This calculation shows that E_x is linear in x through 2nd order, and there are no other terms to that order. As a result the linear x and y motions of a particle in this electric field are uncoupled.


E_x for ellipse axis ratio x/y (a/b) = 2
E_x
x
E_x at y/b = 0
E_x at y/b = 0.2
E_x at y/b = 0.4
E_x at y/b = 0.6
Taylor Series
6
4
2
0
−2
−4
−6
−1.5
−1
−0.5
0
0.5
1
1.5