EMITTANCE GROWTH IN A PLASMA AFTERBURNER

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MCS scattering through a neutral vapor:

The emittance growth of a beam through a neutral vapor can be found from the angular scatter through the vapor. The angular scatter is related to the radiation length. The following formula shows how to find the radiation length (from July 2004 Particle Physics Booklet)

$$X_0 = \frac{716.4 \cdot g \cdot cm^{-2} \cdot A}{Z(Z+1)\ln(287/\sqrt{Z})}$$

Z = atomic number of traversed material, and A = atomic weight of the traversed material. For lithium, Z=3, A=6.941g/mol., so $X_0 = 81 \cdot g \cdot cm^{-2}$. This is turned into a real length by dividing by the density. Thus, the radiation length for a typical PWFA's lithium vapor is $L_{r\ lithium} = 81 \cdot g \cdot cm^{-2} \cdot [(A/N_A) \cdot 2.7 \cdot 10^{17} \cdot cm^{-3}]^{-1} = 2.6 \cdot 10^5$meters ($N_A$ is Avogadro's number).

An ILC scale afterburner may have to use a different gas, so it is worth looking at some others to see how they scale. The following table has examples of some lithium-like elements. The radiation lengths for the following table were calculated for a vapor density of $2.7 \cdot 10^{17} \cdot cm^{-3}$.

Element (Z)	L_r [m]
Li (3)	$2.6 \cdot 10^5$
Na (11)	$2.7 \cdot 10^4$
K (19)	$1.0 \cdot 10^4$
Rb (37)	$3.0 \cdot 10^3$
Cs (55)	$1.4 \cdot 10^3$

The rate of angular scatter of an ultra-relativistic electron through a neutral vapor can be found from the following formula (Wiedemann, H., "Particle Accelerator Physics I" pg. 371) (after projecting the radial angle onto the x axis):

$$\frac{d < \theta^2 >_{vapor}}{dz} = \frac{1}{2L_r}\left(\frac{20MeV}{E}\right)^2$$

Where E is the energy of the incident electron, and z is the distance along the accelerator.

MCS scattering through singly ionized lithium vapor:

This part of the calculation was done following the example of Montague, B.W. and Schnell, W. and their paper title “Multiple Scattering and Synchrotron Radiation in a Plasma Beat-Wave accelerator” and of Montague, B.W. and his paper titled “Emittance Growth From Multiple Scattering in the Plasma Beat-Wave Accelerator”.

Up to this point we have considered how to find the scattering through neutral vapor; however, scattering through a singly ionized vapor is different. With the above calculation, if the incident particle doesn’t come within the atomic dimensions of an atom, then the nucleus of the atom is shielded by the electrons. This means it doesn’t produce significant scatter. In order to add the fact that there are singly charged ions, we include the coulomb scattering from the atomic size, R_a, out to the blow out radius, R_b, of the plasma.

First we have to calculate the deflection of an incident particle as it passes a singly charge ion. We will make the approximation that the deflection of the incident particle and the ion are small during the time that they interact.

$$\frac{d\mathbf{P}}{dt} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Q is the charge of the ion. Since the ion is static then the magnetic field term drops out. Also, since the ion is static, the z component (direction of incident particle trajectory) of the electric field is odd and integrates to zero. The only term that survives is the radial electric field. In order to find the deflection we can integrate the rate of momentum change over time. The angular defection can then be found by the ratio transverse momentum kick over the initial momentum.

$$\Delta P_r = \int dt\ Q \cdot E_r = Q/v \cdot \int dz\ E_r = \frac{q \cdot Q}{2\pi \cdot v \cdot \varepsilon_0 \cdot b}, \quad \frac{\Delta P_r}{P} = \frac{q \cdot Q}{2\pi \cdot P \cdot v \cdot \varepsilon_0 \cdot b}$$

This kick can then be projected onto the x axis with a cosine.

$$\theta = \theta_x = \frac{\Delta P_x}{P} = \frac{-q \cdot Q \cdot \cos\phi}{2\pi \cdot P \cdot v \cdot \varepsilon_0 \cdot b}$$

Where q is the charge, P is the momentum, and v is the velocity of the incident particle. Where φ is the azimuthal angle, and b is the impact parameter of the ion. We next can find the deflection from a singly charge ion that is randomly placed with $R_a < b < R_b$. Since the incident particle is ultra-relativistic, Pv=E. Since incident particle is an electron, q = -e. Since the ion only looks like a singly charge ion in this region, Q = e.

$$<\theta^2>=\left(\frac{e^2}{2\pi\cdot E\cdot\varepsilon_0}\right)^2\cdot<\frac{\cos^2\phi}{b^2}>=\left(\frac{e^2}{2\pi\cdot E\cdot\varepsilon_0}\right)^2\cdot\frac{2\pi\int db\cdot b\cdot b^{-2}\int d\phi\cdot\cos^2\phi}{2\pi\int db\cdot b}$$

$$<\theta^2>=\left(\frac{e^2}{2\sqrt{2}\cdot\pi\cdot E\cdot\varepsilon_0}\right)^2\cdot\frac{\ln(R_b/R_a)}{(R_b{}^2-R_a{}^2)}$$

This is the mean square angular deflection from an ion randomly placed in the volume with $R_a < b < R_b$. Now we have to include the fact that there are multiple collisions that occur. The angular scatter from the ions adds in quadrature. The reason can be seen by looking at the expectation value for the addition of two angles.

$$<\theta_T^2>=<(\theta_1+\theta_2)^2>=<\theta_1{}^2+2\theta_1\theta_2+\theta_2{}^2>=<\theta_1{}^2>+<\theta_2{}^2>$$

The $2\theta_1\theta_2$ term was dropped because it is just as likely to be positive as it is to be negative, so the expectation value is zero. The total mean square scatter can then be found by multiplying by the number of ions that the incident particle will intercept by the mean square scatter from one.

$$<\theta^2>_{ion}=N\cdot<\theta^2>\Rightarrow\frac{d<\theta^2>_{ion}}{dz}=<\theta^2>\cdot\frac{dN}{dz}=<\theta^2>\cdot n_p\cdot\pi\cdot(R_b{}^2-R_a{}^2)$$

$$\frac{d<\theta^2>_{ion}}{dz}=\frac{n_p\cdot e^4}{4\pi\cdot E^2\cdot\varepsilon_0^2}\cdot\ln(R_b/R_a)$$

n_p is the ion density. Now to find the total rate of angular scatter, we can add the term from the angular scatter from the vapor.

$$\frac{d<\theta_T^2>}{dz}=\frac{d<\theta^2>_{ion}}{dz}+\frac{d<\theta^2>_{vapor}}{dz}=\frac{n_p\cdot e^4}{4\pi\cdot E^2\cdot\varepsilon_0^2}\cdot\ln(R_b/R_a)+\frac{1}{2L_r}\left(\frac{20MeV}{E}\right)^2$$

Now in order to see how this affects the beam as it traverses the plasma, we assume a beam that has been matched into the plasma and is oscillating in the ion column. This equation of motion is as follows:

$$\ddot{x}=\frac{-n_p\cdot e^2}{c^2\cdot 2\cdot\gamma\cdot m\cdot\varepsilon_0}x\Rightarrow x=C\cdot\sin\left(\sqrt{\frac{n_p\cdot e^2}{c^2\cdot 2\cdot\gamma\cdot m\cdot\varepsilon_0}}z+\phi\right)=C\cdot\sin\left(\frac{k_p}{\sqrt{2\cdot\gamma}}z+\phi\right)$$

m is the mass of the electron, γ is the Lorentz factor, and the derivatives are in z (distance along accelerator). For a matched beam in the plasma there is no cross term in the emittance (ε^2 = $<x^2>\cdot<x'^2>$). This is a reasonable estimate as long as the emittance growth in one harmonic oscillator oscillation is negligible compared to the total emittance. For a single oscillator, $<x^2>=C^2/2$. We can extend this to a distribution by taking the expectation value of the amplitude squared.

$$<x^2> = {<C^2>}\big/_{2}\text{, and } <\dot{x}^2> = \frac{<C^2>\cdot k_p^2}{4\cdot\gamma} \Rightarrow <C^2> = \frac{\varepsilon}{k_p}\sqrt{8\cdot\gamma}$$

We set this equal to the earlier formula for angular growth and find the rate of emittance growth from the scatters.

$$\frac{d<\theta_T^2>}{dz} = \frac{d<\dot{x}^2>}{dz} = \frac{d}{dz}\left(\frac{\varepsilon\cdot k_p\cdot c\sqrt{m}}{\sqrt{2\cdot E}}\right) = \frac{\dot{\varepsilon}_{scatter}\cdot k_p\cdot c\sqrt{m}}{\sqrt{2\cdot E}}$$

$$\frac{\dot{\varepsilon}_{scatter}\cdot k_p\cdot c\sqrt{m}}{\sqrt{2\cdot E}} = \frac{n_p\cdot e^4}{4\pi\cdot E^2\cdot\varepsilon_0^2}\cdot\ln({R_b}\big/_{R_a}) + \frac{1}{2L_r}\left(\frac{20MeV}{E}\right)^2$$

$$L_r = \frac{X_0}{\rho} = \frac{X_0\cdot N_A}{n_p\cdot A} = \frac{1}{\alpha\cdot n_p}$$

$$\dot{\varepsilon}_{scatter} = e\cdot\sqrt{\frac{2\cdot n_p\cdot\varepsilon_0}{E^3}}\cdot\left(\frac{e^2}{4\cdot\pi\cdot\varepsilon_0^2}\cdot\ln({R_b}\big/_{R_a}) + \frac{\alpha}{2}\cdot(20MV)^2\right)$$

The equations up to this point assume that the energy of the incident particle stays constant. An increasing energy of the particles actually decreases the area in phase space. This is called adiabatic damping. We can find the rate of change in the emittance from adiabatic damping by noting the normalized emittance is conserved in the absence of scatterers.

$$\frac{d\varepsilon_N}{dz} = 0 = \frac{d(\gamma\cdot\varepsilon)}{dz} = \gamma\frac{d\varepsilon}{dz} + \varepsilon\frac{d\gamma}{dz} \Rightarrow \dot{\varepsilon}_{adiabatic} = -\varepsilon\frac{\dot{\gamma}}{\gamma} = -\varepsilon\frac{\dot{E}}{E}$$

We can then find the total rate of emittance growth by adding the terms from adiabatic damping to the term from the coulomb scattering.

$$\dot{\varepsilon} = \dot{\varepsilon}_{adiabatic} + \dot{\varepsilon}_{scatter} = -\varepsilon\frac{\dot{E}}{E} + e\cdot\sqrt{\frac{2\cdot n_p\cdot\varepsilon_0}{E^3}}\cdot\left(\frac{e^2}{4\cdot\pi\cdot\varepsilon_0^2}\cdot\ln({R_b}\big/_{R_a}) + \frac{\alpha}{2}\cdot(20MV)^2\right)$$

We can now make this into an equation for normalized emittance.

$$\dot{\varepsilon}_N = \gamma\dot{\varepsilon} + \dot{\gamma}\varepsilon = \gamma\cdot e\cdot\sqrt{\frac{2\cdot n_p\cdot\varepsilon_0}{E^3}}\cdot\left(\frac{e^2}{4\cdot\pi\cdot\varepsilon_0^2}\cdot\ln({R_b}\big/_{R_a}) + \frac{\alpha}{2}\cdot(20MV)^2\right) + \dot{\gamma}\varepsilon - \gamma\varepsilon\frac{\dot{E}}{E}$$

$$\dot{\varepsilon}_N = \gamma\dot{\varepsilon} + \dot{\gamma}\varepsilon = \gamma \cdot e \cdot \sqrt{\frac{2 \cdot n_p \cdot \varepsilon_0}{E^3}} \cdot \left(\frac{e^2}{4 \cdot \pi \cdot \varepsilon_0^2} \cdot \ln({}^{R_b}\!/_{R_a}) + \frac{\alpha}{2} \cdot (20MV)^2 \right)$$

$$\dot{\varepsilon}_N = \frac{e}{c^3} \cdot \sqrt{\frac{2 \cdot n_p \cdot \varepsilon_0}{\gamma \cdot m^3}} \cdot \left(\frac{e^2}{4 \cdot \pi \cdot \varepsilon_0^2} \cdot \ln({}^{R_b}\!/_{R_a}) + \frac{\alpha}{2} \cdot (20MV)^2 \right)$$

We can also assume that

$$\gamma = \gamma_i + \dot{\gamma} \cdot z \Rightarrow dz = \frac{d\gamma}{\dot{\gamma}}$$

Thus we find in the end the total change to the emittance is:

$$\Delta\varepsilon_N = \frac{e}{c^3} \cdot \sqrt{\frac{8 \cdot n_p \cdot \varepsilon_0}{m^3}} \cdot \left(\frac{e^2}{4 \cdot \pi \cdot \varepsilon_0^2} \cdot \ln({}^{R_b}\!/_{R_a}) + \frac{\alpha}{2} \cdot (20MV)^2 \right) \frac{\sqrt{\gamma_f} - \sqrt{\gamma_i}}{\dot{\gamma}}$$

Where γ_f is the final Lorentz factor and γi is the initial gamma factor. Note: there appears to be singularity for the particles that don't gain any energy. The singularity is reconciled by taking the limit of γ' to zero. In this limit the change of emittance is

$$\frac{\sqrt{\gamma_f} - \sqrt{\gamma_i}}{\dot{\gamma}} \Rightarrow \frac{L}{2 \cdot \sqrt{\gamma_i}}$$

L is the length of the plasma. The change in emittance can be simplified one step more if we take into account the intrinsic scale for the longitudinal electric fields of a PWFA [$E_z \sim mc\omega_p/e = \sqrt{(n_p mc^2/\varepsilon_0)}$]. This step assumes that you are accelerating the particles to the best of the ability of the PWFA.

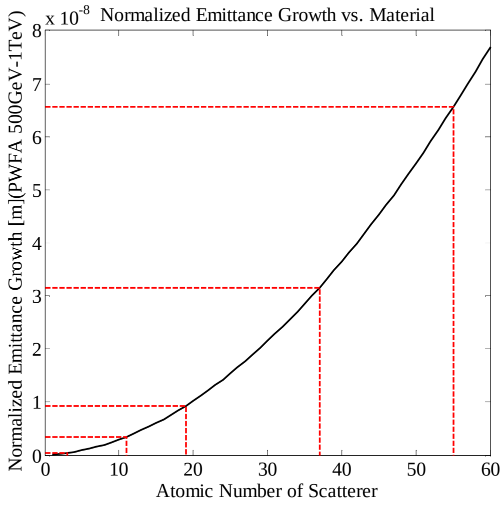
$$\frac{dE}{dt} = e \cdot E_z \cdot c \Rightarrow \dot{E} = e \cdot E_z \Rightarrow \dot{\gamma} = \frac{e \cdot E_z}{m \cdot c^2} \approx \frac{e}{c}\sqrt{\frac{n_p}{m \cdot \varepsilon_0}}$$

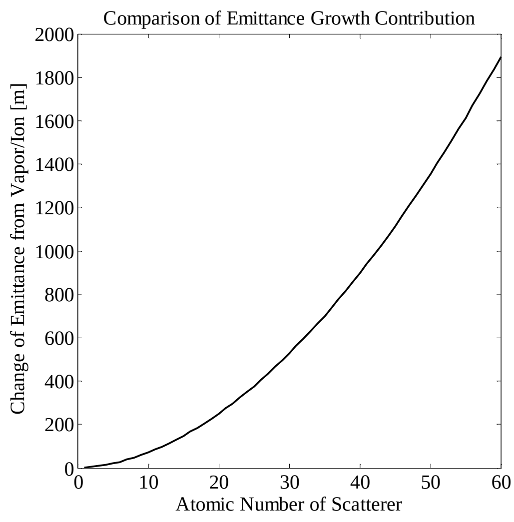
$$\Delta\varepsilon_N = \frac{\varepsilon_0 \cdot \sqrt{8}}{m \cdot c^2} \cdot \left(\frac{e^2}{4 \cdot \pi \cdot \varepsilon_0^2} \cdot \ln({}^{R_b}\!/_{R_a}) + \frac{\alpha}{2} \cdot (20MV)^2 \right) \cdot \left|\sqrt{\gamma_f} - \sqrt{\gamma_i}\right|$$

$$\Delta\varepsilon_N = \frac{\varepsilon_0 \cdot \sqrt{8}}{m \cdot c^2} \cdot \left(\frac{e^2}{4 \cdot \pi \cdot \varepsilon_0^2} \cdot \ln({}^{R_b}\!/_{R_a}) + \frac{Z(Z+1)\ln(287/\sqrt{Z})}{2 \cdot N_A \cdot 716.4 \cdot cm^{-2}} \cdot (20MV)^2 \right) \cdot \left|\sqrt{\gamma_f} - \sqrt{\gamma_i}\right|$$

$$\Delta\varepsilon_N[m] = \left(7.97\cdot10^{-15}\cdot\ln\left(R_b/R_a\right) + 1.42\cdot10^{-14}\cdot Z(Z+1)\ln(287/\sqrt{Z})\right)\cdot\left|\sqrt{\gamma_f}-\sqrt{\gamma_i}\right|$$

We can then use this formula to calculate the emittance growth from energy doubling a 500 GeV electron beam through various materials. The blow out radius was set to $2.5\cdot10^{-6}$ m, and the atomic radius was set to 10^{-10} m.





Note that typical ILC projected emittances are $\varepsilon_{N,y} = 4\cdot10^{-8}$m, and $\varepsilon_{N,x} = 9.6\cdot10^{-6}$m (Raubenheimer, Tor O. "An Afterburner at the ILC: The Collider Viewpoint", AAC 2004). Notice that at Z ~ 40 the y normalized emittance will double after energy doubling. The red dotted lines were put in for Li, Na, K, Rb, and Cs. The normalized emittance growths for these elements after energy doubling a 500GeV beam were:

Element (Z)	$\Delta\varepsilon_N$ (m)
Li (3)	$4.0\cdot10^{-10}$
Na (11)	$3.5\cdot10^{-9}$
K (19)	$9.3\cdot10^{-9}$
Rb (37)	$3.2\cdot10^{-8}$
Cs (55)	$6.6\cdot10^{-8}$

By looking at the ratio of the scattering from the vapor over the scattering from the ion column, we can tell which term dominates. The above graph shows the ratio. The smallest the ratio gets is 1.6 at Z=1. The ratio increases for increasing Z. This shows that the emittance growth is dominated by the scatter through the neutral vapor.