Wakefields in a Dielectric Tube with Frequency Dependent Dielectric Constant

This note is a continuation of ARDB-279 using Georges Dôme's work as the starting point.[1]

The effective retarding gradient experienced by a point charge due to the self-fields from Cherenkov radiation in an infinite dielectric rod with a frequency independent dielectric constant and with a hole of radius R in the center is

$$G = \frac{qcZ_0}{2\pi R^2} \equiv \frac{qcZ_H}{\lambda^2}. \tag{1.1}$$

where

$$Z_H = Z_0 \frac{1}{2\pi\left(R/\lambda\right)^2}. \tag{1.2}$$

The wavelength λ sets a convenient scale for the problem by making Z_H a function of R/λ. The Cherenkov wakefield a distance s behind the source particle is*

$$Z_H(s) = 2Z_H \upsilon_0(s) \qquad \left(s > 0\right) \tag{1.3}$$

Constant Dielectric Constant

Dôme matches boundary conditions at $r = R$, and solves the resultant equations to obtain an expression for $\upsilon_0(s)$. It is

$$\upsilon_0(u) = \frac{1}{\pi}\,\mathrm{Im}\int_0^\infty dz \frac{\exp\left(izu\right)}{z - \dfrac{2\varepsilon_r}{\sqrt{\varepsilon_r - 1}}\dfrac{H_1^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}{H_0^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}} \tag{1.4}$$

where $u = s/R$ and $z = \omega R/c$ and ε_r is the dielectric constant normalized to ε_0. The explicit dependence on ε_r is shown in this equation (thereby making it more convenient for a frequency dependent ε_r) by using slightly different variables from eq. (13) of Dôme. This result is derived in Appendix A (eq (2.7)). That appendix also derives approximations for small u. and contains a discussion of causality that is continued in the analysis of frequency dependent dielectric constants.

Frequency Dependent Dielectric Constant

The same procedure of matching boundary conditions has to be used when the dielectric constant is frequency dependent. From Jackson,[2] eq. (7.51) gives an expression for the frequency dependent dielectric constant

$$\varepsilon_r = \frac{\varepsilon\left(\omega\right)}{\varepsilon_0} = 1 + \frac{Ne^2}{\varepsilon_0 m}\sum_k \frac{f_k}{\omega_k^2 - \omega^2 + i\omega\gamma_k} = 1 + 4\pi N r_e c^2 \sum_k \frac{f_k}{\omega_k^2 - \omega^2 + i\omega\gamma_k} \tag{1.5}$$

where N is the density, and there is a sum rule relating the oscillator strengths, f_k, and the atomic number, Z,

* The sign convention of the Dôme paper is that trailing particles are at $s > 0$.

$$\sum_k f_k = Z \,. \tag{1.6}$$

The sign of the damping term in the denominator is different from Jackson's expression because of the assumed time dependence.** Rewriting

$$\varepsilon_r - 1 = \sum_k \frac{\alpha_k}{1 - \omega^2/\omega_k^2 + i\omega\gamma_k/\omega_k^2} \tag{1.7}$$

Consider a material with only one resonance

$$\varepsilon_r - 1 = \frac{\alpha_1}{1 - \omega^2/\omega_1^2 + i\omega\gamma_1/\omega_1^2} = \frac{\alpha_1}{1 - z^2/z_1^2 + iz\gamma_1/z_1^2} \tag{1.8}$$

where $z = kR = \omega R/c$, $z_1 = \omega_1 R/c$, and γ_1 is normalized to z1 in the right hand expression. At low frequencies

$$\varepsilon_r = 1 + \alpha_1 \tag{1.9}$$

independent of frequency, and at high frequencies the dielectric is a plasma with dielectric constant

$$\varepsilon_r = 1 - \frac{\alpha_1 \omega_1^2}{\omega^2} \,. \tag{1.10}$$

As shown in Appendix B, eq. (1.4) remains a valid expression for the wakefield. There is Cherenkov radiation for frequencies less than ω_1 where the radial wave number

$$k_r = k\sqrt{\varepsilon_r - 1} \tag{1.11}$$

is real. When $|\omega| > \omega_1$, k_r is imaginary, and the Hankel functions become modified Bessel functions. The fields must decay exponentially into the dielectric, which requires that k_r has a negative imaginary value for $z > 0$ in the integral (eq. (1.4)).

Figure 1 shows the results of calculations. There is only weak dependence on the resonant frequency of the dielectric with the wakefield at small u changing by less than 25% over the range $z_1 = \infty$ to $z_1 = 5$. This can be understood by looking at the integrand at $u = 0+$, which is shown in Figure 2. The quantity being plotted is

$$A(z) = \mathrm{Im} \frac{1}{z - \dfrac{2\varepsilon_r}{\sqrt{\varepsilon_r - 1}} \dfrac{H_1^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}{H_0^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}} \tag{1.12}$$

This is the frequency spectrum of the very short distance wakefield. Reasons for the weak dependence on z_1 seen in Figure 1 include:

1. The spectrum is peaked at low frequency, $z_1 \sim 1.5$, and, therefore, cutting off the high frequency portion of the spectrum doesn't strongly affect the integral.
2. There is frequency content even above ω_1.

Initially the latter was a surprising result since there is no Cherenkov radiation at those frequencies. However, in retrospect it is not surprising since the dielectric behaves as a plasma, and plasma waves are excited by the beam. Molecules in the dielectric are unpolarized before

** The constant γ must be positive for damping independent of the assumed time dependence. But the sign in front of $i\omega\gamma_k$ depends on the assumed dependence. Jackson uses $e^{-i\omega t}$ while Dome uses $e^{i\omega t}$. There will be more discussion of this sign and the relation to causality below.

the beam passage, and they are polarized by the beam fields initiating plasma oscillations. Although plasma waves do not propagate as far-field Cherenkov radiation, they do contribute to the near-field wakefields.

Consider application to Eddie Lin's PBGFA[3], which has $R = 0.678\lambda$ where λ is the wavelength of the accelerating mode. Then $z_1 = 4.26\lambda/\lambda_1$ where λ_1 is the wavelength corresponding to the resonance. Therefore, $z_1 = 5$ corresponds to the cutoff being just above, but close to the fundamental mode frequency.

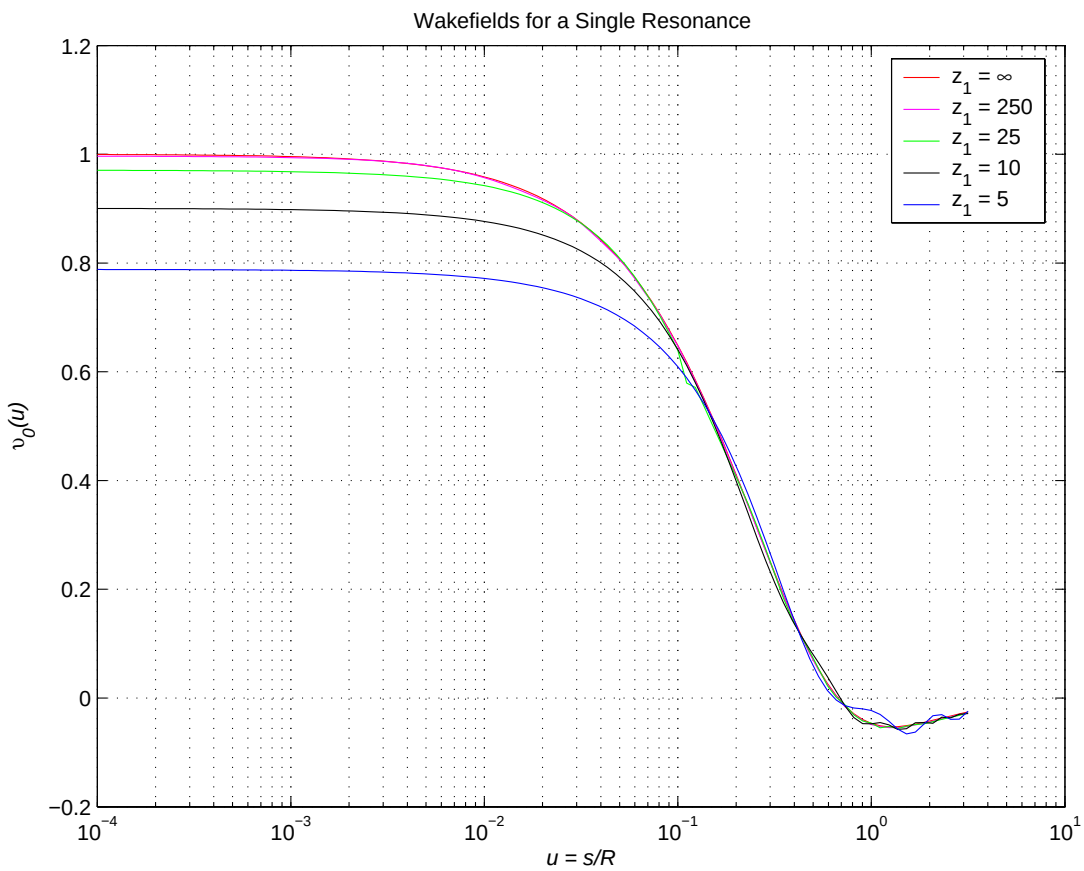

Figure 1: Wakefields for a single resonance in the dielectric constant. These curves were calculated for $\alpha_1 = 2$, $\gamma_1 = 0.001$.

Longitudinal Impedance

The expression for $\upsilon_0(s)$ is related to the impedance per unit length, $Z_0^{\parallel}(\omega)/L$, by

$$\upsilon_0(s>0) = \frac{R^2}{2cZ_0}\int_{-\infty}^{\infty} d\omega e^{i\omega s/c}\frac{Z_0^{\parallel}(\omega)}{L}. \tag{1.13}$$

In terms of the quantity *A(z)*, eq. (1.12),

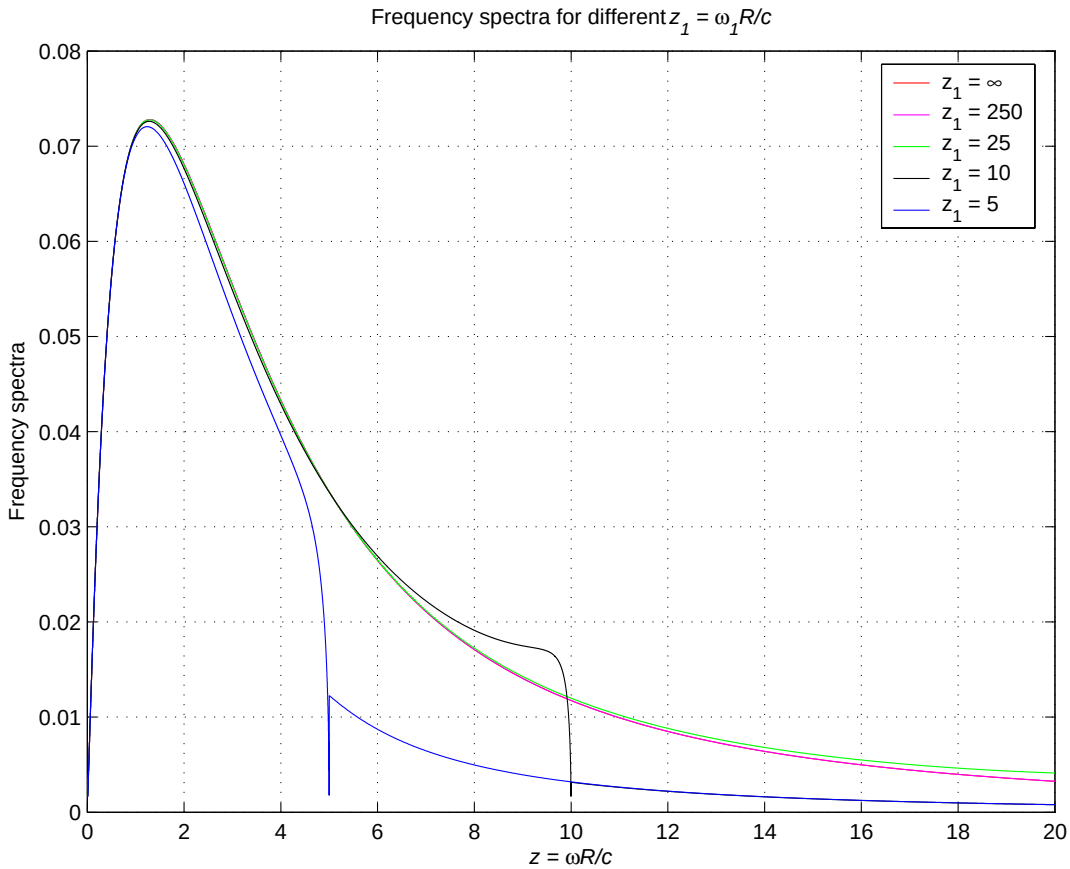

Figure 2: Frequency spectra for different values of z_1. $\alpha_1 = 2$, $\gamma_1 = 0.001$ were used.

$$A(z) = \operatorname{Im} \bar{A}(z)$$

$$\bar{A}(z) = \frac{1}{z - \dfrac{2\varepsilon_r}{\sqrt{\varepsilon_r - 1}} \dfrac{H_1^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}{H_0^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}} \tag{1.14}$$

we have

$$\frac{Z_0^{\parallel}(\omega)}{L} = -i \frac{Z_0}{\pi R} \bar{A}\left(\frac{\omega R}{c}\right) \tag{1.15}$$

where ω is considered to contain an infinitesimal negative imaginary part.

After the impedance is obtained, it should have two properties:

- When ω is confined to the real axis, $\operatorname{Re} Z_0^{\parallel}(\omega)$ should be an even function of ω, and $\operatorname{Im} Z_0^{\parallel}(\omega)$ is an odd function of ω. This assures the wake function is real.
- When ω is extended to cover the entire complex ω-plane, $Z_0^{\parallel}(\omega)$ should have no singularities in the lower half of the complex ω-plane. This assures causality.

Appendix A: Frequency Independent Dielectric Constant

Dôme makes changes in variables that are convenient when ε_r is constant. These are reexamined, and the resultant equation allows later inclusion of a variable dielectric constant. Begin with his eq. (10) in the limit $b \to \infty$

$$\upsilon_m(s) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dk}{k} e^{iks/\beta} \frac{1}{1 - \frac{A_m}{(k_r R)^2} + \frac{B_m}{k_r R}\frac{H_m^{(2)\prime}(k_r R)}{H_m^{(2)}(k_r R)}} \tag{2.1}$$

In this equation

$$k = \frac{\omega}{\beta c}, \tag{2.2}$$

and the radial wave number

$$k_r = k\sqrt{\varepsilon_r \mu_r - 1/\beta^2} \tag{2.3}$$

contains all of the dependence on the dielectric constant.

Specialize to $m = 0$. Then $A_0 = 0$, $B_0 = 2\varepsilon_r$ and $H_0^{(2)\prime}(k_r R) = -H_1^{(2)}(k_r R)$. Then

$$\upsilon_0(s) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk e^{iks/\beta} \frac{1}{k - \frac{2\varepsilon_r k}{k_r R}\frac{H_1^{(2)}(k_r R)}{H_0^{(2)}(k_r R)}} \tag{2.4}$$

Multiply numerator and denominator by R and let $z = kR$ and $u = s/R$. Take the limit $\beta \to 1$ and set $\mu_r = 1$ to get

$$\upsilon_0(u) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dz \frac{\exp(izu)}{z - \frac{2\varepsilon_r}{\sqrt{\varepsilon_r - 1}}\frac{H_1^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}{H_0^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}}. \tag{2.5}$$

The physical result is the real part of this expression, so

$$\upsilon_0(u) = \frac{1}{2\pi}\operatorname{Im}\int_{-\infty}^{\infty} dz \frac{\exp(izu)}{z - \frac{2\varepsilon_r}{\sqrt{\varepsilon_r - 1}}\frac{H_1^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}{H_0^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}} \tag{2.6}$$

The sign convention of the Dôme paper is that trailing particles are at $u > 0$, so causality requires that the integral be performed with z being considered as a complex quantity with a small negative imaginary part. In the lower half of the complex plane, the real part of the ratio $H_1^{(2)}/H_0^{(2)}$ is an odd function of the argument and the imaginary part is an even function. The real and imaginary parts of $\exp(iyu)$ have the opposite symmetry, so the integral can be performed from 0 to ∞, so

$$\upsilon_0(u) = \frac{1}{\pi}\operatorname{Im}\int_{0}^{\infty} dz \frac{\exp(izu)}{z - \frac{2\varepsilon_r}{\sqrt{\varepsilon_r - 1}}\frac{H_1^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}{H_0^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}} \tag{2.7}$$

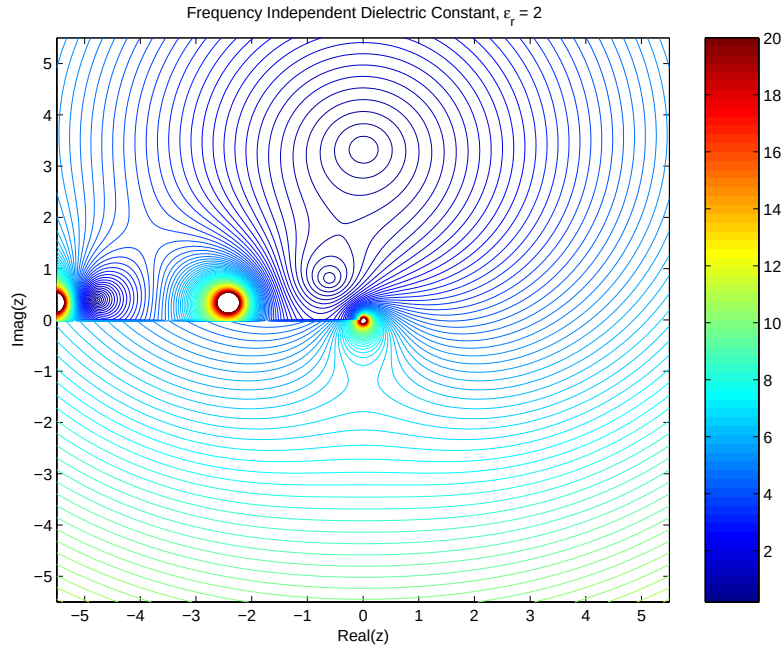

Figure A1: Denominator of the exact expression in eq. (2.5)

The Complex Plane

Eq. (2.5) is the primary result. The complex plane for the denominator of this equation is shown for $\varepsilon_r = 2$. There is a zero on the imaginary axis at *Imag(z) ~ 3.5*, another zero near the imaginary axis at *Imag(z) ~ 1*, a set of zeros and ∞'s just above the negative real axis and a cut on the negative real axis. There are no zeros in the lower half plane. This figure shows the justification for the choice of performing the integral below the real axis. By doing so, the cut on the negative real axis is avoided, and since there are no zeros in the dominator (and hence no poles in the function), causality is satisfied. Note that this choice of integration path was critical in deriving eq. (2.7) from eq. (2.5).

Expansions at Small u

Using the asymptotic expression for the Hankel functions that holds when $\left|z\sqrt{\varepsilon_r - 1}\right| \to \infty$

$$\frac{H_1^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}{H_0^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)} = i \tag{2.8}$$

The denominator has a single zero at

$$z = \frac{2i\varepsilon_r}{\sqrt{\varepsilon_r - 1}} \tag{2.9}$$

This corresponds to the zero on the imaginary axis. In addition, at this level of approximation the denominator does not have a cut. The integral for trailing particles has to be closed in the upper half plane. The result is

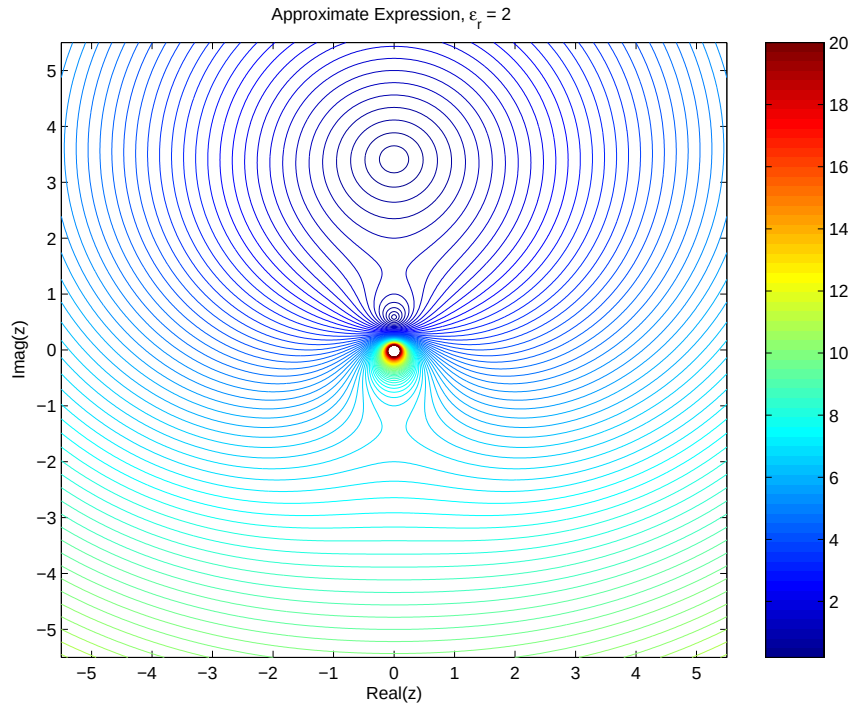

Figure A2: Complex plane for the ratio of Hankel functions approximated by eq. (2.11)

$$\upsilon_0(u) = \exp\left(\frac{-2u\varepsilon_r}{\sqrt{\varepsilon_r - 1}}\right) \approx 1 - \frac{2u\varepsilon_r}{\sqrt{\varepsilon_r - 1}} + \frac{2u^2\varepsilon_r^2}{\varepsilon_r - 1} \tag{2.10}$$

This linear term agrees with Dôme's expression in his eq. (14).

Keeping the next term in the asymptotic expansion

$$\frac{H_1^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}{H_0^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)} = i + \frac{1}{2z\sqrt{\varepsilon_r - 1}} \tag{2.11}$$

leads to two zeros in the denominator at

$$z_{1,2} = i\left(\frac{\varepsilon_r}{\sqrt{\varepsilon_r - 1}}\right)\left(1 \pm \sqrt{1 - \frac{\sqrt{1-\varepsilon_r}}{\varepsilon_r}}\right) \tag{2.12}$$

These are both located on the positive imaginary axis. (Note there is still no cut on the negative real axis at this level of approximation.) Using these expressions and closing the contour integral in the upper half plane and evaluating the residues gives

$$\upsilon_0(u) = \frac{\sqrt{\varepsilon_r - 1}z_1^2 e^{iz_1 u}}{\varepsilon_r + \sqrt{\varepsilon_r - 1}z_1^2} + \frac{\sqrt{\varepsilon_r - 1}z_2^2 e^{iz_2 u}}{\varepsilon_r + \sqrt{\varepsilon_r - 1}z_2^2} \approx 1 - \frac{2u\varepsilon_r}{\sqrt{\varepsilon_r - 1}} + u^2\left(\frac{2\varepsilon_r^2}{\varepsilon_r - 1} - \frac{\varepsilon_r}{2\sqrt{\varepsilon_r - 1}}\right) \tag{2.13}$$

This expansion on the right-hand side is plotted below.

Return to the Complex Plane

Return to the complex plane drawn in Figure A1. The exact expression for the denominator has the two zeros that determine the small u behavior as well as a cut on the

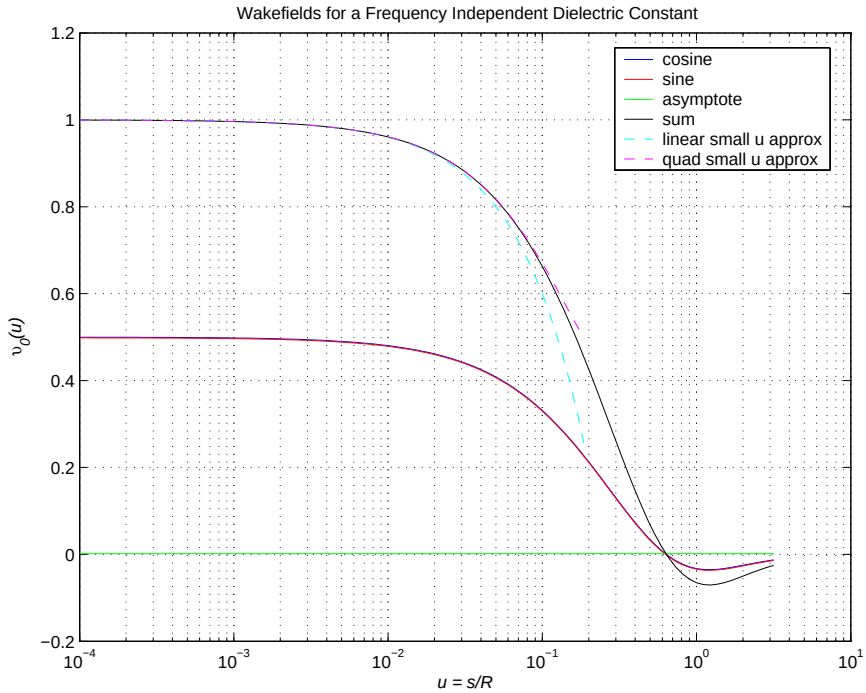

Figure A3: Evaluation of the wakefield for a frequency independent dielectric constant $\varepsilon_r = 2$. The line labeled "sum" is eq. (2.7), which is evaluated below in the discussion leading to eq. (2.14). This equation is written as a sum of three terms in eq. (2.14), and the lines labeled "cosine", "sine" and "asymptote" correspond to those three terms. The "asymptote" curve is identically zero because $\zeta = 2\pi/u\, Ceil\left(100/\pi\right)$ (*Ceil* is round to the next highest integer function), and the argument of E_1 is on the real axis making $\mathrm{Im}(E_1) = 0$. The "linear small *u* approx" is eq. (2.10), and the "quad small *u* approx" is eq. (2.13).

negative real axis and an infinite number of zeros above the negative real axis. Eq. (2.5) must be integrated just below the real axis, and for trailing particles it must be closed in the upper half plane.

Contours of integration are shown below. For leading particles the contour is just below the real axis and is closed in the lower half plane. This gives zero wakefield since there are no poles or cuts in the lower half plane. For trailing particles the contour must be just below the real axis and closed in the upper half plane. The integral for the contour above the real axis can be evaluated by the residue theorem since the integrand is analytic in the upper half plane. Doing so would require evaluating the residues at the two poles near the imaginary axis and the infinite number of poles just above the negative real axis. In addition the contribution of the cut must be evaluated. That can be done by numerically integrating around the cut as indicated. The sum of the integrals would give the wakefield for trailing particles.

Fortunately one does not have to perform these integrals, which involve an infinite number of poles and the cut, because eq. (2.7) already has the expression for wakefield.

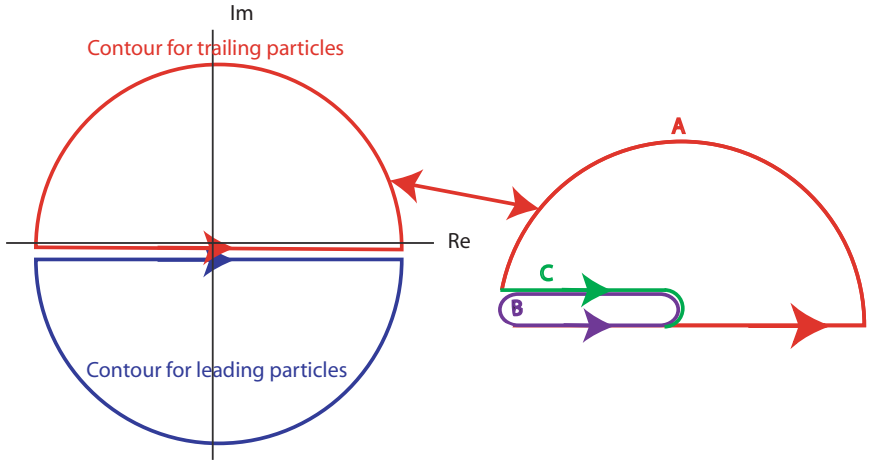

Figure A4: Contours of integration. Causality requires that the integrals be performed with z having a small negative imaginary part. The contour for leading particles is along the blue contour. The integral for trailing particles follows the red contour, which can be evaluated as the sum of two contour integrals. The paths A & C together enclose all of the poles in the upper half plane. Contour B encloses the cut on the negative real axis, and the portion above the negative real axis cancels C.

Numerical Evaluation

Following ARDB-279, eq. (2.7) can be broken up into two integrals, one from 0 to ζ and the other from ζ to ∞.

$$\upsilon_0(u) = \frac{1}{\pi}\mathrm{Im}\int_0^{\zeta} dz \frac{\cos(zu) + i\sin(zu)}{z - \dfrac{2\varepsilon_r}{\sqrt{\varepsilon_r - 1}}\dfrac{H_1^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}{H_0^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}} + \frac{1}{\pi}\mathrm{Im}\int_{\zeta}^{\infty} dz \frac{\exp(izu)}{z - \dfrac{2i\varepsilon_r}{\sqrt{\varepsilon_r - 1}}} \tag{2.14}$$

where the asymptotic expression for the Hankel function ratio has been used. Make the change of variables $v = z - 2i\varepsilon_r \big/ \sqrt{\varepsilon_r - 1}$. The second of the integrals in eq. (2.14) becomes

$$\frac{1}{\pi}\mathrm{Im}\int_{\zeta}^{\infty} dz \frac{\exp(izu)}{z - \dfrac{2i\varepsilon_r}{\sqrt{\varepsilon_r - 1}}} = \frac{1}{\pi}\exp\left(-\frac{2\varepsilon_r}{\sqrt{\varepsilon_r - 1}}\right)\mathrm{Im}\int_{\zeta - 2i\varepsilon_r/\sqrt{\varepsilon_r - 1}}^{\infty} dv \frac{\exp(ivu)}{v} \tag{2.15}$$

Making a second change of variables $t = -ivu$ gives

$$\begin{aligned}\frac{1}{\pi}\mathrm{Im}\int_{\zeta}^{\infty} dz \frac{\exp(izu)}{z - \dfrac{2i\varepsilon_r}{\sqrt{\varepsilon_r - 1}}} &= \frac{1}{\pi}\exp\left(-\frac{2\varepsilon_r}{\sqrt{\varepsilon_r - 1}}\right)\mathrm{Im}\int_{(-ju)\left(\zeta - 2i\varepsilon_r/\sqrt{\varepsilon_r - 1}\right)}^{\infty} dt \frac{\exp(-t)}{t} \\ &= \frac{1}{\pi}\exp\left(-\frac{2\varepsilon_r}{\sqrt{\varepsilon_r - 1}}\right)\mathrm{Im}\, E_1\left(-iu\zeta - 2u\,\varepsilon_r \big/ \sqrt{\varepsilon_r - 1}\right)\end{aligned} \tag{2.16}$$

where E_1 is the exponential integral. Note that versions of ARDB-279 prior to July 14, 2004 were based on a similar derivation and contained a sign error. See the 7/14/2004 version of that note for comments.

Appendix B: Frequency Dependent Dielectric Constant

Eq. (2.6) is based on matching boundary conditions at all frequencies, and it remains valid when the dielectric constant is a function of frequency. When $\omega < \omega_1$, $\varepsilon_r > 1$, and Cherenkov radiation is emitted. When $\omega > \omega_1$, $\varepsilon_r < 1$ the Hankel functions become modified Bessel functions and the fields decay exponentially in the dielectric.
Begin with eq. (2.6)

$$\upsilon_0(u) = \frac{1}{2\pi}\,\mathrm{Im}\int_{-\infty}^{\infty} dz \frac{\exp(izu)}{z - \dfrac{2\varepsilon_r}{\sqrt{\varepsilon_r - 1}}\dfrac{H_1^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}{H_0^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}} \tag{3.1}$$

and the expression for the frequency dependent dielectric constant (eq. (1.8))

$$\varepsilon_r - 1 = \frac{\alpha_1}{1 - \omega^2/\omega_1^2 + i\omega\gamma_1/\omega_1^2} = \frac{\alpha_1}{1 - z^2/z_1^2 + iz\gamma_1/z_1^2} \tag{3.2}$$

where $z = kR = \omega R/c$, $z_1 = \omega_1 R/c$, and γ_1 is normalized to z_1. For $|\omega| < \omega_1$, the positive square root is used to have outward propagating radiation. For $|\omega| > \omega_1$, the argument of the Hankel functions must be near the negative imaginary axis for the fields to decay away. This means that the appropriate solutions are

$$\frac{\sqrt{\varepsilon_r - 1}}{\sqrt{|\varepsilon_r - 1|}} = \left.\begin{matrix} -i\exp\left(\dfrac{i\theta}{2}\right) & z > z_1 \\ i\exp\left(\dfrac{i\theta}{2}\right) & z < -z_1 \end{matrix}\right\} \tag{3.3}$$

where

$$\theta = \tan^{-1}\left(\frac{\gamma_1 z}{z^2 - z_1^2}\right) \tag{3.4}$$

The integral, eq. (3.1), must be performed with z being considered as a complex quantity with a small negative imaginary part. With the signs of the square roots, the features seen for a frequency independent ε_r are still present: denominator has the dominant zeros near the positive imaginary axis in addition to the zeros and infinities near the negative real axis. The latter are compressed into the region $-z_1 < z < 0$. Two additional zeros appear on the real axis near $|z| > z_1$. See Figure B1.

The approximate values of these zeros are given by two of the solutions to the quadratic equation

$$z^4\left(\frac{1}{z_1^2} - \frac{4\alpha_1}{z_1^4}\right) + z^2\left(\frac{8(1+\alpha_1)}{\alpha_1 z_1^2} - 1\right) + \frac{4}{\alpha_1}\left(1 + \alpha_1^2 + 2\alpha_1\right) = 0 \tag{3.5}$$

This equation was obtained by putting $\gamma_1 = 0$, and using the approximation in eq. (2.8) for the ratio of the Hankel functions. It can be shown by performing a Taylor expansion of the denominator of eq. (3.1) that the zeros move into the upper half plane for $\gamma_1 > 0$.

The result of these considerations is that there are no poles or cuts in the lower half plane, and, therefore, causality is satisfied. In addition, the real part of the denominator is an odd

function of z, and the imaginary part is an even function. Therefore, the imaginary part of the integrand is an even function of z, so just as in the case of a frequency independent dielectric constant

$$\upsilon_0(u) = \frac{1}{\pi}\,\mathrm{Im}\int_0^\infty dz\,\frac{\exp(izu)}{z - \dfrac{2\varepsilon_r}{\sqrt{\varepsilon_r - 1}}\dfrac{H_1^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}{H_0^{(2)}\left(z\sqrt{\varepsilon_r - 1}\right)}} \tag{3.6}$$

where eq. (3.3) is to be used for $z > z_1$.

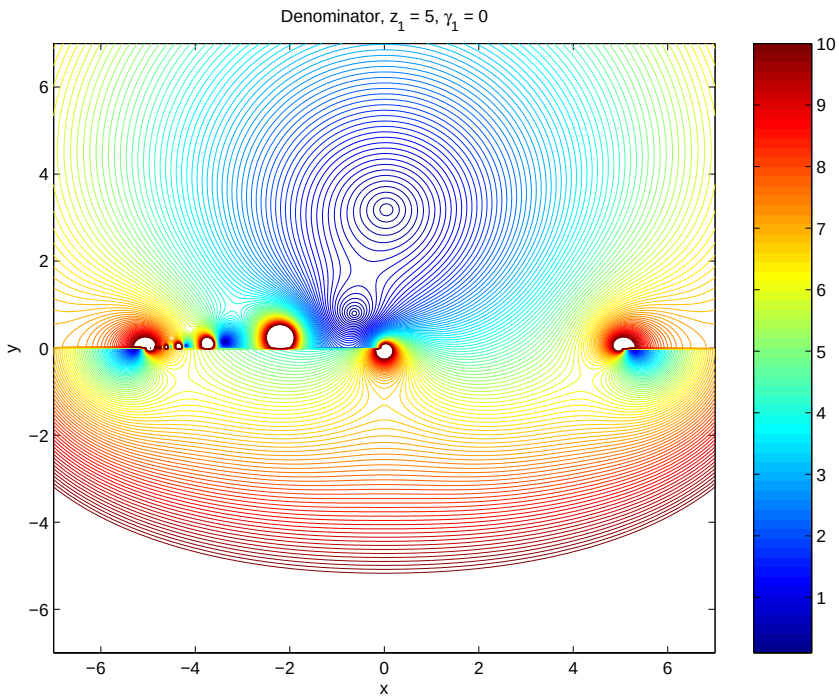

Figure B1: Denominator of eq. for $z_1 = 5$, $\gamma_1 = 0.001$, $\alpha_1 = 1$.

The individual contributions to the wakefield are shown in Figure B2. There are contributions to the short range wake from both and below and above ω_1.

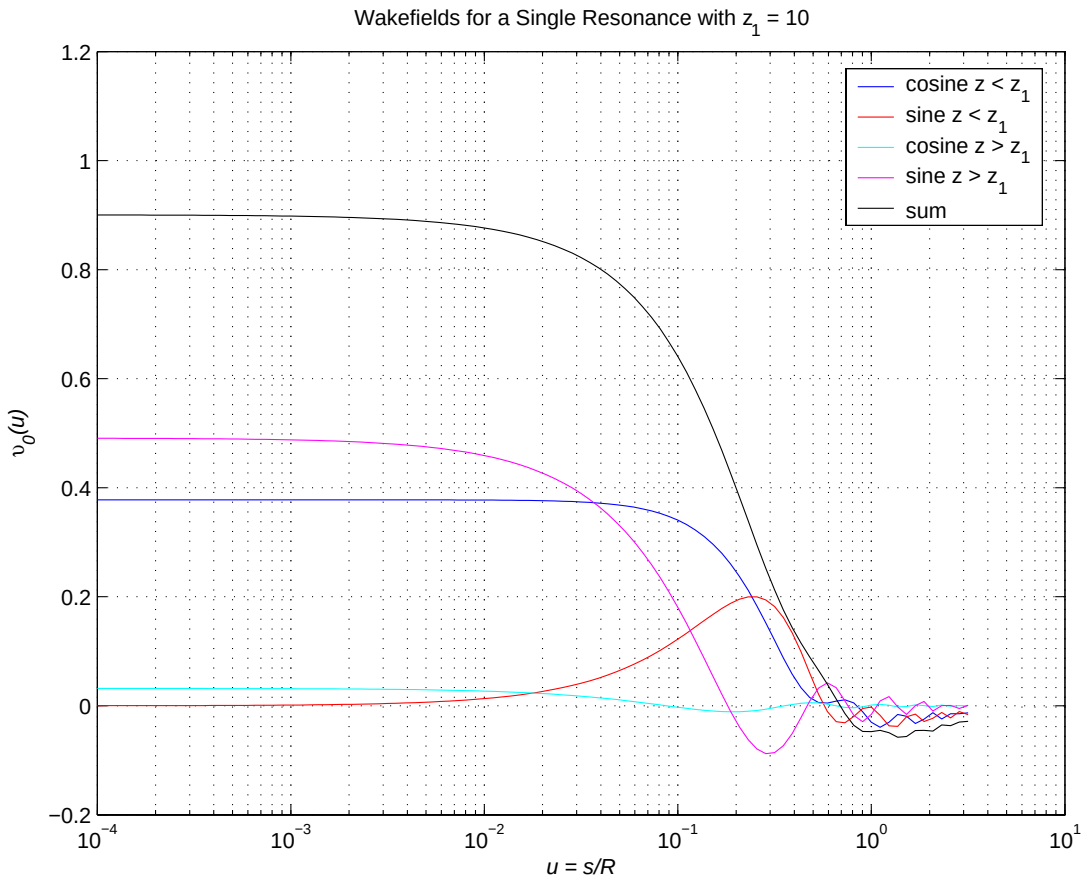

Figure B2: Wakefield for $z_1 = 10$, $\gamma_1 = 0.001$, $\alpha_1 = 2$. The various curves are the contributions of the cosine and sine portions of the integral (eq.) for the regions above and below z_1.

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[1] G. Dôme, Proceedings of the Second European Accelerator Conference 628 (1990).
[2] J. D. Jackson, Classical Electrodynamics, Third Edition.
[3] Xintian Eddie Lin, "Photonic Band Gap Fiber Accelerator" *Physical Review Special Topics – Accelerators and Beams* **4**, 051301 (2001).