Betatron Motion in Action-Angle Variables

The action angle variables for betatron motion are {J, φ}. The Hamiltonian for transverse motion in terms of these variables is

$$H = \frac{J}{\beta} \tag{1.1}$$

which gives the equations of motion

$$\frac{d\varphi}{ds} = \frac{\partial H}{\partial J} = \frac{1}{\beta}; \frac{dJ}{ds} = -\frac{\partial H}{\partial \varphi} = 0 \tag{1.2}$$

The position and angle are related to the action-angle variables through the Twiss parameters by

$$x = \sqrt{2J\beta}\cos\varphi \tag{1.3}$$

and

$$x' = -\sqrt{\frac{2J}{\beta}}\left(\sin\varphi + \alpha\cos\varphi\right) \tag{1.4}$$

The Twiss parameters, α, β and γ are related by

$$\beta\gamma = 1 + \alpha^2 \tag{1.5}$$

and they give an ellipse in phase space

$$\beta x'^2 + 2\alpha x x' + \gamma x^2 = 2J \tag{1.6}$$

α is related to β by

$$\alpha = -\frac{1}{2}\frac{d\beta}{ds} \tag{1.7}$$

A Gaussian beam is distributed exponentially in action

$$f(J) = \exp\left(-\frac{J}{\varepsilon}\right) \tag{1.8}$$

and uniformly in angle over the range $0 \le \varphi \le 2\pi$. The mean value of the action is

$$\langle J \rangle = \frac{\int f(J) J dJ}{\int f(J) dJ} = \varepsilon \tag{1.9}$$

Monte Carlo Generation of Gaussian Beams

Generate action-angle pairs with the action exponentially distributed and the angle uniformly distributed. If R_1 and R_2 are uniformly distributed random numbers $0 < R_1, R_2 \le 1$, then

$$J = -\varepsilon \ln R_1 \quad \& \quad \varphi = 2\pi R_2 \tag{2.1}$$

Transform to position and angle using equations (1.3) and (1.4) to get the appropriate distributions.

Determining Beam Parameters from Particle Distributions

Beam parameters can be extracted using the results below

$$\left\langle x^2 \right\rangle = 2\beta \left\langle J \cos^2 \varphi \right\rangle = 2\beta\varepsilon \frac{1}{2} = \beta\varepsilon \tag{3.1}$$

$$\left\langle x'^2 \right\rangle = \frac{2}{\beta}\left\langle J\left(\sin^2\varphi + \alpha^2\cos^2\varphi + 2\alpha\cos\varphi\sin\varphi\right)\right\rangle = \frac{2\varepsilon}{\beta}\frac{1}{2}\left(1+\alpha^2\right) = \gamma\varepsilon \tag{3.2}$$

$$\left\langle xx' \right\rangle = -2\left\langle J\left(\sin\varphi\cos\varphi + \alpha\cos^2\varphi\right)\right\rangle = -2\varepsilon\frac{\alpha}{2} = -\alpha\varepsilon \tag{3.3}$$

Combining them

$$\left\langle x^2 \right\rangle\left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle^2 = \varepsilon^2 \tag{3.4}$$

Beam Parameter Changes from Multiple Coulomb Scattering

Multiple scattering in a thin foil does not change the beam size

$$\left\langle x^2 \right\rangle = \beta\varepsilon = \beta_0\varepsilon_0 \tag{4.1}$$

where the subscript *0* denotes the value before the foil. The rms angle increases by the multiple scattering angle

$$\left\langle x'^2 \right\rangle = \gamma\varepsilon = \gamma_0\varepsilon_0 + \theta_{rms}^2 \tag{4.2}$$

From the *Particle Physics Booklet*

$$\sqrt{\theta_{rms}^2} = \frac{13.6MeV}{pc}\left(1 + .038\ln\left(\frac{X}{X_0}\right)\right)\sqrt{\frac{X}{X_0}} \tag{4.3}$$

where X is the foil thickness and X_0 is the radiation length for the foil material. The $x - x'$ correlation after the foil is given by

$$xx' = -2J_0\left(\alpha_0\cos^2\varphi_0 + \cos\varphi_0\sin\varphi_0\right) + \theta\sqrt{\frac{2J_0}{\beta_0}}\cos\varphi_0 \tag{4.4}$$

This must be averaged over all values of action, angle and multiple scattering angle. The last term will average to zero $\langle\theta\rangle = 0$; $\langle\cos\varphi_0\sin\varphi_0\rangle = 0$; $\langle\cos^2\varphi_0\rangle = 1/2$, and $\langle J_0\rangle = \varepsilon_0$, so

$$\left\langle xx' \right\rangle = -\varepsilon_0\alpha_0 \tag{4.5}$$

Combining these expressions

$$\begin{gathered}\varepsilon^2 = \left\langle x^2 \right\rangle\left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle^2 = \beta_0\varepsilon_0\left(\gamma_0\varepsilon_0 + \theta_{rms}^2\right) - \alpha_0^2\varepsilon_0^2 \\ \varepsilon^2 = \varepsilon_0^2 + \beta_0\varepsilon_0\theta_{rms}^2\end{gathered} \tag{4.6}$$

The new value of α is

$$\alpha = sign\left(\alpha_0\right)\sqrt{\beta\gamma - 1} \tag{4.7}$$

The $sign\left(\alpha_0\right)$ term says that multiple scattering does not change the orientation of the phase space ellipse.

The results are given by equations (4.1), (4.2), (4.6) and (4.7). These are the results presented in J. B. Rosenzweig and P. Chen, Phys. Rev D **39**, 2039 (1989).

Connection to Quasioptics and Ray Optics

The "ABCD Law" (Chapter 3 of P. F. Goldsmith, Quasioptical Systems) for ray optics can be cast in the framework of betatron oscillations. The size of a Gaussian beam propagates from a waist as

$$w = w_0\sqrt{1+\left(\frac{\lambda s}{\pi w_0^2}\right)^2} \tag{6.1}$$

Comparing this to betatron motion

$$\sqrt{\beta\varepsilon} = \sqrt{\beta_0\varepsilon}\sqrt{1+\left(\frac{s}{\beta_0}\right)^2} \tag{6.2}$$

for propagation from a focus gives the following equivalences

$$\beta_0 = \frac{\pi w_0^2}{\lambda}; \quad \varepsilon = \frac{\lambda}{\pi} \tag{6.3}$$

The radius of curvature of the wavefront is

$$R(s) = s + \frac{1}{s}\left(\frac{\pi w_0^2}{\lambda}\right)^2 = s + \frac{\beta_0^2}{s} \tag{6.4}$$

The Guoy phase shift leaving the focus is

$$\varphi(s) = \tan^{-1}\left(\frac{\lambda s}{\pi w_0^2}\right) = \tan^{-1}\left(\frac{s}{\beta_0}\right) \tag{6.5}$$

This is the same result that is obtained by integrating the inverse of the β-function.

There is a change in effective emittance as the light passes from an index n_1 to index n_2. Subscripts 1 and 2 denote values in the two materials. The beam size does not change at the interface

$$\left\langle x^2\right\rangle = \beta_1\varepsilon_1 = \beta_2\varepsilon_2 \tag{6.6}$$

The angle changes according to Snell's law

$$x_2' = -\sqrt{\frac{2J_1}{\beta_1}}\frac{n_2}{n_1}\left(\sin\varphi_1 + \alpha_1\cos\varphi_1\right) = -\sqrt{\frac{2J_2}{\beta_2}}\left(\sin\varphi_2 + \alpha_2\cos\varphi_2\right) \tag{6.7}$$

The rms angular spread is

$$\left\langle x_2'^2\right\rangle = \gamma_2\varepsilon_2 = \left(\frac{n_2}{n_1}\right)^2\gamma_1\varepsilon_1 \tag{6.8}$$

The correlation term is

$$\left\langle x_2x_2'\right\rangle = \left\langle -2J_1\frac{n_2}{n_1}\cos\varphi_1\left(\sin\varphi_1 + \alpha_1\cos\varphi_1\right)\right\rangle = -\varepsilon_1\alpha_1\frac{n_2}{n_1} \tag{6.9}$$

Combining terms

$$\varepsilon_2^2 = \left\langle x_2^{\,2}\right\rangle\left\langle x_2'^2\right\rangle - \left\langle x_2x_2'\right\rangle^2 = \beta_1\varepsilon_1\gamma_1\varepsilon_1\left(\frac{n_2}{n_1}\right)^2 - \alpha_1^2\varepsilon_1^2\left(\frac{n_2}{n_1}\right)^2 = \left(\frac{n_2}{n_1}\right)^2\varepsilon_1^2 \tag{6.10}$$

Substituting this result into eqs. (6.6) and (6.8) gives

$$\beta_2 = \beta_1 \frac{n_1}{n_2}; \quad \gamma_2 = \gamma_1 \frac{n_2}{n_1}; \quad \alpha_2 = \alpha_1 \tag{6.11}$$