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Principles of Harmonic Acceleration for a Prescribed Bunch Profile

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The final focus system for a 5 TeV collider may be greatly shortened, from 40 km to 0.1-1 km, provided chromatic correction is abandoned, and replaced with ultra-low energy spread of 10^{-3}%. A conventional single-frequency linac cannot produce bunches of the requisite length and charge, with an energy spread this small. In this note we elaborate on previous discussion of harmonic acceleration for the purpose of energy spread compensation.

Introduction

The final focus system as described by Zimmermann,[1] for a W-Band 5-TeV collider includes no chromatic correction section, relying on the linac to produce a beam with small energy spread, less than 10 ppm in rms. We accomplish this by employing rf accelerator sections operated at harmonics of the fundamental. Basic concerns are: how many and which harmonics, energy budget, energy spread achievable, sensitivity to the wakefield model. In this note we put aside detailed excursions into the matter of the wakefield model.

Consider a voltage waveform representing the energy kick imparted by a linac with harmonic acceleration. It takes the form

$$V = \sum_n V_n \cos(n\omega_1 t + \phi_n) - \int^t dt' \, I(t') G(t - t'),$$

with V_n, ϕ_n the voltage and phase describing the net voltage phasor for the entire linac, and the function G describing the longitudinal wakefield of the linac. For the sake of definiteness let us consider a model wakefield, varying with time as $G(t) \propto t^{-\nu}$, and we suppose $0 < \nu < 1$. We consider a "flat-top" current profile turning on at t=0, and extending to t=T, with charge Q. It is convenient to express quantities in terms of the total loss-factor k_l, defined such that the total energy loss of the beam in traversing the linac with rf power off is $k_l Q^2$.

With this model wakefield and beam we may write

$$V = \sum_n V_n \cos(n\omega_1 t + \phi_n) - k_l Q(2-\nu)\left(\frac{t}{T}\right)^{1-\nu}.$$

To simplify expressions, let us resort to normalized variables, measuring voltages in units of the no-load fundamental amplitude

$$\hat{V} = \frac{V}{V_1} = \sum_n \hat{V}_n \cos(n\Omega_1 \tau + \phi_n) - \hat{Q}\tau^{\mu},$$

and introducing $\Omega_1 = \omega_1 T$, $\tau = t/T$ varying from 0 at the beam head to 1 at the tail, and

[1] F. Zimmermann, *et al.* "Final focus system and collision schemes for a 5 TeV collider", submitted to *Proceedings of the 2nd International Workshop on Electron-Electron Interactions at TeV Energies.*

$$\hat{Q} = \frac{k_l Q(2-\nu)}{V_1}.$$

We abbreviate $\mu = 1-\nu$.

Our problem is to determine, given a finite selection of harmonics (0, 1 or 2 for this work), amplitude and phase-settings that minimize rms energy spread. Perhaps one's first thought is to provide for a zeroing of slope, curvature, and then higher derivatives of the voltage waveform. However, given the character of the wakefield we are employing, derivatives are not well-behaved. Perhaps one's second thought is then to take finite differences. We may choose to phase according to

$$\hat{V}(tail) = \hat{V}(1) = \hat{V}(0) = head,$$

or

$$\sum_n \hat{V}_n \cos(\phi_n) = \sum_n \hat{V}_n \cos(n\Omega_1 + \phi_n) - \hat{Q}$$

A simple measure of energy spread is then

$$\begin{aligned}\delta\hat{V} &= \hat{V}\left(\tfrac{1}{2}\right) - \hat{V}(0) \\ &= \sum_n \hat{V}_n \cos\left(\tfrac{1}{2} n\Omega_1 + \phi_n\right) - \sum_n \hat{V}_n \cos(\phi_n) - 2^{-\mu}\hat{Q} \\ &= \sum_n -2\hat{V}_n \sin\left(\tfrac{1}{4} n\Omega_1\right) \sin\left(\tfrac{1}{4} n\Omega_1 + \phi_n\right) - 2^{-\mu}\hat{Q}.\end{aligned}$$

One could then consider zeroing the voltage offset at additional points within the beam. We will find that this algorithm is not optimal. One could guess the selection of points is implied to be uniformly distributed, and thus does not respect the form of the wakefield, nor the particulars of the contributions to the rms of the various portions of the beam. However, let us pursue it a bit farther before abandoning it.

Single-Frequency Linac

For example, for the conventional, single-frequency linac, this algorithm reduces to

$$\cos(\phi_1) = \cos(\Omega_1 + \phi_1) - \hat{Q},$$

or

$$-\phi_1 = \tfrac{1}{2}\Omega_1 + \sin^{-1}\left[\frac{\hat{Q}}{2\sin\left(\tfrac{1}{2}\Omega_1\right)}\right].$$

Energy spread is gauged by

$$\begin{aligned}\delta\hat{V} &= -2\sin\left(\tfrac{1}{4}\Omega_1\right)\sin\left(\tfrac{1}{4}\Omega_1 + \phi_1\right) - 2^{-\mu}\hat{Q} = 2\sin\left(\tfrac{1}{4}\Omega_1\right)\sin\left(\tfrac{1}{4}\Omega_1 + \sin^{-1}\varepsilon\right) - 2^{-\mu}\hat{Q} \\ &= 2\sin\left(\tfrac{1}{4}\Omega_1\right)\left[\sqrt{1-\varepsilon^2}\sin\left(\tfrac{1}{4}\Omega_1\right) + \varepsilon\cos\left(\tfrac{1}{4}\Omega_1\right)\right] - 2^{-\mu}\hat{Q}\end{aligned}$$

where we abbreviate

$$\varepsilon = \frac{\hat{Q}}{2\sin\left(\frac{1}{2}\Omega_1\right)}.$$

Denoting $\theta = \frac{1}{2}\Omega_1$, we have

$$\delta\hat{V} = \sqrt{1-\varepsilon^2}\,(1-\cos\theta) + \varepsilon\sin\theta - 2^{-\mu}\hat{Q}$$
$$= \sqrt{1-\varepsilon^2}\left(\tfrac{1}{2}\theta^2 - \tfrac{1}{24}\theta^4 + \ldots\right) + \varepsilon\left(\theta - \tfrac{1}{6}\theta^3 + \tfrac{1}{120}\theta^5 + \ldots\right) - 2^{-\mu}\hat{Q}.$$

In approximate form this is

$$\delta\hat{V} = \frac{1}{8}\Omega_1^2 - \frac{1}{2}\hat{Q}\left(2^{\nu} - 1\right).$$

and in more practical units

$$\frac{\delta V}{V_{NL}} = \frac{1}{8}\left(\omega_1 T\right)^2 - \left(1 - \tfrac{\nu}{2}\right)\left(2^{\nu} - 1\right)\frac{k_l Q}{V_{NL}}.$$

Notice that the wakefield has helped to lower the energy spread, tending to flatten out the accelerating waveform, provided the beam has been properly phased ahead of crest, on the rising portion of the voltage waveform. From the Gaussian analog of this result, in the absence of wakefields, one can demonstrate the well-known result

$$\sigma_\phi(^\circ RF) = 6.8\sqrt{\delta_{rms}(\%)}\,.$$

For example, a 1.1 mm bunch in a 10.5 cm wavelength accelerator will approach a 0.3% energy spread asymptotically. Lower energy spread is possible only with the help of wakefields.

To illustrate the scalings let us consider a 60-pC bunch with a loss-factor corresponding to the nominal fundamental mode loss at W-Band (3.128 mm wavelength), 32 V/pC/cell, multiplied by 5 to account for higher mode losses. (This multiplier is a function of beam port radius normalized to free-space wavelength). This corresponds to 9600V/cell. For cells 1/3 of a wavelength long, and 2500 m of linac, we have 2.3×10^6 cells, and 22 GeV for the $k_l Q$ product. Meanwhile $V_1 \approx 2490$ GeV so that

$$\hat{Q} \approx \frac{22\text{ GeV}}{2490\text{ GeV}} \times (2-\nu) \approx 1.3\times10^{-2},$$

taking $\nu \approx 0.5$ for illustration. Taking an rms bunch length of 30 μm, $T \approx \sqrt{12}\times 30\mu\text{m} \approx 104\mu\text{m}$, we have

$$\Omega_1 = \omega_1 T \approx 2\pi\frac{104\,\mu\text{m}}{3280\,\mu\text{m}} \approx 0.20,$$

and $\varepsilon \approx 6.5\times10^{-2}$. The fundamental phase is given by

$$-\phi_1 \approx \tfrac{1}{2}0.20 + \sin^{-1}\left[6.5\times10^{-2}\right] \approx 9.5^\circ,$$

corresponding to $\cos(9.5°) \approx 0.986$ and a 1.4% reduction in voltage below the maximum no-load voltage. Energy spread is gauged by

$$\delta\hat{V} = \frac{1}{8}\Omega_1^2 - \frac{1}{2}\hat{Q}\left(2^\nu - 1\right) \approx \frac{1}{8}(0.20)^2 - \frac{1}{2}\left(1.3\times10^{-2}\right)\left(2^{1/2} - 1\right)$$
$$\approx 2.3\times10^{-3}$$

This figure is 5×10^{-3} without the wakefield term. (As a point of reference, to produce an energy spread of 1×10^{-5}, one would require $T \approx 1.3\ \mu\text{m}$, far shorter than we envision.) From these considerations one can see that more is required to accomplish the desired 10ppm rms energy spread.

Let us end this section with several illustrations of the voltage profile along the bunch, with these parameters, and several different values of ν. Figs.1-3 depict plots of

$$\hat{V}(\tau) = \cos\left(\Omega_1\tau + \phi_1\right) - \hat{Q}\left\{\tau^{1-\nu} - (\tau-1)^{1-\nu} H(\tau-1)\right\},$$

for the phase minimizing the center-beam energy spread,

$$-\phi_1 = \tfrac{1}{2}\Omega_1 + \sin^{-1}\left[\frac{\hat{Q}}{2\sin\left(\tfrac{1}{2}\Omega_1\right)}\right],$$

and for the phase minimizing the actual rms energy spread, at fixed bunch charge. These calculations were performed simply with a spreadsheet. One appreciates from these figures that the exponent and the actual phasing algorithm can make a difference at the factor of 2 level in energy spread (a significant difference). However, in all cases, energy spread is far larger than we require for the compact final focus scheme.

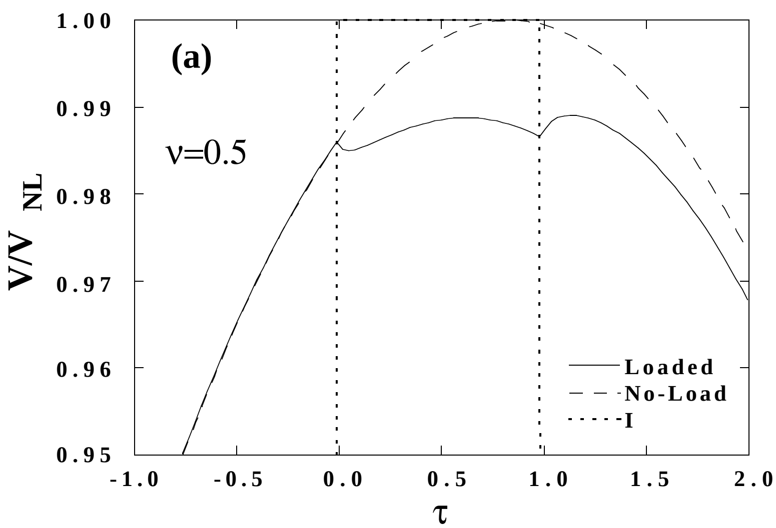

FIGURE 1 (a). Depicting the loaded and unloaded voltage waveforms for ν=0.5, and the center-bunch phasing φ=-9.5°, corresponding to rms energy spread of $1.2\text{x}10^{-3}$. Also shown is the current waveform. In practical units, the bunch full length is 11.4° of RF phase, and the horizontal scale in this figure corresponds to 3 x that length or about 34° of

RF phase. The vertical scale in this figure can be viewed simply as the energy scale for accelerated particles, normalized to the no-load (single-particle, on crest) energy.

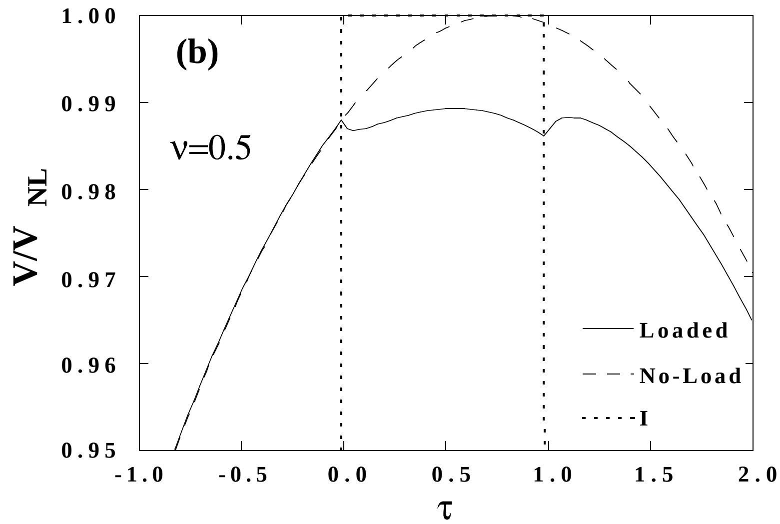

FIGURE 1 (b). Depicting the loaded and unloaded voltage waveforms for ν=0.5, and the rms phasing φ=-8.8°, corresponding to rms energy spread of 0.96x10^{-3}.

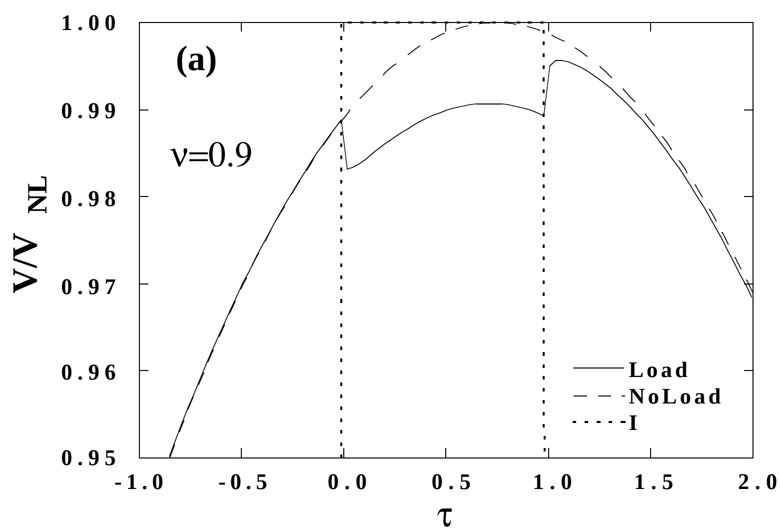

FIGURE 2 (a). Depicting the loaded and unloaded voltage waveforms for ν=0.9, and the center-bunch phasing φ=-8.5°, corresponding to rms energy spread of 2.4x10^{-3}.

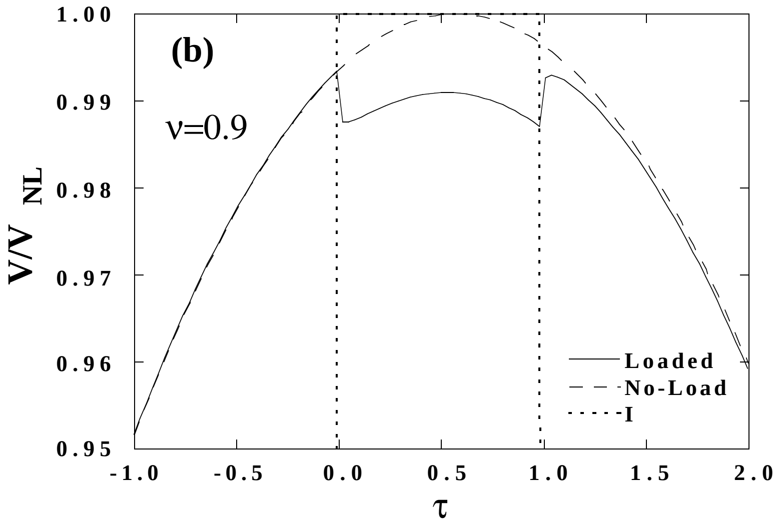

FIGURE 2 (b). Depicting the loaded and unloaded voltage waveforms for ν=0.9, and the rms phasing φ=-6.5°, corresponding to rms energy spread of 1.2x10^{-3}.

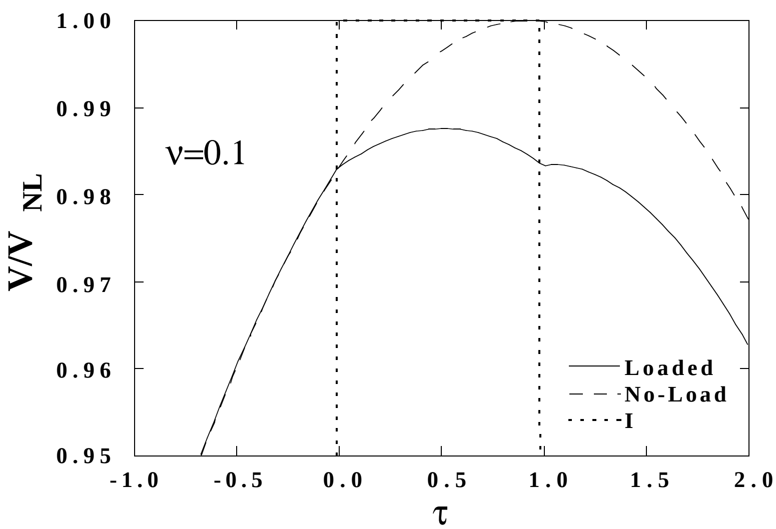

FIGURE 3. Depicting the loaded and unloaded voltage waveforms for ν=0.1, and the center-bunch phasing φ=-10.5°, corresponding to rms energy spread of 1.3x10^{-3}. This phasing corresponds approximately to the minimum rms phasing.

One-Harmonic

To illustrate the usefulness of harmonic acceleration let us consider the simplest case, of one harmonic. We consider how to set the amplitude and phase for the harmonic section. Our conditions for approximate energy spread minimization are:

$$\cos(\phi_1) + \hat{V}_h \cos(\phi_h) = \cos(\Omega_1 + \phi_1) + \hat{V}_h \cos(h\Omega_1 + \phi_h) - \hat{Q},$$

and energy spread is gauged by

$$\delta\hat{V} = -2\sin\left(\tfrac{1}{4}\Omega_1\right)\sin\left(\tfrac{1}{4}\Omega_1 + \phi_1\right) - 2\hat{V}_h \sin\left(\tfrac{1}{4}h\Omega_1\right)\sin\left(\tfrac{1}{4}h\Omega_1 + \phi_h\right) - 2^{-\mu}\hat{Q}.$$

We suppose the linac is setup to provide minimum energy spread (as a function of fundamental mode phase) with the harmonic sections off. This is condition is not strictly necessary, and, therefore one suspects that it will lead to a less than optimal solution. However, we are attracted to this assumption, since it implies a linac that it is relatively insensitive to fluctuations in the harmonic rf system.

Accepting this condition, we must require

$$\hat{V}_h \cos(\phi_h) = \hat{V}_h \cos(h\Omega_1 + \phi_h),$$

so that

$$\phi_h = -\tfrac{1}{2}h\Omega_1.$$

For a harmonic number that is not too high, $h\Omega_1/2 \le \pi/2$ ($h \le 15$ for our parameters, or harmonic frequency lower than 1.3 THz), this constraint implies that the harmonic waveform takes on a parabolic shape, centered on the beam. One pictures the loaded fundamental waveform also as a parabolic form, and thus the harmonic could in principle provide some useful cancellation, leaving a higher-frequency residual (that we could later cancel in part with a harmonic h' satisfying $h'\Omega_1/2 \ge \pi$ ($h' \ge 30$ for our parameters, or a free-space wavelenth on the order of 100 μm, comparable to the bunch length). This reasoning is qualitative, and eventually we will be left looking for a more rigorous algorithm --- but let's continue on.

As before we abbreviate

$$\varepsilon = \frac{\hat{Q}}{2\sin\left(\tfrac{1}{2}\Omega_1\right)}, \quad \theta = \tfrac{1}{2}\Omega_1,$$

so that the center-beam energy offset (from head or tail) is

$$\begin{aligned}\delta\hat{V} &= \sqrt{1-\varepsilon^2}(1-\cos\theta) + \varepsilon\sin\theta + \hat{V}_h(1-\cos h\theta) - 2^{-\mu}\hat{Q} \\ &= \sqrt{1-\varepsilon^2}\left(\tfrac{1}{2}\theta^2 - \tfrac{1}{24}\theta^4 + \ldots\right) + \varepsilon\left(\theta - \tfrac{1}{6}\theta^3 + \tfrac{1}{120}\theta^5 + \ldots\right) + \hat{V}_h\left(\tfrac{1}{2}h^2\theta^2 - \tfrac{1}{24}h^4\theta^4 + \ldots\right) - 2^{-\mu}\hat{Q}.\end{aligned}$$

This we set to zero by adjusting $\hat{V}_h$,

$$\begin{aligned}\hat{V}_h &= -\frac{1}{(1-\cos h\theta)}\left\{\sqrt{1-\varepsilon^2}(1-\cos\theta) + \varepsilon\sin\theta - 2^{-\mu}\hat{Q}\right\} \\ &= -\frac{1}{h^2}\left[\frac{h\theta/2}{\sin(h\theta/2)}\right]^2\left\{\sqrt{1-\varepsilon^2}\left[\frac{\sin(\theta/2)}{\theta/2}\right]^2 - \frac{1}{4}\frac{\hat{Q}}{(\theta/2)^2}\left(2^{\nu}-1\right)\right\}.\end{aligned}$$

In order of magnitude, one requires $\hat{V}_h \approx -1/h^2$ to minimize the energy spread at this order in bunch length. The actual coefficient depends on the particulars of the wakefield, and is lower at higher charge. For the sake of conservatism in power estimates, we will employ the higher figure.

With these harmonic settings, the residual energy spread can be gauged by computing $\hat{V}(1/4)-\hat{V}(0)$, or more simply and rigorously by numerically evaluating the rms. Where the single-frequency calculations could easily be performed on a spreadsheet, it is more efficient to perform the two-frequency and later calculations with a fortran program. We have written a short code `harm.for` to examine the scalings. Such a code comes in handy since the energy spread from head to mid-beam or other points on the beam is at best a gauge of energy spread, and energy spread is more easily and precisely evaluated and minimized numerically. The input file `harm.in` includes the following settings

```
$INPUTNML
      harm1=4.,          !first harmonic number
      harm2=30.,         !second harmonic number
      eps=0.0,           !compute energy spread on eps to 1
      !
      energy=2.5E12,     !V, energy gain in linac
      gradient=1.e9,     !V/M
      wave0=3.28e-3,     !m, RF wavelength
      xnu=0.5,           !wakefield parameter
      charge=60.e-12,    !Coulomb, bunch charge
      sigz=30.e-6,       !m, bunch length
      !
      checking=F, !T=> checks made
 $END
```

We employ ν=0.5 for illustration, and examine the function

$$\hat{V}(\tau)=\cos\left(\Omega_1\tau+\phi_1\right)+\hat{V}_h\cos\left(h\Omega_1\tau+\phi_h\right)-\hat{Q}\left\{\tau^{1-\nu}-(\tau-1)^{1-\nu}H(\tau-1)\right\}.$$

We take

$$-\phi_1=\tfrac{1}{2}\Omega_1+\sin^{-1}\left[\frac{\hat{Q}}{2\sin\left(\tfrac{1}{2}\Omega_1\right)}\right],\qquad \phi_h=-\tfrac{1}{2}h\Omega_1,$$

and adjust $\hat{V}_h$ to minimize the rms energy spread.

Typical code output to screen looks like this

```
----------Reading inputs from harm.in
...using loss factor per cell = (V/pC)    160.0609
...loss in linac (V)   2.1959580E+10
...qhat =   1.3175748E-02
...T(m) =   1.0392304E-04
...Omega1 =   0.1990755
...epsilon =   6.6294089E-02
...ideal phase 1 (deg)   -9.504252
******Examining Single - Frequency & "Ideal" Phasing
              writing plot file vnl.dat
...1 frequency avg, rms =   0.9873692      1.2132282E-03
******Examining Single - Frequency & Scanning Phase
              writing plot file scan1.dat
...min drms, dPhi1 (deg)   9.9106098E-04  0.9677410
```

```
...davg, net Phi1 (deg)   0.9883353      -8.536511
              writing plot file vnls.dat
...after phase1 tweak 1 frequency avg, rms =   0.9883353      9.9106098E-04
******Examining Two - Frequency, "Ideal" Settings
using harmonic ampl & phase (deg)  -6.8946439E-03  -22.81237
              writing plot file vnlh1.dat
..."ideal",  2 frequency avg, rms =   0.9816228      8.4167795E-04
...
```

The code first looks at the single-frequency linac scaling, with the output here merely confirming our previous spreadsheet calculation. Next the code pursues the simple algorithm we have suggested to minimize energy spread, putting the beam head and tail at the same energy with harmonic voltage off, and then maintaining this constraint with harmonic voltage on, and adding the additional constraint that the beam midpoint should have energy the same as head and tail. The result of 8.4×10^{-4} for rms energy spread is not satisfactory at all, only 10% lower than with the harmonic voltage off. The code then scans harmonic phase and finds no particular improvement. In fact, using the phase-settings given above the actual rms is 1.2×10^{-3}, no better than without the harmonic term.

A more rigorous minimization proceeds as follows. We consider the waveform

$$\hat{V}(\tau)=\hat{V}_0(\tau)+A\cos\phi(\tau),$$

with $\hat{V}_0(\tau)$ the loaded fundamental mode waveform. Denoting

$$\delta\hat{V}(\tau)=\hat{V}_0(\tau)-\left\langle\hat{V}_0(\tau)\right\rangle,$$

with $\langle\ldots\rangle$ representing a charge-weighted average over the beam (for our flat-top example, uniformly weighted). We wish to minimize

$$\sigma_v^2=\left\langle\delta\hat{V}^2\right\rangle.$$

Abbreviating

$$\delta C(\tau)=\cos\phi(\tau)-\langle\cos\phi\rangle,\qquad \delta S(\tau)=\sin\phi(\tau)-\langle\sin\phi\rangle,$$

we require

$$\frac{\partial\sigma_v^2}{\partial A}=0\Rightarrow A=-\frac{\left\langle\delta\hat{V}\delta C\right\rangle}{\left\langle\delta C^2\right\rangle}.$$

In addition, we have a requirement on phase. We express

$$\phi(\tau)=\phi_0(\tau)+\alpha,$$

and require that

$$\frac{\partial\sigma_v^2}{\partial\alpha}=-2A\left\langle\delta\hat{V}\delta S\right\rangle-2A^2\langle\delta C\delta S\rangle=0.$$

This condition may be expressed as

$$\mu = \left\langle \delta\hat{V}\delta S \right\rangle \left\langle \delta C_2 \right\rangle - \left\langle \delta\hat{V}\delta C \right\rangle \left\langle \delta C \delta S \right\rangle = 0.$$

Forming

$$\delta E = \delta C + i\delta S,$$

we may write as the imaginary part of a product,

$$\mu = \Im \left\langle \delta\hat{V}\delta E \right\rangle \left\langle \delta C \delta E^* \right\rangle.$$

Next we observe that if $\alpha \to \alpha + \beta$, then $\delta E \to \delta E e^{i\beta}$, while

$$\delta C \to \delta C \cos\beta - \delta S \sin\beta.$$

and therefore

$$\begin{aligned} \mu &\to \Im \left\langle \delta\hat{V}\delta E \right\rangle \left\langle (\delta C \cos\beta - \delta S \sin\beta)\delta E^* \right\rangle \\ &= \cos\beta\, \Im \left\langle \delta\hat{V}\delta E \right\rangle \left\langle \delta C \delta E^* \right\rangle - \sin\beta \Im \left\langle \delta\hat{V}\delta E \right\rangle \left\langle \delta S \delta E^* \right\rangle. \end{aligned}$$

Evidently then, starting from any choice of phase, we may compute the phase for minimization according to

$$\beta = \tan^{-1} \frac{\Im \left\langle \delta\hat{V}\delta E \right\rangle \left\langle \delta C \delta E^* \right\rangle}{\Im \left\langle \delta\hat{V}\delta E \right\rangle \left\langle \delta S \delta E^* \right\rangle}.$$

Adopting this phase, and recomputing the cosine, sine, and voltage products, we may then adopt

$$A = -\frac{\left\langle \delta\hat{V}\delta C \right\rangle}{\left\langle \delta C^2 \right\rangle},$$

and arrive at the minimum energy spread.

This algorithm may be combined with a scan over fundamental mode phase to rapidly optimize the three parameters: fundamental mode phase, harmonic phase, and harmonic amplitude. Results are summarized in Fig. 4 (a)-(d). For example, for *h*=10, it is possible to achieve an rms energy spread of 1.6×10^{-4} and 97.4% of the unloaded average voltage, with a normalized harmonic amplitude of 1.2×10^{-2}.

Eventually we may decide that we would prefer not to scan over fundamental phase, for in the case of a harmonic section located separately at the end of the linac (presumably a good location in consideration of transverse wakefields), the beam must pass through the linac with whatever energy spread arises with the fundamental + wakefields alone. Thus fundamental phase may have a pre-existing constraint on it, either due to a need for minimum energy spread to avoid filamentation, or, perhaps due to a need for a precise energy chirp to control beam break-up. In a previous technical note,[2] we have found beam break-up to be acceptable without BNS energy chirp, and so, eventually, we will decide instead to fix the fundamental mode phase for minimum energy spread in the main linac.

[2] D. H. Whittum and M. Hill, "Beam Dynamics in a 2.5 TeV Planar Linac" ARDB TN 40.

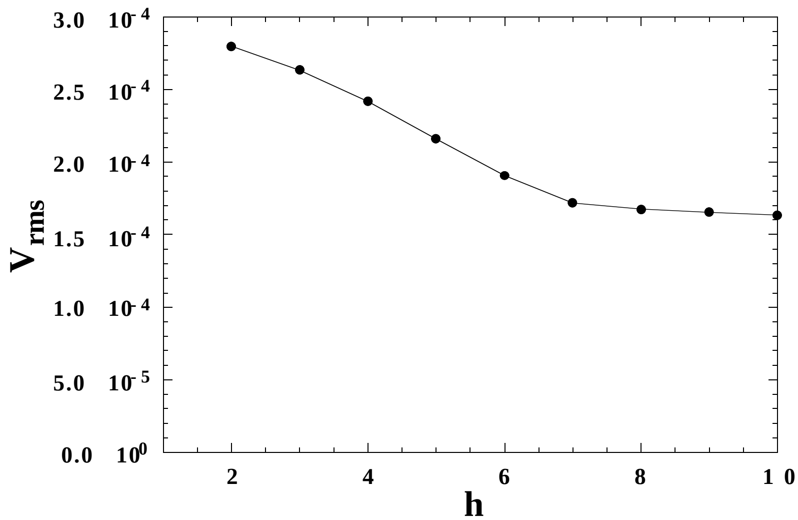

FIGURE 4. **(a)** Variation in minimum rms energy spread versus harmonic number.

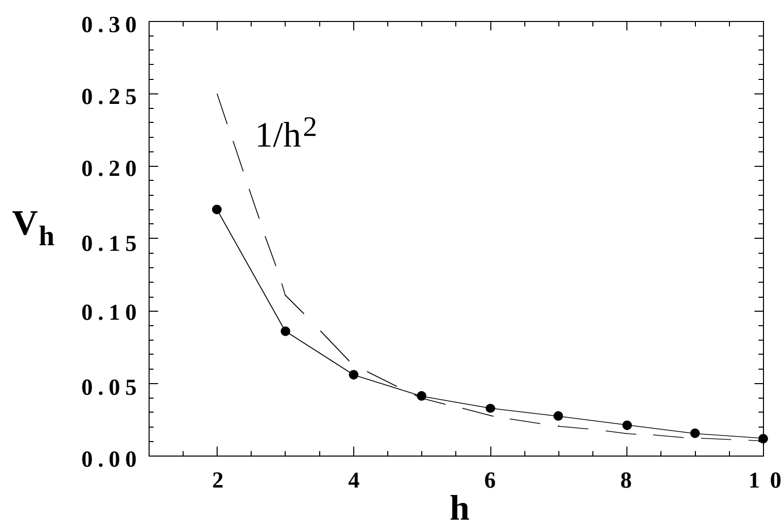

FIGURE 4. **(b)** Variation in harmonic amplitude required for minimum rms energy spread, versus harmonic number. Also shown is the curve $1/h^2$ for comparison.

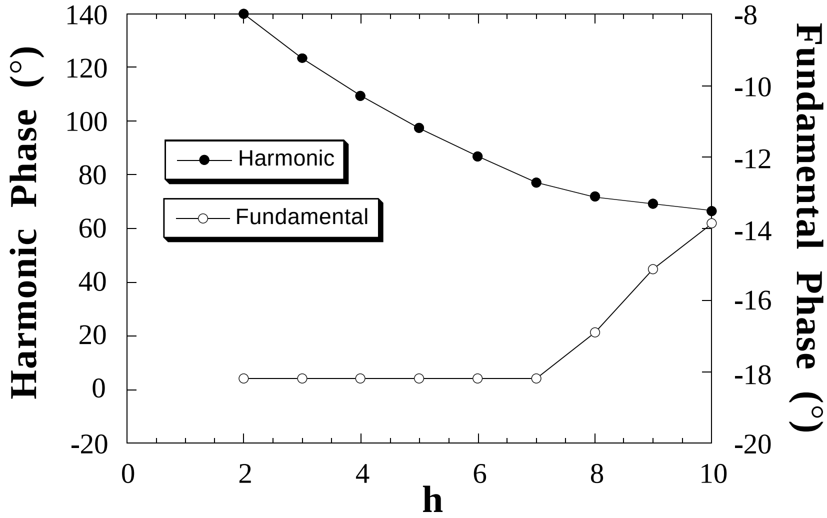

FIGURE 4. **(c)** Variation of fundamental and harmonic phase at the operating point for minimum rms energy spread, versus harmonic number. The fundamental phase levels off for lower harmonic numbers, since we have set a constraint that the average no-load voltage over the bunch should not drop too low, in this case, below 95% of the peak no-load voltage.

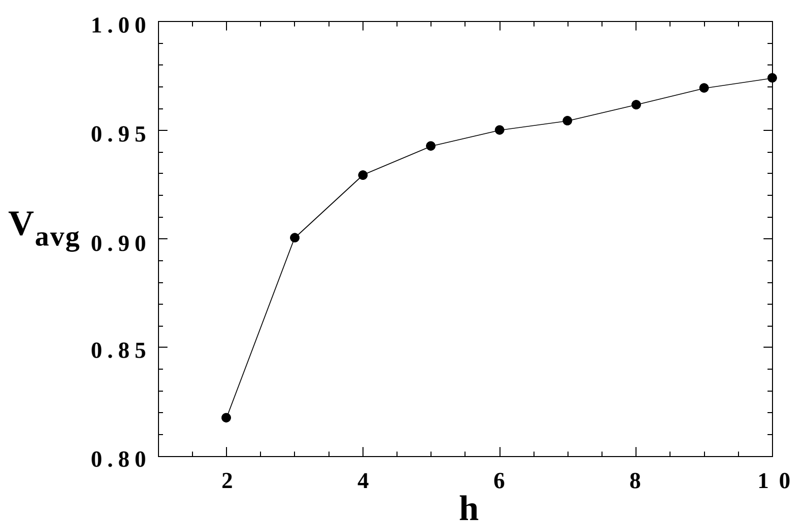

FIGURE 4. **(d)** Variation of the average voltage over the bunch versus harmonic number, at the operating point for minimum rms energy spread.

While quite small, the rms energy spread of 1.6×10^{-4} is still an order of magnitude higher than we require. This motivates us to inspect carefully the results of the optimization algorithm. The unloaded and loaded waveforms are shown in Fig. 5. Inspection shows that much of the energy spread in the beam occurs at the beam head. More precisely, if one excludes the first 5% of the beam, the rms energy spread drops to 3.9×10^{-5}. Excluding 10% of the beam, the result is 1.4×10^{-5}, close to the required 10 pppm. While close, this suggest that an additional knob is required, an additional harmonic waveform.

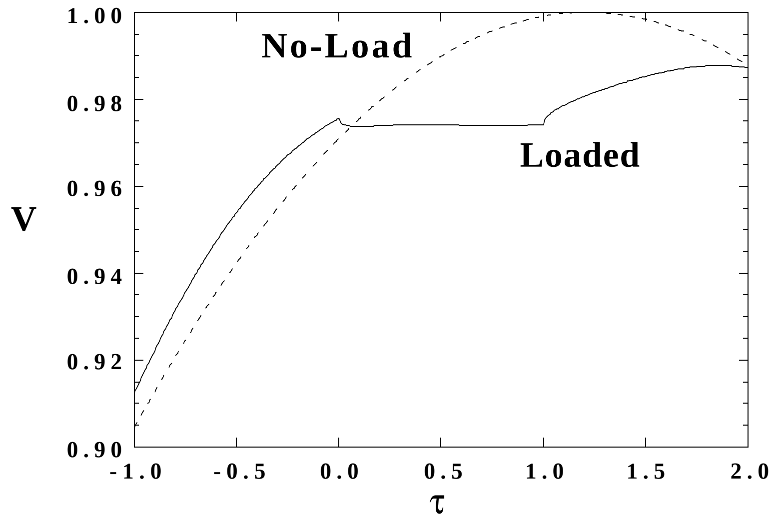

FIGURE 5. (a) No-load and loaded waveforms in a two-frequency linac, and **(b)** detail of the voltage deviation across the bunch.

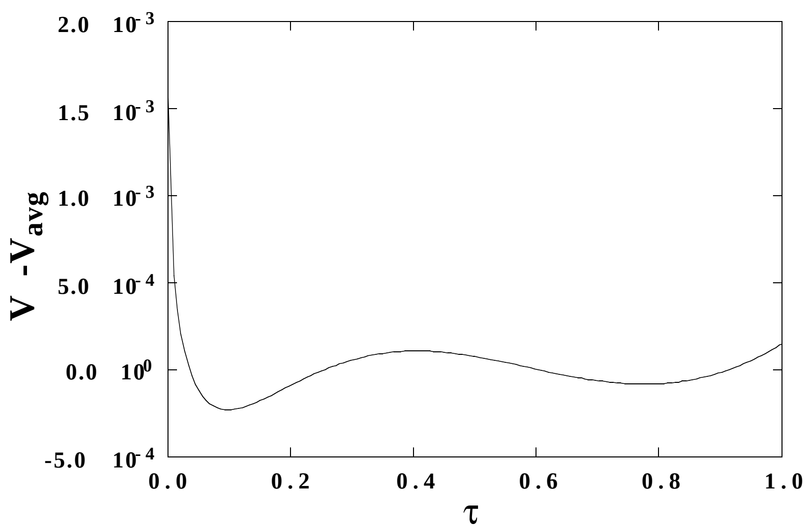

FIGURE 5. (b) Detail of the voltage deviation across the bunch for the results depicted in Fig. 5 (a).

Two Harmonics

We consider a two-harmonic scheme, with flat-top current profile, and the model wakefield as before. The total voltage waveform takes the form.

$$\hat{V} = \cos(\Omega_1\tau + \phi_1) + \hat{V}_h \cos(h\Omega_1\tau + \phi_h) + \hat{V}_{h'} \cos(h'\Omega_1\tau + \phi_{h'}) - \hat{Q}\tau^{\mu}.$$

For purposes of this note, we will optimize energy spread as follows. We accept the two-frequency optimized parameters, and then optimize the 2nd frequency amplitude and phase by means of the least-squares algorithm described previously, treating the two-frequency waveform as given. This is surely not optimal, but for present purposes it is adequate, as one might suspect viewing the residual depicted in Fig. 5 (b), with its sine-like appearance, after the initial transient. The shape of the residual suggests using h'=60 or so.

We find that the transient at the beam head inhibits good fitting of the residual with the second harmonic and thus consider excluding a fraction near the beam head. Excluding the front 10% of the beam, we arrive in this way at an energy spread of 1.1×10^{-5}, and excluding the front 20%, we obtain 2.4×10^{-6}, well below the stated requirement. The voltage waveform is depicted in Fig. 6, for the case of 1 frequency, 2 frequencies and 3 frequencies --- to summarize our work thus far.

This initial experience with two-harmonics emphasizes that the detailed form of the beam profile and the wake are critical inputs to the design of the second harmonic section (choice of frequency even) --- and suggests a new importance for the detailed understanding of the wakefields typical of the structure to be employed.

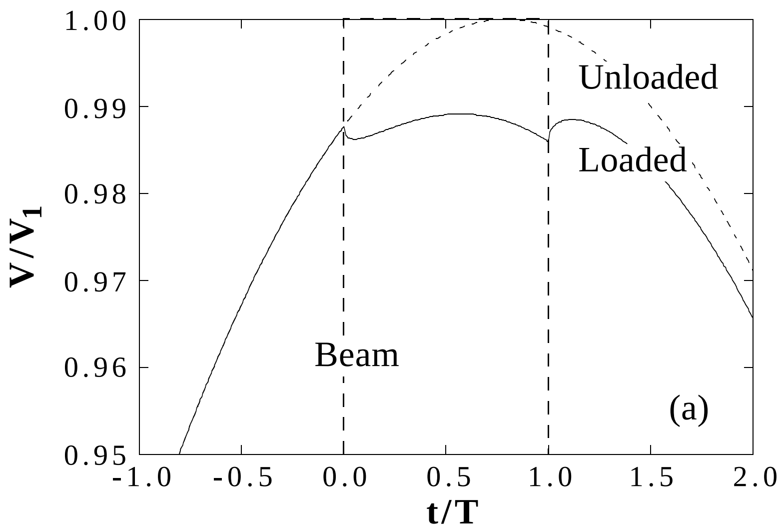

FIGURE 6. (a) The one-frequency voltage waveform.

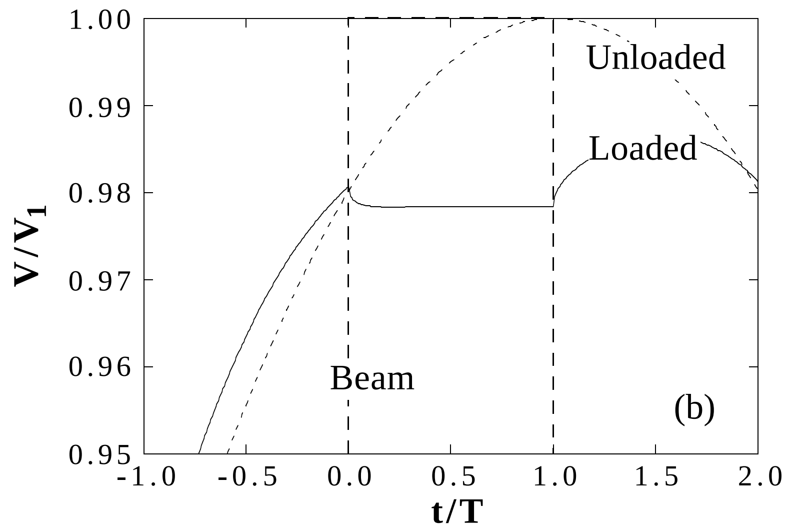

FIGURE 6. (b) The two-frequency voltage waveform.

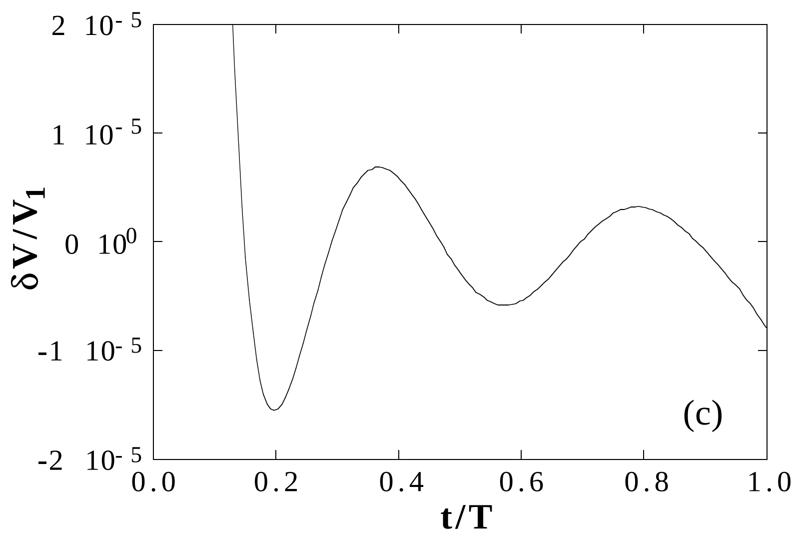

FIGURE 6. (c) Microscopic view of the three-frequency voltage waveform.

In a later section, below, we will consider multiple harmonics in more detail. In the meantime, let us consider an important practical matter for harmonic acceleration.

Energy Budget

A number of important issues must be resolved in considering the implementation of a harmonic acceleration scheme: wakefields, energy budget, rf sources, etc. Let us consider here only the energetics of the one-harmonic problem. Let us suppose that harmonic sections are run at a gradient G_h, and the fundamental mode sections are run at gradient G_1,

$$G_h = \eta G_1.$$

We need a net harmonic voltage

$$V_h = \frac{1}{h^2} V_1,$$

so that the length of harmonic sections then is

$$L_h = \frac{1}{h^2 \eta} L_1.$$

Let us denote the length ratio

$$f_h = \frac{L_h}{L_1} = \frac{1}{h^2 \eta}.$$

Clearly we would prefer this ratio to be small, to avoid a significant reduction in average gradient in the linac (and absent a workable scheme to combine both frequencies in one structure).

Meanwhile, the energy per unit length (time-integrated rf input power per unit length for one machine pulse) is scaling as $u \propto G^2 / f^2$, so we may express

$$u_h = \frac{1}{\varepsilon} \frac{G_h^2}{h^2} \frac{1}{G_1^2} u_1,$$

The factor ε allows for a reduction in the [*R/Q*] per cell in the harmonic sections (see below). the total rf input energy per pulse for all harmonic sections together is

$$U_h = L_h u_h = \frac{1}{\varepsilon} \frac{1}{h^6 f_h} U_1.$$

To these considerations we must add one more. Loss factor for the harmonic section scales as loss factor per cell $\left(\propto h\omega_1\right)$, multiplied by the number of cells $\left(\propto L_1 h\omega_1\right)$. This implies that the total loss factor is scaling as

$$k_l \approx k_{lF}\left(1 + \varepsilon \frac{L_h}{L_1} h^2\right) \approx k_{lF}\left(1 + \frac{\varepsilon}{\eta}\right),$$

with k_{lF} the loss factor from the main linac, and the factor ε is the factor by which [*R/Q*] per cell in the harmonic section is reduced below that for the main linac. Evidently to avoid having the harmonic linac contribute significantly to loss factor, we must require $\varepsilon << \eta$. (This is not completely necessary, but it seems prudent). If we fix

$$\frac{\varepsilon}{\eta} \approx 0.1,$$

our energy scaling becomes

$$\frac{U_h}{U_1} \approx \frac{10}{h^4}.$$

If we accept a 5% reduction in average gradient due to the additional length required for harmonic acceleration, then $f_h = 0.05$, and we obtain

$$\eta = \frac{20}{h^2}.$$

For *h*=10, this implies harmonic gradients 20% of the fundamental, or 200 MeV/m for a 1 GeV/m linac. In this case, $U_h / U_1 \approx 10^{-3}$.

Multiple Harmonics

In this section we set down a simple algorithm for the case of multiple harmonics. For convenience we will amend our notation a bit. Let us denote by

$$\vec{\delta} \equiv \hat{V}_0 + \hat{V}_b,$$

the *N*-tuple with *j*-th element just $\hat{V}_0 + \hat{V}_b$ evaluated at τ_j, chosen from a representative sampling of points $\{\tau_j\}$. Here, "representative" will mean distributed uniformly by charge. In this case dot-products represent a current-weighted average, so, for example, $\sqrt{\vec{\delta} \bullet \vec{\delta}}$ is the rms energy spread. Our problem is to find harmonic amplitudes $\hat{V}_h$ and phases ϕ_h such that

$$\vec{V} = \vec{\delta} + \sum_h \hat{V}_h \vec{C}_h$$

has minimum length. Here $\left(\vec{C}_h\right)_j \equiv \cos\left(h\omega_1\tau_j + \phi_h\right)$, and we will also be making use of $\left(\vec{S}_h\right)_j \equiv \sin\left(h\omega_1\tau_j + \phi_h\right)$. Evidently

$$\frac{\partial\vec{V}}{\partial\hat{V}_h} = \vec{C}_h, \qquad \frac{\partial\vec{V}}{\partial\phi_h} = -\hat{V}_h\vec{S}_h,$$

so that the minimization conditions may be expressed as

$$0 = \frac{\partial\vec{V}^2}{\partial\hat{V}_h} = 2\vec{V} \bullet \vec{C}_h, \qquad 0 = \frac{\partial\vec{V}^2}{\partial\phi_h} = -2\hat{V}_h\vec{V} \bullet \vec{S}_h.$$

In short,

$$\vec{V} \bullet \vec{E}_h = 0,$$

where $\vec{E}_h = \vec{C}_h + i\vec{S}_h$. In expanded form,

$$0 = \vec{\delta} \bullet \vec{C}_{h'} + \sum_h \hat{V}_h\left(\vec{C}_h \bullet \vec{C}_{h'}\right), \qquad 0 = \vec{\delta} \bullet \vec{S}_{h'} + \sum_h \hat{V}_h\left(\vec{C}_h \bullet \vec{S}_{h'}\right).$$

For M harmonics, we have M amplitude variables $\hat{V}_h$, and M phase variables ϕ_h, and $2M$ equations to satisfy. Finally we can cast our problem in a different and simpler form by expressing

$$\vec{V} = \vec{\delta} + \sum_h a_h \vec{C}_h + b_h \vec{S}_h,$$

with $\hat{V}_h = \sqrt{a_h^2 + b_h^2}$, and $\tan\phi_h = -b_h / a_h$. In the simple case of a flat current profile, and a harmonic frequencies a multiple of the inverse bunch length, this expansion amounts to a truncated Fourier series. In general one can obtain coefficients in a straightforward manner, by singular value decomposition (SVD).[3]

To implement SVD we form vectors with zero mean,

$$\delta\vec{C}_h = \vec{C}_h - \langle \vec{C}_h \rangle, \quad \delta\vec{S}_h = \vec{S}_h - \langle \vec{S}_h \rangle, \quad \vec{\Delta} = \vec{\delta} - \langle \vec{\delta} \rangle,$$

and unit length

$$\delta\hat{C}_h = \frac{\delta\vec{C}_h}{\sqrt{\delta\vec{C}_h^2}}, \quad \delta\hat{S}_h = \frac{\delta\hat{S}_h}{\sqrt{\delta\hat{S}_h^2}}, \quad \hat{\Delta} = \frac{\vec{\Delta}}{\sqrt{\vec{\Delta}^2}}.$$

Next we abbreviate

$$\Psi_l . \ni . \Psi_1 = \hat{\Delta}, \Psi_2 = \delta\hat{C}_{h_1}, \Psi_3 = \delta\hat{S}_{h_1}, \ldots \Psi_{2k} = \delta\hat{C}_{h_k}, \Psi_{2k+1} = \delta\hat{S}_{h_k},$$

and form the correlation matrix

$$\mathbf{C} = \begin{pmatrix} \langle \Psi_1\Psi_1 \rangle & \langle \Psi_1\Psi_2 \rangle & \cdots & \langle \Psi_1\Psi_{2M+1} \rangle \\ \langle \Psi_2\Psi_1 \rangle & \langle \Psi_2\Psi_2 \rangle & \cdots & \langle \Psi_2\Psi_{2M+1} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \Psi_{2M+1}\Psi_1 \rangle & \langle \Psi_{2M+1}\Psi_2 \rangle & \cdots & \langle \Psi_{2M+1}\Psi_{2M+1} \rangle \end{pmatrix}.$$

This we decompose as $C = \mathbf{UWU}^t$, with $\mathbf{W}$ a diagonal matrix,

$$\mathbf{C} = \mathbf{U} \begin{pmatrix} W_1 & 0 & 0 & 0 \\ 0 & W_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & W_M \end{pmatrix} \mathbf{U^t}.$$

We then identify the eigenvalue smallest in magnitude, and let us suppose it is W_1. Then first column of $\mathbf{U}$ provides the coefficients u_k, such that

$$\varepsilon = \sum_k u_k \Psi_k$$

---

[3] D. H. Whittum and F. Zimmermann, "Use Singular Value Decomposition in Ten Minutes or Your Money Back", ARDB TN.

has minimum rms with respect to variations within the range of the correlation matrix. Thus

$$\hat{\Delta} = -\tfrac{1}{u_1}\varepsilon - \tfrac{1}{u_1}\sum_{k=1}^{M}\left(u_{2k}\delta\hat{C}_{h_k} + u_{2k}\delta\hat{S}_{h_k}\right),$$

where the residual ε cannot be improved by variation of the coefficients. This result may be expressed as

$$\frac{\vec{\Delta}}{\sqrt{\vec{\Delta}^2}} = -\tfrac{1}{u_1}\varepsilon - \tfrac{1}{u_1}\sum_{k=1}^{M}\left(u_{2k}\frac{\delta\vec{C}_{h_k}}{\sqrt{\delta\vec{C}_{h_k}^2}} + u_{2k+1}\frac{\delta\hat{S}_{h_k}}{\sqrt{\delta\hat{S}_{h_k}^2}}\right),$$

or

$$\vec{\delta} - \left\langle\vec{\delta}\right\rangle = -\tfrac{\sqrt{\vec{\Delta}^2}}{u_1}\varepsilon - \tfrac{\sqrt{\vec{\Delta}^2}}{u_1}\sum_{k=1}^{M}\left(u_{2k}\frac{\vec{C}_{h_k} - \backslash\vec{C}_{h_k}\rangle}{\sqrt{\delta\vec{C}_{h_k}^2}} + u_{2k+1}\frac{\vec{S}_{h_k} - \left\langle\vec{S}_{h_k}\right\rangle}{\sqrt{\delta\hat{S}_{h_k}^2}}\right).$$

Thus we may identify coefficients in our harmonic compensation algorithm, based on the SVD decomposition.

$$\vec{V} = \vec{\delta} + \sum_h a_h\vec{C}_h + b_h\vec{S}_h \equiv \vec{\delta} + \sum_{k=1}^{M}\left(\frac{\sqrt{\vec{\Delta}^2}}{\sqrt{\delta\vec{C}_{h_k}^2}}\frac{u_{2k}}{u_1}\vec{C}_h + \frac{\sqrt{\vec{\Delta}^2}}{\sqrt{\delta\hat{S}_{h_k}^2}}\frac{u_{2k+1}}{u_1}\vec{S}_h\right),$$

and be assured that no variation of the coefficients will reduce the residual rms. For our work we will use a routine from Numerical Recipes.[4] In Fortran, the subroutine is called as

```
call svdcmp(c,m,m,m,m,w,u)
```

where `c`,`u` are `m`×`m` arrays, `w` is an array of dimension `m` (`c` is replaced on return).

This algorithm is implemented in the most recent version of `harm.for`, `harm5.for`. This version includes the previous algorithms as well and permits scans of various kinds over combinations of harmonics, extending to include 3 harmonics (4 frequencies in all), as well as a user-defined beam-profile. For example, with a Gaussian beam, we can attain an rms of 0.97×10^{-5}, with 97.0% of the unloaded gradient, and with the fundamental mode phased to minimum energy spread (1.2×10^{-3}) with harmonic sections turned off, making us of harmonics 7, 21, and 42.

Having reached this point let us finish this note for now.

Discussion

A number of matters weren't addressed here that should be in a future work. We haven't discussed bunch shaping and this will surely make the harmonic section easier. On the other hand, the purpose of the harmonic section should be to make the rest of the design easier, so perhaps it is just as well for now. Ultimately one wants to take the linac requirements, and analyze and optimize the harmonic section accounting not only for energy spread, but for sensitivity to beam profile variations from the (perhaps sophisticated) "design" waveform. Ease of measuring the beam profile would be a

[4] Matlab is also a popular tool.

significant practical matter.

The matter of harmonic acceleration raises a number of other practical questions. It seems that a low harmonic is helpful, if the fundamental is phased for minimum energy spread with harmonics off. Clearly the choice of harmonic is a major issue, not one that could be changed easily after implementation as it amounts to 100 m or so of linac. Thus to confirm any future harmonic plan, one would need good data on actual longitudinal wakefields in the structures to be employed, and with the bunch shape(s) to be employed. For the harmonic rf system one has questions too, concerning the source, the structure, and the beam dynamics in them. Is an active structure preferred? It isn't strictly necessary. It seems fair to say that one cannot rule out that frequencies like 270GHz, 360GHz and so on would be of interest. For the high-harmonics plasma and laser accelerators would be essential. Perhaps the inverse free-electron laser. Notice that the application of these "exotic" linacs here does not require the high source efficiency as a collider linac rf system. The matter of transverse wakefields favors a shorter length of harmonic section. In fact, unless the harmonic section is quite short, or designed with a wide beam aperture (and therefore lower [*R/Q*], higher power requirement), the loss-factor contribution from the harmonic section may tend to dominate the loss-factor calculation.

For lower frequency work, it is tantalizing to realize that perhaps the 30 GHz and other such Ku and Ka band work could have application in an upgraded X-Band NLC, in that, at the 5 TeV energy range, the usual chromatic correction techniques would require many 10's of kilometers. Research into harmonic energy spread compensation could possibly be a much less-demanding rf system problem than the challenge of a full linac rf system. Once one recognizes the extreme scaling of the chromatic-correction section of a conventional final focus, and the need for energy spread compensation, one realizes that there is some advantage to employing a lower-frequency fundamental mode.

Other issues include use of a correlated energy residual for a travelling focus, versus minimization of the rms for a static focus.

Finally, one wonders, if a linac operates at an already high frequency fundamental (as for a plasma-based collider) --- how will one obtain harmonic energy spread cancellation? For a laser-linac it would appear that harmonics at UV and X-Ray wavelengths would be important. If so, then intense coherent X-Ray sources could actually become important to the future of high-energy physics.

Appendix - Beam Spreading

We have not yet, in this series of technical notes, presented an estimate of beam-spreading due to energy-spread. While it is a small effect, and not really along the lines of this note, let us mention it here for the record.

The longitudinal motion of an electron traversing the linac is described by

$$\frac{dz}{d(ct)} \approx 1 - \frac{1}{2\gamma^2},$$

and an equation for energy variation. We would like to check here that, despite the very short bunch lengths of interest, the usual linac picture of "frozen" longitudinal position is still appropriate. We consider the separation of two electrons $\Delta = z_1 - z_2$, satisfying

$$\frac{d\Delta}{ds} \approx \frac{1}{2\gamma_2^2} - \frac{1}{2\gamma_1^2}.$$

We suppose there is a constant energy offset, and a gradient error between the two particles,

$$\gamma_k = \gamma_{ik} + g_k s,$$

k= 1,2, and obtain

$$\Delta = \frac{L}{2\gamma_{i2}(\gamma_{i2} + g_2 L)} - \frac{L}{2\gamma_{i1}(\gamma_{i1} + g_1 L)} = \frac{1}{2g_2}\left(\frac{1}{\gamma_{i2}} - \frac{1}{\gamma_{f2}}\right) - \frac{1}{2g_1}\left(\frac{1}{\gamma_{i1}} - \frac{1}{\gamma_{f1}}\right),$$

with

$$\gamma_{fk} = \gamma_{ik} + g_k L,$$

and L the linac length. In the limit $\gamma_f >> \gamma_i$, one can show that

$$\left.\frac{\partial\Delta}{\partial g_2}\right|_{g_2=g,\gamma_{i2}=\gamma_i} \approx \frac{1}{2\gamma_i g^2}, \qquad \left.\frac{\partial\Delta}{\partial\gamma_{i2}}\right|_{g_2=g,\gamma_{i2}=\gamma_i} \approx -\frac{1}{2\gamma_i^2 g}.$$

Thus for small deviations, such that

$$\gamma_{i2} = \gamma_i + \tfrac{1}{2}\delta\gamma_i, \quad \gamma_{i1} = \gamma_i - \tfrac{1}{2}\delta\gamma_i, \quad g_2 = g + \tfrac{1}{2}\delta g, \quad g_1 = g - \tfrac{1}{2}\delta g,$$

we find

$$\Delta \approx \frac{1}{2g\gamma_i}\left(\frac{\delta g}{g} - \frac{\delta\gamma_i}{\gamma_i}\right).$$

As discussed in previous notes our working parameters correspond to 1 GeV/m and injection at 10 GeV, so that $g \approx 2.0 \times 10^3\ \mathrm{m}^{-1}$ and $\gamma_i \approx 2.0 \times 10^4$. Thus

$$\Delta \approx 1.3 \times 10^{-8}\,\mathrm{m}\left(\frac{\delta g}{g} - \frac{\delta\gamma_i}{\gamma_i}\right).$$

Measured in units of our nominal rms bunch length, $\sigma_z \approx 30\,\mu\mathrm{m}$,

$$\frac{\Delta}{\sigma_z} \approx 4 \times 10^{-4}\left(\frac{\delta g}{g} - \frac{\delta\gamma_i}{\gamma_i}\right).$$

Thus longitudinal slippage due to injection energy spread, or gradient variation is completely negligible, and we are justified in employing the picture of electrons as "frozen" in relative longitudinal position within the bunch.

In passing let us note the result applicable to the SLC, with injection at 1.2 GeV, and an average gradient of 20 MeV/m or $\gamma_i \approx 2.3 \times 10^3$ and $g \approx 39\,\mathrm{m}^{-1}$. We have

$$\Delta \approx 5.5 \times 10^{-6}\,\mathrm{m}\left(\frac{\delta g}{g} - \frac{\delta\gamma_i}{\gamma_i}\right),$$

and with $\sigma_z \approx 1.1\,\mathrm{mm}$, we find

$$\frac{\Delta}{\sigma_z} \approx 5 \times 10^{-3}\left(\frac{\delta g}{g} - \frac{\delta\gamma_i}{\gamma_i}\right).$$

Thus, as far as the slippage effect is concerned, the W-Band parameters we have in mind involve less slippage than in the SLC. Had we taken injection energies to be the same, they would be comparable.