Stability Criterion for All-Beam Collisions in a Plasma Channel

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Here we formulate the condition on plasma density, for stability in the crab-crossing W-Band final focus.

Introduction

The interest in multiple beam collisions is described in other technical notes. Here we consider the problem of disruption as it appears for a single e- bunch traversing a gauntlet of 50 other oncoming e- bunches. In vacuum the system is unstable, due to disruption, and for this reason we consider interposing a background of plasma to provide focusing between collisions. The concept is illustrated in Fig. 1, depicting a series of collisions interspersed with plasma focusing.

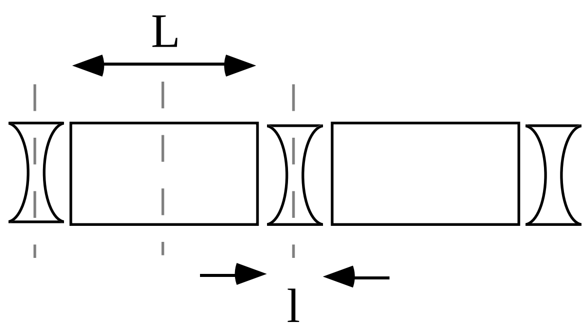

FIGURE 1. We consider optics in a periodic sequence of lens, with one period consisting of a short, strong defocusing lens, followed by a long, weaker focusing lens.

Linear Beam-Beam Approximation

The effective length of each oncoming e- beam is, at zeroth order, that of a defocusing lens, of effective length $l = \sqrt{2\pi}\,\sigma_z$, and effective gradient $K_- = -\kappa_-^2 = -k_b^2/2\gamma$, with $k_b^2 = 4\pi n_b r_e$. The classical electron radius is $r_e = e^2/mc^2$ and n_b is the maximum beam density,

$$n_b = \frac{N}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \approx 1.25\times10^{23}\,\mathrm{cm}^{-3}.$$

We have in mind $N \approx 3.75\times10^8$ particles per bunch (60 pC), with $\sigma_x \approx \sigma_y \approx 1.7\,\mathrm{nm}$, and $\sigma_z \approx 33\,\mu\mathrm{m}$.

The character of the focusing provided by the plasma depends on the regime of propagation. In a previous note we have shown that the ion-focused regime (IFR) is the most appropriate.[1] In this case focusing is uniform (neglecting the small transient region near the beam head and tail), and characterized by focusing strength $K_+ = +\kappa_+^2 = +k_p^2/2\gamma$.

We write out the map for one-half of one plasma section, of length $L/2$,

[1] "Recombining Final Focus Based on All-Beam Collisions", ARDB Tech. Note.

$$R_{1/2F} = \begin{bmatrix} \cos\theta_+ & \dfrac{\sin\theta_+}{\kappa_+} \\ -\kappa_+ \sin\theta_+ & \cos\theta_+ \end{bmatrix},$$

with $\kappa_+ = \sqrt{K} \equiv "k_\beta"$, and $\theta_+ = \kappa_+ L/2$. For the defocusing collision, the map for one-half of the "lens" is

$$R_{1/2D} = \begin{bmatrix} \cosh\theta_- & \dfrac{\sinh\theta_-}{\kappa_-} \\ \kappa_- \sinh\theta_- & \cosh\theta_- \end{bmatrix},$$

with $\theta_- = \kappa_- l/2$. We employ the thick-lens formula here since we will be interested in the limit of disruption of order unity or larger, with $D \approx 2.3$ for nominal parameters.

Transport from lens center to lens center (1/2 period) is then governed by $R_{D-F} = R_{1/2F} R_{1/2D}$, or

$$R_{D-F} = \begin{bmatrix} \cos\theta_+ & \dfrac{\sin\theta_+}{\kappa_+} \\ -\kappa_+ \sin\theta_+ & \cos\theta_+ \end{bmatrix} \begin{bmatrix} \cosh\theta_- & \dfrac{\sinh\theta_-}{\kappa_-} \\ \kappa_- \sinh\theta_- & \cosh\theta_- \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta_+ \cosh\theta_- + \dfrac{\kappa_-}{\kappa_+} \sin\theta_+ \sinh\theta_- & \dfrac{1}{\kappa_-} \cos\theta_+ \sinh\theta_- + \dfrac{1}{\kappa_+} \sin\theta_+ \cosh\theta_- \\ \kappa_- \cos\theta_+ \sinh\theta_- - \kappa_+ \sin\theta_+ \cosh\theta_- & \cos\theta_+ \cosh\theta_- - \dfrac{\kappa_+}{\kappa_-} \sin\theta_+ \sinh\theta_- \end{bmatrix}.$$

The beta functions may be obtained by computing this product of matrices and comparing to

$$R_{D-F} = \begin{bmatrix} \left(\frac{\beta_+}{\beta_-}\right)^{1/2} \cos\left(\frac{\phi}{2}\right) & \left(\beta_+ \beta_-\right)^{1/2} \sin\left(\frac{\phi}{2}\right) \\ -\frac{1}{\left(\beta_+ \beta_-\right)^{1/2}} \sin\left(\frac{\phi}{2}\right) & \left(\frac{\beta_-}{\beta_+}\right)^{1/2} \cos\left(\frac{\phi}{2}\right) \end{bmatrix}.$$

Stability corresponds to $0 < \cos^2(\phi/2) < 1$, with $\cos^2(\phi/2) = R_{11} R_{12}$. The beta functions may be computed from $\beta_+ / \beta_- = R_{11} / R_{22}$ and $\beta_+ \beta_- = -R_{12} / R_{21}$.

One can check that the map through the remaining 1/2 period, $R_{F-D} = R_{1/2D} R_{1/2F}$, is given by

$$R_{F-D} = \begin{bmatrix} \cosh\theta_- & \dfrac{\sinh\theta_-}{\kappa_-} \\ \kappa_- \sinh\theta_- & \cosh\theta_- \end{bmatrix} \begin{bmatrix} \cos\theta_+ & \dfrac{\sin\theta_+}{\kappa_+} \\ -\kappa_+ \sin\theta_+ & \cos\theta_+ \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta_+ \cosh\theta_- - \dfrac{\kappa_+}{\kappa_-} \sin\theta_+ \sinh\theta_- & \dfrac{1}{\kappa_-} \cos\theta_+ \sinh\theta_- + \dfrac{1}{\kappa_+} \sin\theta_+ \cosh\theta_- \\ \kappa_- \cos\theta_+ \sinh\theta_- - \kappa_+ \sin\theta_+ \cosh\theta_- & \cos\theta_+ \cosh\theta_- + \dfrac{\kappa_-}{\kappa_+} \sin\theta_+ \sinh\theta_- \end{bmatrix},$$

and that this corresponds to

$$R_{F-D} = \begin{bmatrix} \left(\frac{\beta_-}{\beta_+}\right)^{1/2} \cos\left(\frac{\phi}{2}\right) & (\beta_+\beta_-)^{1/2} \sin\left(\frac{\phi}{2}\right) \\ -\frac{1}{(\beta_+\beta_-)^{1/2}} \sin\left(\frac{\phi}{2}\right) & \left(\frac{\beta_+}{\beta_-}\right)^{1/2} \cos\left(\frac{\phi}{2}\right) \end{bmatrix}.$$

Numerical values of interest are $\gamma \approx 4.9 \times 10^6$ (2.5 TeV), and

$$l \approx \sqrt{2\pi}\ 33\,\mu\text{m} \approx 8.3 \times 10^{-3}\,\text{cm}.$$

Other parameters are

$$k_b = \sqrt{4\pi \times 1.25 \times 10^{23}\,\text{cm}^{-3} \times 2.82 \times 10^{-13}\,\text{cm}} \approx 6.66 \times 10^5\,\text{cm}^{-1},$$

so that $\kappa_- = k_b / \sqrt{2\gamma} \approx 2.1 \times 10^2\,\text{cm}^{-1}$, and

$$\theta_- \approx 2.1 \times 10^2\,\text{cm}^{-1} \times 8.3 \times 10^{-3}\,\text{cm} / 2 \approx 0.875 \text{ rad} \approx 50^\circ.$$

The length traversed between collisions is[2] $L \approx 2.1 - 4.2\,\text{mm}$ (for matrix secondary period in the range $\lambda/4$ - $\lambda/2$). We select $L \approx 2.1\,\text{mm}$ for illustration.

Next, we plot phase-angle ϕ, versus plasma density, with other parameters fixed, to locate the stable regions. The first such stable region is depicted in Fig. 2, corresponding to under 180° phase-advance per period. For example, for $n_p = 5.51 \times 10^{20}\,\text{cm}^{-3}$, we have $\beta_+ / \beta_- \approx 258$, and $\beta_- \approx 1.75 \times 10^{-2}\,\text{cm}$. This produces the assumed spot-size at the center of the defocusing lens,

$$\sigma = \sqrt{\frac{\beta_- \varepsilon_n}{\gamma}} \approx \sqrt{\frac{1.75 \times 10^{-4}\,\text{m} \times 5.8 \times 10^{-8}\,\text{m - rad}}{4.9 \times 10^6}} \approx 1.4\,\text{nm}.$$

Our luminosity estimate of $10^{35}\,\text{cm}^{-2}\text{s}^{-1}$ is based on an effective spot-size of $1.7\,\text{nm}$, and corresponds to a derating by a factor of 1.5.

Notice that it would be natural to match into such a final focus by placing a waist with $\beta \approx 4.5\,\text{cm}$ at the mid-point of an initial "matching" plasma cell. This β value is about 1/20-1/10 the typical value in the linac. With a demagnification of less than $\times 5$, the conventional final focus triplet would not be greatly stressed.However, it is probably not desirable to use this plasma density between each collision as the variations in β are quite large. We return to this shortly in connection with tracking.

[2] "Geometrical and RF Considerations for All Beam Collisions via Crab-Crossing", ARDB TN 151.

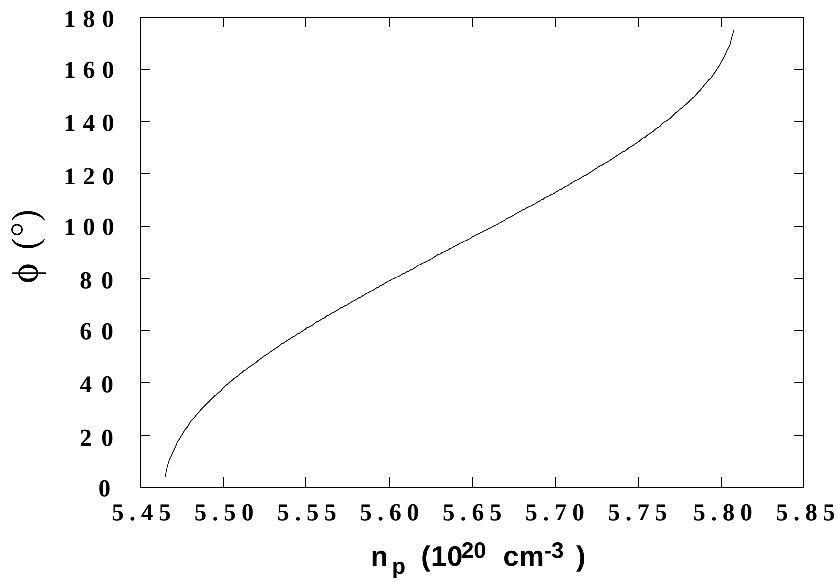

FIGURE 2.(a) Phase-advance per full period in the first stable region as a function of plasma density.

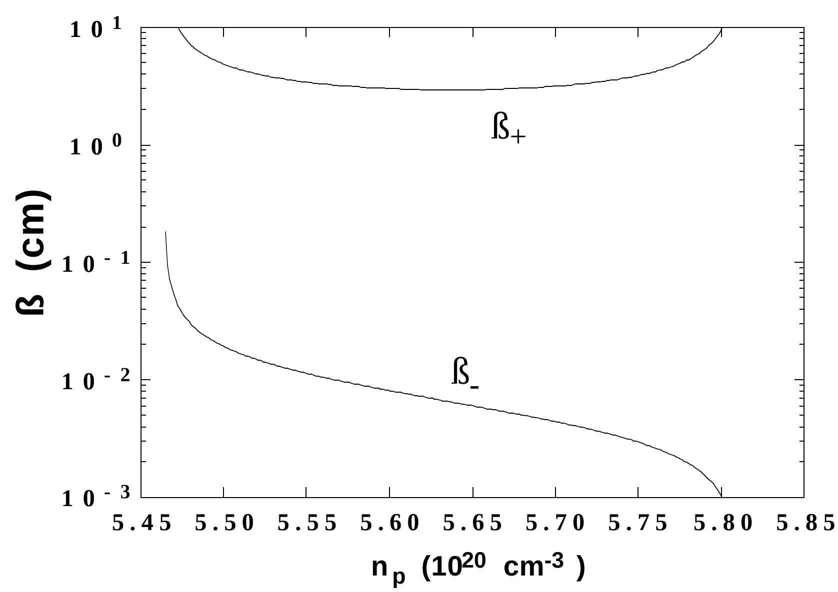

FIGURE 2.(b) beta-function values in collision (β_-) and at the center of the plasma (β_+) in the first interval of plasma density corresponding to stability.

Gaussian Beam-Beam Approximation

The foregoing considerations modelled the effect of the oncoming beam as a linear lens. Let us consider next the simplest model for a non-linear lens, treating the oncoming beam as a Gaussian. This approximation neglects distortion of the beam profile during the course of multiple collisions, still a significant approximation, given that disruption is of order unity, and numerous collisions are envisioned. We make the further approximation that the beam is a thin lens. In this case the kick to a particle may be expressed as

$$\Delta x' = -K\_l \frac{\left(1-e^{-u}\right)}{u} x, \qquad \Delta y' = -K\_l \frac{\left(1-e^{-u}\right)}{u} y,$$

with other parameters as before, and $u = r^2/2\sigma^2$, with $r^2 = x^2 + y^2$, and $\sigma^2 = \sigma_x^2 + \sigma_y^2$.

This map can be applied to a Gaussian ensemble of macroparticles, followed by the linear map through the plasma,

$$R_F = \begin{bmatrix} \cos 2\theta_+ & \dfrac{\sin 2\theta_+}{\kappa_+} \\ -\kappa_+ \sin 2\theta_+ & \cos 2\theta_+ \end{bmatrix},$$

and the procedure may be iterated 50 times to arrive at an illustration of the optics throughout the interaction region.

To simplify matters, we may start the sequence of maps in mid-collision, with the beam at a waist corresponding to the desired 2 nm spot, and the nominal $\varepsilon_n \approx 5.8 \times 10^{-8}\,\mathrm{m-rad}$ rms normalized emittance. We then apply the beam-beam kick corresponding to one-half of a collision, and commence the periodic sequence of plasma and beam-beam.

[to be continued]

Particle-in-Cell Simulation

To obtain a still more realistic picture of collisions, we employ a particle-in-cell simulation, described in a previous note.[3]

[to be continued]

[3] D.H. Whittum, "Transverse two-stream instability of a beam with a Bennett profile", Phys. Plasma **4** (1997) 1154.