Detouring and Tapering a Wave Guide - A small mathematical problem -

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Abstract

This Paper presents the mathematical background for TN 140 . It shows the way to develop a wave guide taper by considering different geometrical boundary conditions.

I. Definition of the problem

The Problem we are talking about is presented in [1]. To solve it, let us first determine the inner symmetrical line as shown in figure 1. The two outer lines are the boundaries of this wave guide, figure 1.a). So it is allowed to define such an inner symmetrical line. We assume further that this line is point symmetrical to point Ps, figure 1.b). So that one half of the defined line simplifies to a line which can be approximated by a simple circle and a straight section as shown in figure 1.c).

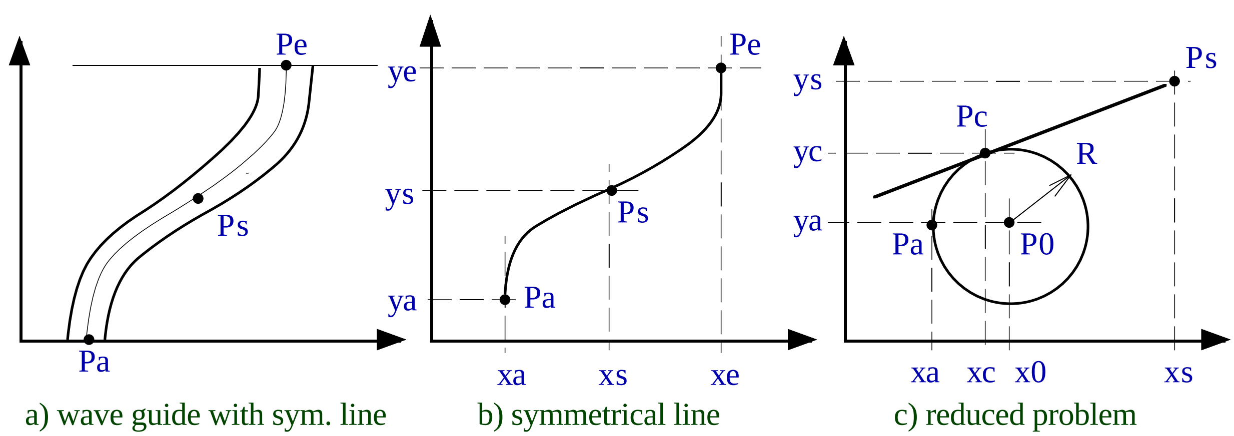

FIGURE 1. Reducing the problem.

The boundary condition of the symmetrical line are the start point Pa and the end point Pe. Known coordinates are therefore: xa, ya, xe, ye and assuming the mentioned symmetry the symmetry point Ps is further known:

$$xs = (xc+xa) / 2, \quad ys = (ye + ya) / 2.$$

That means that we need to determine the coordinates of the crossing point Pc, the coordinates of the circle origin P0 and the radian of the circle R: xc, yc, x0, y0, R. These are five unknown values and we need five independent conditions to solve the problem.
The five conditions are:

- C.I.) the function of circle should start perpendicular,
- C.II.) the circle has to pass the points Pa and Pc,
- C.III.) the straight section has to pass the points Pc and Ps,
- C.IV.) to get a smooth transition from circle to the straight section, the slope of both function has to be equal in the crossing point Pc,
- C.V.) the complete symmetrical line should have a minimized length.

Remark to C.V.):
A further condition is needed because we need a 5th one. However, which kind of condition should we take? The idea was that we want to transmit waves and actually we don‘t want a lot of losses and reflection. Both can be satisfied by minimizing the wave guide surface. Let us see if it works!
Another idea was to do a mode matching. What means using orthogonal development to match the problem for one frequency. This would lead to no reflection for this matched frequency. The question is if we really get this aimed frequency after fabrication with unavoidable tolerances. So the decision was to realize the first one which is for a broadband design.

II. Determine the symmetrical line

To determine the symmetrical line let us fulfill the conditions C.I. to C.V. A circle is described by equation 1, P0(x0,y0) the origin of the circle:

$$(y - y0)^2 + (x - x0)^2 = R^2. \tag{1}$$

The circle in point Pa should be perpendicular to the x-axis. This is fulfilled if the first derivative of (1) becomes infinity. Solve for y:

$$y = \sqrt{R^2 - (x - x0)^2} + y0. \tag{2}$$

First derivative in point Pa:

$$\frac{dy}{dx}_{\text{in Pa}} = \frac{x0 - xa}{\sqrt{R^2 - (x0 - xa)^2}}. \tag{3}$$

Equation (3) will be infinity for:

$$R = x0 - xa. \tag{4}$$

We also can take the inverse function x = f(y) and demand that the first derivative in point Pa gets equal zero. Inverse function:

$$x = \sqrt{R^2 - (y - y0)^2} + x0 . \tag{5}$$

First derivative in point Pa:

$$\left.\frac{dx}{dy}\right|_{\text{in Pa}} = \frac{y0 - ya}{\sqrt{R^2 - (y0 - ya)^2}} . \tag{6}$$

Equation (6) will be zero for:

$$y0 = ya. \tag{7}$$

Equation (4) and (7) in (1), noted for y:

$$f_{circle}(x) = y = \sqrt{(x0 - xa)^2 - (x - x0)^2} + ya . \tag{8}$$

This circle has to cross the point Pa(xa,ya), which is satisfied through equation (4) and (7), and to cross the point Pc(xc,yc). This follows:

$$yc = \sqrt{(x0 - xa)^2 - (xc - x0)^2} + ya . \tag{9}$$

The function of the simple straight section will be described by equation (10):

$$y = mx + b \tag{10}$$

This function has to cross the points Pc(xc,yc) and Ps(xs,ys). This follows:

$$f_{straight}(x) = y = \frac{ys - yc}{xs - xc}(x - xs) + ys \tag{11}$$

The next condition is to get the same slope in the crossing point. That means to get both first derivatives equal. This follows:

$$\frac{ys - yc}{xs - xc} = \frac{x0 - xc}{\sqrt{(x0 - xa)^2 - (x0 - xc)^2}} \tag{12}$$

After fulfilling the first four conditions we get this two independent equations:

$$yc = \sqrt{(x0 - xa)^2 - (xc - x0)^2} + ya \tag{I}$$

$$\frac{ys - yc}{xs - xc} = \frac{x0 - xc}{\sqrt{(x0 - xa)^2 - (x0 - xc)^2}} \tag{II}$$

Known are xa, ya, xs, ys and unknown are xc, yc, x0. We need the 5th condition to solve it. So let us try to minimize the length of the symmetrical line. To write this condition mathematically, we get a variational problem.

The length of one half of the symmetrical line is given by:

$$L = \int_x f(x)\,dx \rightarrow \min. \tag{13}$$

$$L = \int_{xa}^{xc} f_{circle}(x)\,dx + \int_{xc}^{xs} f_{straight}(x)\,dx \rightarrow \min \tag{14}$$

To solve this functional we need to apply some variational principles. The problem here is that the integral border xc is a unknown value. However, the solution of this problem is the standard Lagrange problem which leads to the result, that the shortest connection between to points is a straight section (2D space)!! This means xc becomes xa and the first integral avoids. What a pity, so let us forget this minimizing idea!

WE NEED A FURTHER CONDITION, WHAT CAN WE DO ????

As said in the title, it is a „small“ mathematical problem. Let us find other ways. Time is short. A quick and dirty solution is to solve it numerically. We could predict some values and interpret the result and do as much attempts as necessary, but we don‘t want to do that. An alternative way is a semi analytical way. It is shown in figure 2. We simply do a substitution and express xc and yc through an unknown angle p and reduce the number of unknown values.

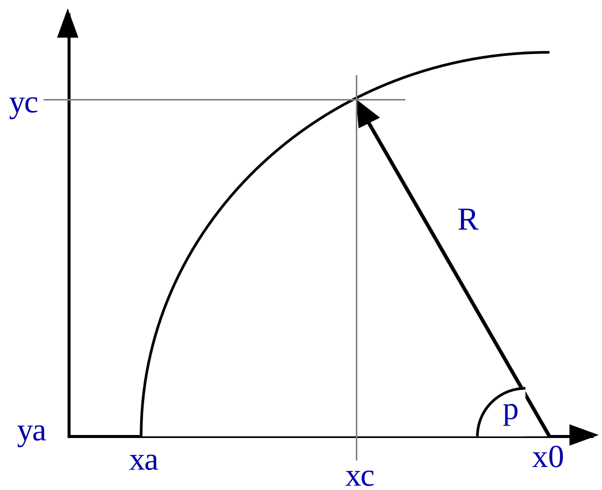

FIGURE 2. Xc and yc substituted through an angle p.

$$yc = (\ x0 - xa\ )\ \sin(p) + ya. \tag{15}$$

$$xc = x0\ (\ 1 - \cos(p)\ ) + xa\ \cos(p) \tag{16}$$

If we substitute (15) and (16) in (I) and (II) and add this equations, we can find a function like:

$$\mathbf{x0 = f(p).} \tag{17}$$

A small program was written to find a good one of infinity angles p. One of the best results which fulfill all the mathematical conditions (accept # C.V.) and technological limitations, which means a realizable one, is an angle of 30 degree. Yes, simply 30 degree! After x0 is determined, it is simple to find xc and yc with equation (15) and (16).

Now, one half of the symmetrical line is determined and we can calculate the whole by using geometrical relations as shown in figure 3.

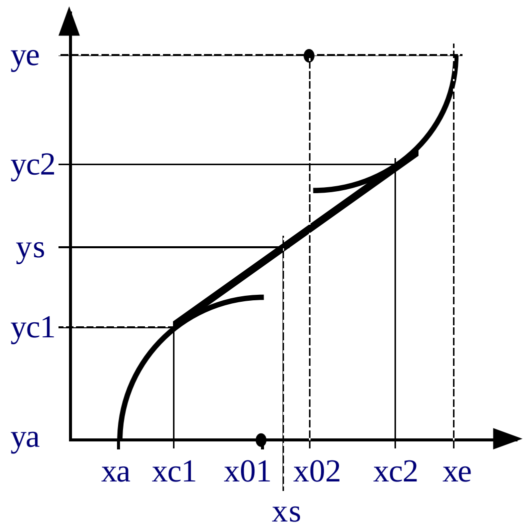

FIGURE 3. Definition of the symmetrical line.

$$x01 = x0,\ \ y01 = y0,\ \ xc1 = xc,\ \ yc1 = yc \quad (18)$$

$$x02 = xe - x01 + xa \quad (19)$$

$$y02 = ye \quad (20)$$

$$xc2 = xe + xa - xc1 \quad (21)$$

$$yc2 = ye + ya - yc1 \quad (22)$$

$$f_{circle\ 2}(x') = y - f_{circle\ 1}(x''),\ x',\ x'' \text{ are relative coordinates} \quad (23)$$

III. Wave guide shape

After the symmetrical is determined, we can start to find the wave guide shape. Further we need to consider a taper in the middle section. The width at the bottom is w1 = 0.8634 mm and the width at the top is a standard WR 10 size w2 = 1.27 mm, see figure 4.

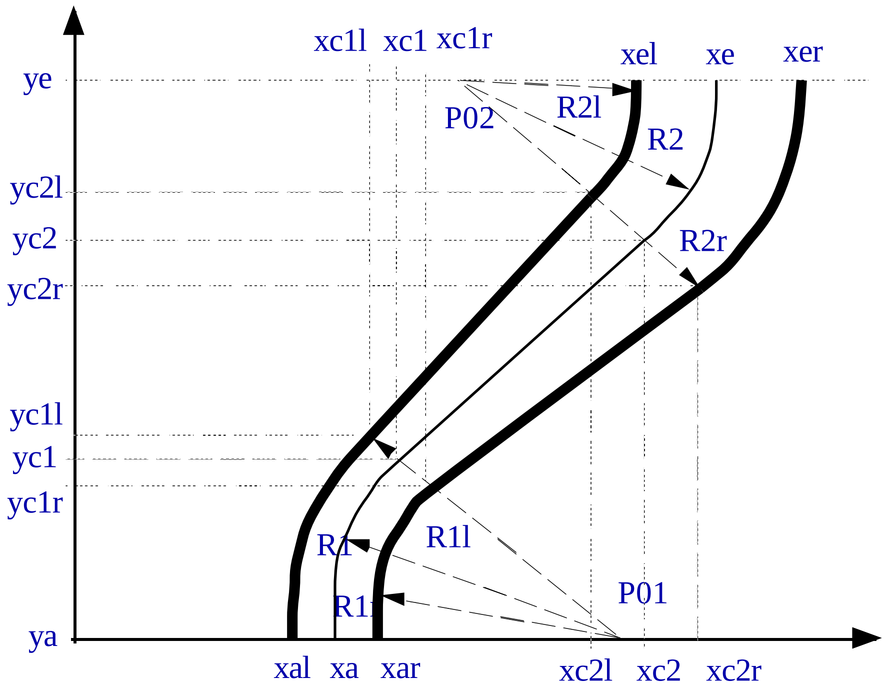

FIGURE 4. Definition of wave guide shape with taper.

All the coordinates may be found with simple geometrical relationships:

$$xal = xa - w1 / 2 \qquad (24)$$

$$xar = xa + w1 / 2 \qquad (25)$$

$$R1 = (x01 - xa) \qquad (26)$$

$$R1r = R1 - w1 / 2 \qquad (27)$$

$$R1l = R1 + w1 / 2 \qquad (28)$$

$$xc1l = x01 - R1l \cos(p) \qquad (29)$$

$$xc1r = x01 - R1r \cos(p) \qquad (30)$$

$$yc1l = R1l \sin(p) + ya \qquad (31)$$

$$yc1r = R1r \sin(p) + ya \qquad (32)$$

$$xel = xe - w2 / 2 \qquad (33)$$

$$xer = xe + w2 / 2 \tag{34}$$

$$R2 = (xe - x02) \tag{35}$$

$$R2r = R2 + w2 / 2 \tag{36}$$

$$R2l = R2 - w2 / 2 \tag{37}$$

$$xc2l = R2l \cos(p) + x02 \tag{38}$$

$$xc2r = R2r \cos(p) + x02 \tag{39}$$

$$yc2l = ye - R2l \sin(p) \tag{40}$$

$$yc2r = ye - R2r \sin(p) \tag{41}$$

<u>Some remarks to the taper shape:</u>

Figure 5 shows the taper section and the definition of some new angles. It is true that the method we used to find the shape is not a way to find an exact smooth transition from circle to taper and taper to circle. The symmetrical line is matched and has a smooth transition, but not the outer wave guide shape. This doesn‘t matter, what means that because of the real length of the tapering section and the small distance which is to enlarge, the tapering angle b‘ is so small, that the found solution is good enough.

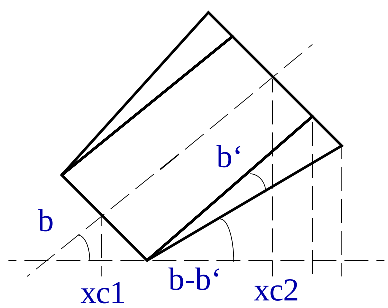

FIGURE 5. Definition of the taper.

The length of the tapering section is given by:

$$L_t = \sqrt{(xc2 - xc1)^2 + (yc2 - yc1)^2}\,, \tag{41}$$

and the tapering angle by:

$$b' = \arctan\left(\frac{w2 - w1}{2\sqrt{(xc2 - xc1)^2 + (yc2 - yc1)^2}}\right). \tag{42}$$

The other angles can be found simply by:

$$b = \arctan\left(\frac{yc2 - yc1}{xc2 - xc1}\right). \tag{43}$$

Now we can determine the functions of the outer wave guide shape:

$$f_{tap,r}(x) = \tan(b - b') (x - xc2r) + yc2r \quad (44)$$

$$f_{tap,l}(x) = \tan(b + b') (x - xc2l) + yc2l \quad (45)$$

IV. Results

All the formulas above are integrated in the mentioned written program. The input data are the relative coordinates of the start and end point of the outer wave guide shape (this is what we have exactly):

$$ya, ye, xar, xal, xer, xel. \quad (46)$$

The output data is a plot of the optimized geometry, shown in figure 6 and the absolute values of all angles and crossing point coordinates, shown in table 1.

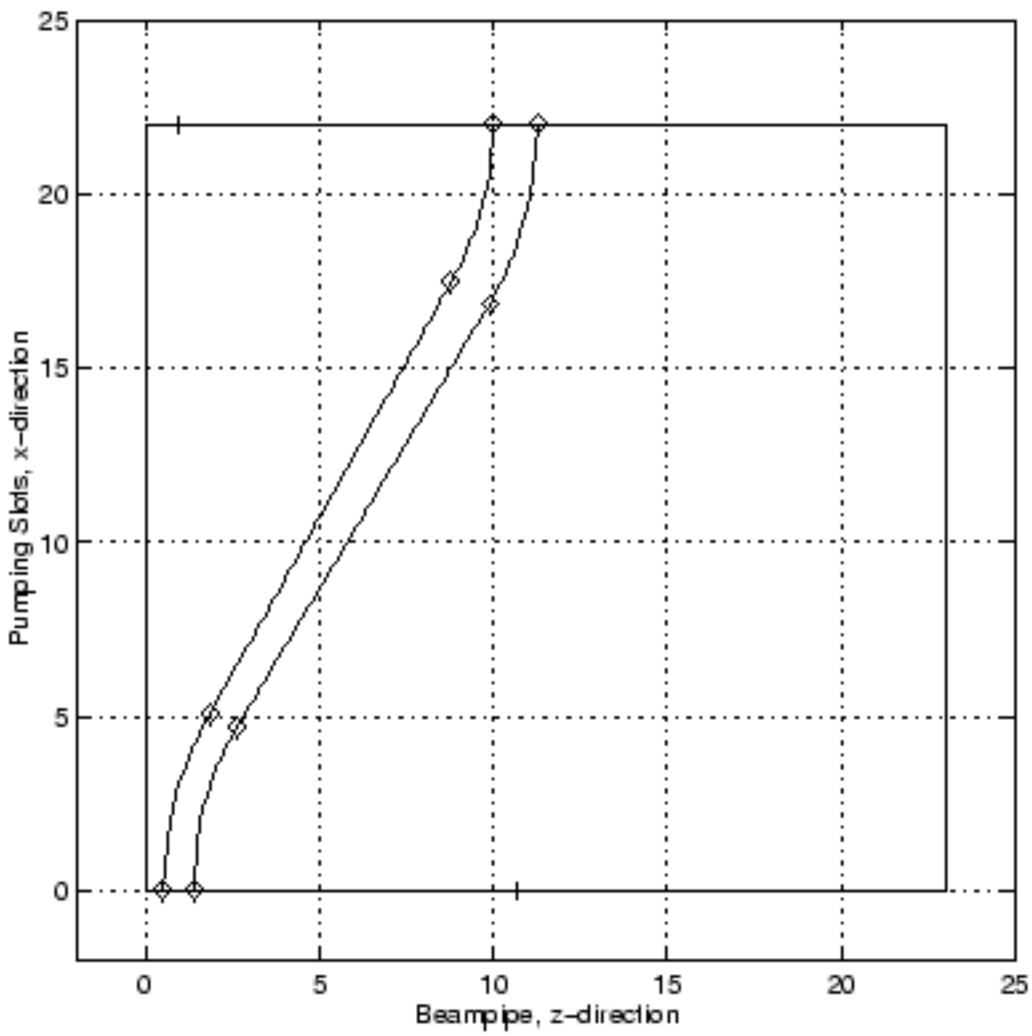

FIGURE 6. Geometry output of optimizing program.

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----------------------------------------------------
RESULTS   ( all in mm )
----------------------------------------------------
Point Pa,  x/y :  ( 0.9317150116 /  0.0000000000 )
Point Pe,  x/y :  ( 10.63500023  /  22.00000000  )
Point Ps,  x/y :  ( 5.783357620  /  11.00000000  )
Point Pc1, x/y :  ( 2.200000048  /  4.793442249  )
Point Pc2, x/y :  ( 9.366715431  /  17.20655823  )
-----------------------------------------------------
Angle from origin to crossing point: p = 30.00000000
Angle of straight section         :  b = 60.00000000
Angle of taper                    :  b‘= 0.812547266
Angle of right straight section   :  59.18745422
Angle of left  straight section   :  60.81254578
-----------------------------------------------------
Origin of bottom circle x01/y01 : ( 10.62418 / 0.00 )
Radian of interior bottom circle,  R1 =  9.692472458
Radian of right bottom circle,     R1r =  9.260757446
Radian of left  bottom circle,     R1l =  10.12418747
-----------------------------------------------------
Point Pc1r,  x/y :  ( 2.604136229  /  4.630378723  )
Point Pc1l,  x/y :  ( 1.856384039  /  5.062093735  )
-----------------------------------------------------
Origin of top circle x02/y02 : ( 0.9425277 / 22.000 )
Radian of interior top circle,    R2 =  9.692472458
Radian of right top circle,       R2r =  10.32747269
Radian of left  top circle,       R2l =  9.057472229
-----------------------------------------------------
Point Pc2r,  x/y :  ( 9.886381149  /  16.83626366  )
Point Pc2l,  x/y :  ( 8.786528587  /  17.47126389  )

-----------------------------------------------------
```

TABLE 1. Value output for all variables of optimizing program.

V. References

[1] R. Merte, "Improved Shape of an Input/Output Wave Guide with Integrated Taper for W-Band Muffin Tin WBAND-003", Tech. Note 140, ARDB, SLAC, Stanford.