10

Resonances

In this chapter, we use a simple model of perturbations to examine the conditions for resonant oscillations of betatron motion which are driven by various multipole errors of the magnetic guide field.

10.1 Integer Resonance

Magnetic fields in the elements constituting any lattice cannot be perfect with some small imperfections appearing here and there, giving rise to extra terms. For instance, a radial misalignment of ΔR in the bending magnets yields a term, constant with respect to x:

$$\frac{d^2x}{d\theta^2} + {Q_\mathrm{H}}^2 x = (1-n)\Delta R. \tag{10.1}$$

Here we have chosen the horizontal equation; however, similar arguments can be made for the vertical and even longitudinal motions. The constant term may be expanded in a Fourier series with terms like $\varepsilon\cos(m\theta)$ where ε is the strength of the m-th harmonic. For a single periodic error the equation of motion becomes

$$\frac{d^2x}{d\theta^2} + {Q_\mathrm{H}}^2 x = \varepsilon\cos(m\theta). \tag{10.2}$$

The most general solution of Eq. (10.2) may be written as

$$x = \tilde{x} + \bar{x}, \tag{10.3}$$

where the homogeneous solution is

$$\tilde{x} = A\cos(Q_\mathrm{H}\theta) + B\sin(Q_\mathrm{H}\theta), \tag{10.4}$$

and the particular solution is

$$\bar{x} = \frac{\varepsilon}{{Q_\mathrm{H}}^2 - m^2}[\cos(m\theta) - \cos(Q_\mathrm{H}\theta)], \tag{10.5}$$

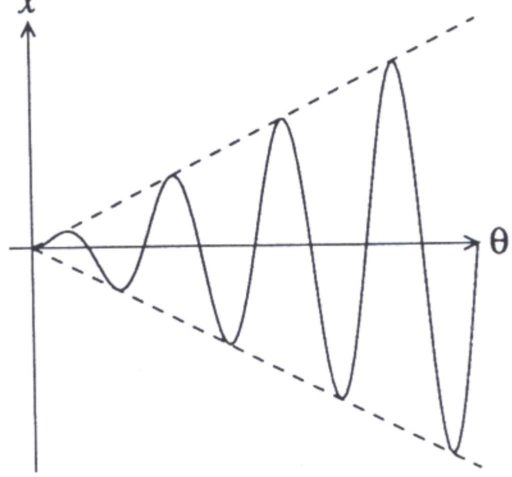

Figure. 10.1 A plot of the particular solution $\bar{x}$.

with A and B given by the initial conditions. By use of a little trigonometry, this last equation may be transformed to

$$\bar{x} = \frac{\varepsilon\theta}{Q_{\mathrm{H}} + m} \sin\left(\frac{Q_{\mathrm{H}} + m}{2}\theta\right) \frac{2}{(Q_{\mathrm{H}} - m)\theta} \sin\left(\frac{Q_{\mathrm{H}} - m}{2}\theta\right), \tag{10.6}$$

which reduces to

$$\bar{x} \simeq \frac{\varepsilon\theta}{2Q_{\mathrm{H}}} \sin(Q_{\mathrm{H}}\theta), \tag{10.7}$$

for $Q_{\mathrm{H}} = m$. If the error occurs only once in the ring, as might be caused by a random error, then m can take all integral values; whereas, if the error is a systematic error occurring in each periodic cell, and there are N periodic cells in the ring, then m takes the values Nj, where j is any integer.

Fig. 10.1 illustrates the steady increase of the amplitude of the betatron oscillations referred to the particular solution, $\bar{x}$. Of course the general solution,

$$x(\theta) = A\cos(Q_{\mathrm{H}}\theta) + \left(B + \frac{\varepsilon}{2Q_{\mathrm{H}}}\theta\right) \sin(Q_{\mathrm{H}}\theta), \tag{10.8}$$

also exhibits the unbounded nature for $Q_{\mathrm{H}} = m$.

10.2 Linear coupling[2]

The sharp separation between horizontal and vertical motions so far considered, is of course an approximation. Indeed some amount of coupling must exist, giving rise to a pair of modified betatron equations:

$$\frac{d^2x}{d\theta^2} + {Q_{\mathrm{H}}}^2 = \varepsilon\cos(m\theta)\, y, \quad \text{and} \tag{10.9}$$

$$\frac{d^2y}{d\theta^2} + {Q_{\mathrm{V}}}^2 = \varepsilon\cos(m\theta)\, x. \tag{10.10}$$

If ε is very small, the solutions of the homogeneous equations for x and y may be substituted into the corresponding inhomogeneous terms on the right-hand side:

$$\frac{d^2x}{d\theta^2} + {Q_\mathrm{H}}^2 = \frac{1}{2}\varepsilon_y \left[\cos(Q_\mathrm{V} + m)\theta + \cos(m - Q_\mathrm{V})\theta\right], \quad \text{and} \tag{10.11}$$

$$\frac{d^2y}{d\theta^2} + {Q_\mathrm{V}}^2 = \frac{1}{2}\varepsilon_x \left[\cos(Q_\mathrm{H} + m)\theta + \cos(m - Q_\mathrm{H})\theta\right], \tag{10.12}$$

where ε_x and ε_y contain the respective amplitude information of the homogeneous solutions. The same arguments as in the previous section lead to the resonance conditions

$$Q_\mathrm{H} + Q_\mathrm{V} = m, \quad \text{and} \tag{10.13}$$

$$|Q_\mathrm{H} - Q_\mathrm{V}| = m. \tag{10.14}$$

10.3 Assessment of resonances

For a perturbing magnetic field $\vec{B} = (B_x, B_y, 0)$, the equations are of the form:

$$\frac{d^2x}{d\theta^2} + {Q_\mathrm{H}}^2 x = \varepsilon \left(\frac{\partial B_x}{\partial y}\right) x\cos(m\theta), \quad \text{and} \tag{10.15}$$

$$\frac{d^2y}{d\theta^2} + {Q_\mathrm{V}}^2 y = \varepsilon \left(\frac{\partial B_x}{\partial y}\right) y\cos(m\theta), \tag{10.16}$$

since $\partial B_x/\partial y = \partial B_y/\partial x$. We may use Eq. (4.8) to express the perturbing field gradient in a multipole expansion:

$$\left(\frac{\partial B_x}{\partial y}\right) = \mathrm{Re}\left[i\sum_{n=0}^{\infty} n(a_n - ib_n)(x + iy)^{n-1}\right], \tag{10.17}$$

where we have incorporated the B_0 and a^n into the multipole constants b_n and a_n.

If the perturbing field is only due to the normal $2(n+1)$-multipole, the gradient becomes

$$\left(\frac{\partial B_x}{\partial y}\right) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} nb_n(-1)^j \binom{n-1}{2j} x^{n-1-2j}y^{2j}. \tag{10.18}$$

Substituting solutions,

$$x = A_1\cos(Q_\mathrm{H}\theta), \quad \text{and} \quad y = A_2\cos(Q_\mathrm{V}\theta), \tag{10.19}$$

of the linear homogeneous equations into the right-hand sides of Eqs. (10.15 and 10.16), yields

$$\frac{d^2x}{d\theta^2}+{Q_{\rm H}}^2x=\varepsilon n b_n\cos(m\theta)\sum_{j=0}^{\left[\frac{n-1}{2}\right]}\binom{n-1}{2j}A_1^{n-2j}A_2^{2j}\cos^{n-2j}(Q_{\rm H}\theta)\cos^{2j}(Q_{\rm V}\theta), \tag{10.20}$$

and

$$\frac{d^2y}{d\theta^2}+{Q_{\rm V}}^2y=\varepsilon n b_n\cos(m\theta)\sum_{j=0}^{\left[\frac{n-1}{2}\right]}\binom{n-1}{2j}A_1^{n-1-2j}A_2^{2j+1}\cos^{n-1-2j}(Q_{\rm H}\theta)\cos^{2j+1}(Q_{\rm V}\theta). \tag{10.21}$$

A given product of cosines may be written as

$$\cos(m\theta)\cos^p(Q_{\rm H})\cos^q(Q_{\rm V}) = 2^{-(p+q)}\sum_{k=0}^{p}\sum_{l=0}^{q}\binom{p}{k}\binom{q}{l}\cos\{[(p-2k)Q_{\rm H}+(q-2l)Q_{\rm V}-m]\theta\}. \tag{10.22}$$

For the x-equation, Eq. (10.20), $p=n-2j$, and $q=2j$, giving resonances when

$$[n\pm1-2(j+k)]Q_{\rm H}+2(j-l)Q_{\rm V}=\pm m. \tag{10.23}$$

For the y-equation, Eq. (10.21), $p=n-1-2j$, and $q=2j+1$, giving the additional conditions

$$[n-1-2(j+k)]Q_{\rm H}+[1\pm1+2(j-l)]Q_{\rm V}=\pm m. \tag{10.24}$$

From these relations, it is easy to show that a normal quadrupole error excites the half-integer resonances,

$$2Q_{\rm H}=\pm m, \quad \text{and} \quad 2Q_{\rm V}=\pm m. \tag{10.25}$$

A normal octopole component excites the resonances

$$\begin{aligned}\pm4Q_{\rm H}&=m,\\ \pm4Q_{\rm V}&=m,\\ \pm2Q_{\rm H}&=m,\\ \pm2Q_{\rm V}&=m, \quad \text{and}\\ \pm2Q_{\rm H}\pm2Q_{\rm V}&=m.\end{aligned} \tag{10.26}$$

Notice that the resonances driven by the normal quadrupole are also driven by the octopole. Fig. 10.2a gives a graphical representation of these resonance relations for the quadrupole and octopole.

A normal sextupole drives the resonances given by

$$\begin{aligned} \pm 3Q_{\mathrm{H}} &= m, \\ \pm Q_{\mathrm{H}} &= m, \quad \text{and} \\ \pm Q_{\mathrm{H}} \pm 2Q_{\mathrm{V}} &= m. \end{aligned} \tag{10.27}$$

A normal decapole drive the resonances given by

$$\begin{aligned} \pm 5Q_{\mathrm{H}} \pm 2Q_{\mathrm{V}} &= m, \\ \pm 5Q_{\mathrm{H}} &= m, \\ \pm 3Q_{\mathrm{H}} \pm 2Q_{\mathrm{V}} &= m, \\ \pm 3Q_{\mathrm{H}} &= m, \\ \pm Q_{\mathrm{H}} \pm 4Q_{\mathrm{V}} &= m, \\ \pm Q_{\mathrm{H}} \pm 2Q_{\mathrm{V}} &= m, \quad \text{and} \\ \pm Q_{\mathrm{H}} &= m. \end{aligned} \tag{10.28}$$

Similar to the case of the odd multipoles, we see that an even multipole will drive the same resonances as a lower order even multipole. The resonance relations for the normal sextupole and octopole are shown in Fig. 10.2b.

In fact, if the perturbation is strong enough, we must look to solutions which are of second order or higher in ε. For example, it has been shown[9] in second order that a sextupole can drive a quarter-integer resonance.

10.4 Krylov-Bogoliubov method[3]

Before proceeding, it is of primary interest to introduce a method which will enable us to solve the differential equations describing linear and nonlinear oscillators, driven by nonlinear forcing terms close to one or more resonances. A general equation for such an oscillator is

$$\frac{d^2x}{d\theta^2} + (Q_{\mathrm{res}}^2 + \delta)x = \varepsilon F(x, \theta), \tag{10.29}$$

with

$$\delta = {Q_{\mathrm{H}}}^2 - Q_{\mathrm{res}}^2 \simeq 2Q_{\mathrm{res}}(Q_{\mathrm{H}} - Q_{\mathrm{res}}). \tag{10.30}$$

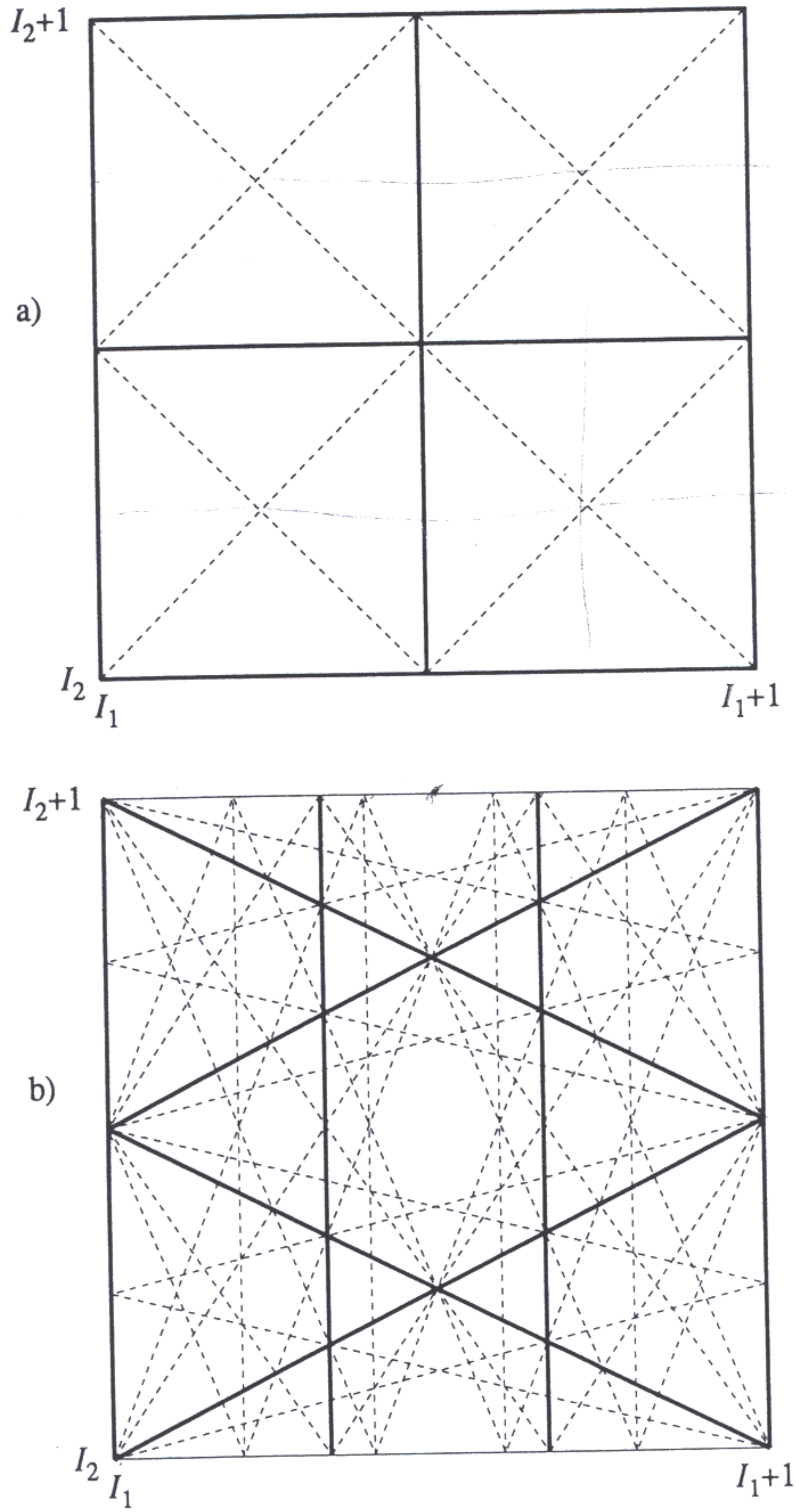

Figure. 10.2 a) The a tune plot showing the resonance lines driven by a normal quadrupole perturbation (heavy lines), and a normal octopole perturbation (all lines). I_1 and I_2 are arbitrary integers. b) A tune plot showing the resonance lines driven by a normal sextupole (heavy lines), and a normal decapole (heavy and dashed lines).