RELATIVISTIC GRAVITY AND SOME ASTROPHYSICAL APPLICATIONS

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ABSTRACT

These lectures are intended to provide a framework for the others at this institute. We will first discuss the foundations and structure of metric theories of gravity, developed from the consequences of various extremal action principles. Then we restrict our consideration to scalar-tensor theories of gravity, and introduce various astrophysical applications. The weak-field applications are the index of refraction of gravity and gravitational waves. The strong-field applications are spherically symmetric compact objects, their surrounding particle orbits, and cosmology. Except for waves and cosmology, effects of the scalar field are assumed negligible, reducing the theory to general relativity.

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1 Key Concepts and Principles

Gravitation is the most fundamental interaction, affecting all forms of mass-energy. This allows its geometrical description, at least within the classical (non-quantum) regime that we shall consider. The scope of this regime is indicated in figure 1. This series of three lectures is intended to provide a foundation for the others in this school, most of which deal with various astrophysical, cosmological, and quantum mechanical manifestations of gravity. Useful results rather than detailed derivations will be emphasized. The approach that we will take and some applications that we will briefly consider are outlined in figure 2.

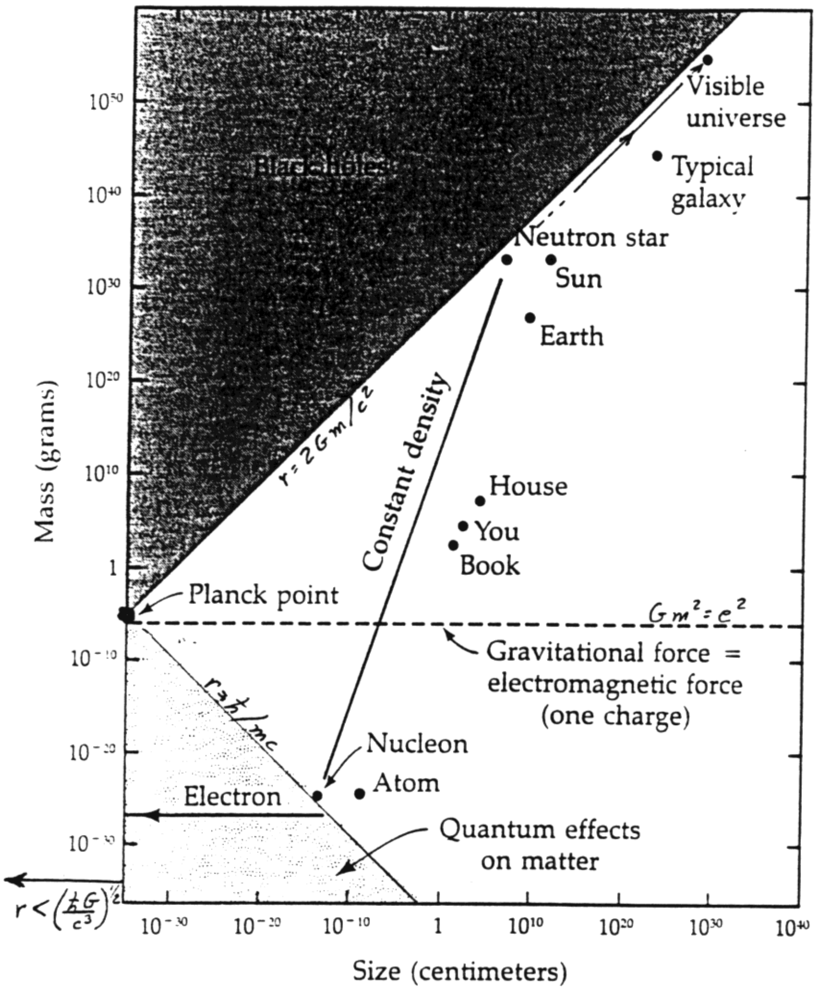

Fig. 1. The domains of our knowledge of the physical world, with the mass and size of representative objects indicated. A theory of quantum gravity is needed at distances less than the Planck length $L_P \equiv \sqrt{\hbar G/c^3}$.

The textbooks that are closest to our viewpoint are by Schutz[1] at the introductory level and by Misner, Thorne, and Wheeler[2] at the comprehensive level. Except for only setting the speed of light $c = 1$, but not the gravitational constant G, we adopt their notation and conventions.

We shall first consider the broad class of *metric theories of gravitation.* This class is defined by the specification of how gravity affects matter. The manner in which matter generates gravity is separately specified by each theory within this class. Metric theories are based upon a few key concepts and principles:

1) **Universality of Free–Fall (UFF)**

This principle states that if a test particle is placed at an initial event and is given an initial velocity there, its subsequent worldline through spacetime will be independent of its structure (i. e., all forms of energy 'fall' at the same rate). A test particle is conveniently defined as one whose charge, mass, and size are reduced until experimental results are unchanged.

2) **Coordinate Frame**

A coordinate (or reference) frame can be visualized as a continuous set of spatially labeled 'clocks' filling spacetime. Such generalized clocks merely provide the time label of events. Infinitesimal distances are best measured by the radar method. The proper distance is $c/2$ times the round trip travel time, as measured by an ideal clock carried by the fiducial observer. The time associated with the measurement is the average of the photon emission and reception times.

3) **Inertial Reference Frame**

This is a local coordinate frame in which any free test particle is unaccelerated (to a specified accuracy) within a small specified region of spacetime. It can always be constructed at *any* point (event) in spacetime (if UFF is valid). A physical realization is a nonrotating lab in free-fall, small enough that the effects of tidal gravitational forces are negligible.

4) **Einstein Equivalence Principle (EEP)**

This principle states that in all inertial frames, the *nongravitational* laws of physics are those formulated within special relativity (when local Lorentz coordinates are employed).

Employing the EEP, we will see how *gravity emerges from a local analysis.* Demanding that the laws of physics retain their form under general coordinate transformations $x^{\mu'} = x^{\mu'}(x^\alpha)$ (general covariance) will then allow us to determine how matter couples to gravity: via a metric tensor. This metric tensor has components $g_{\mu\nu}(x^\alpha)$ which can be put in the Minkowski form $\eta_{\mu\nu}$, with $\partial g_{\mu\nu}/\partial x^\alpha = 0$, in every inertial frame (which is then called a local Lorentz frame).

Note that all tensor equations ($\mathbf{T} = \mathbf{0}$) are generally covariant. Such an equation will thus be true in all coordinate systems if it is known to be true in any one (such as a local Lorentz frame).

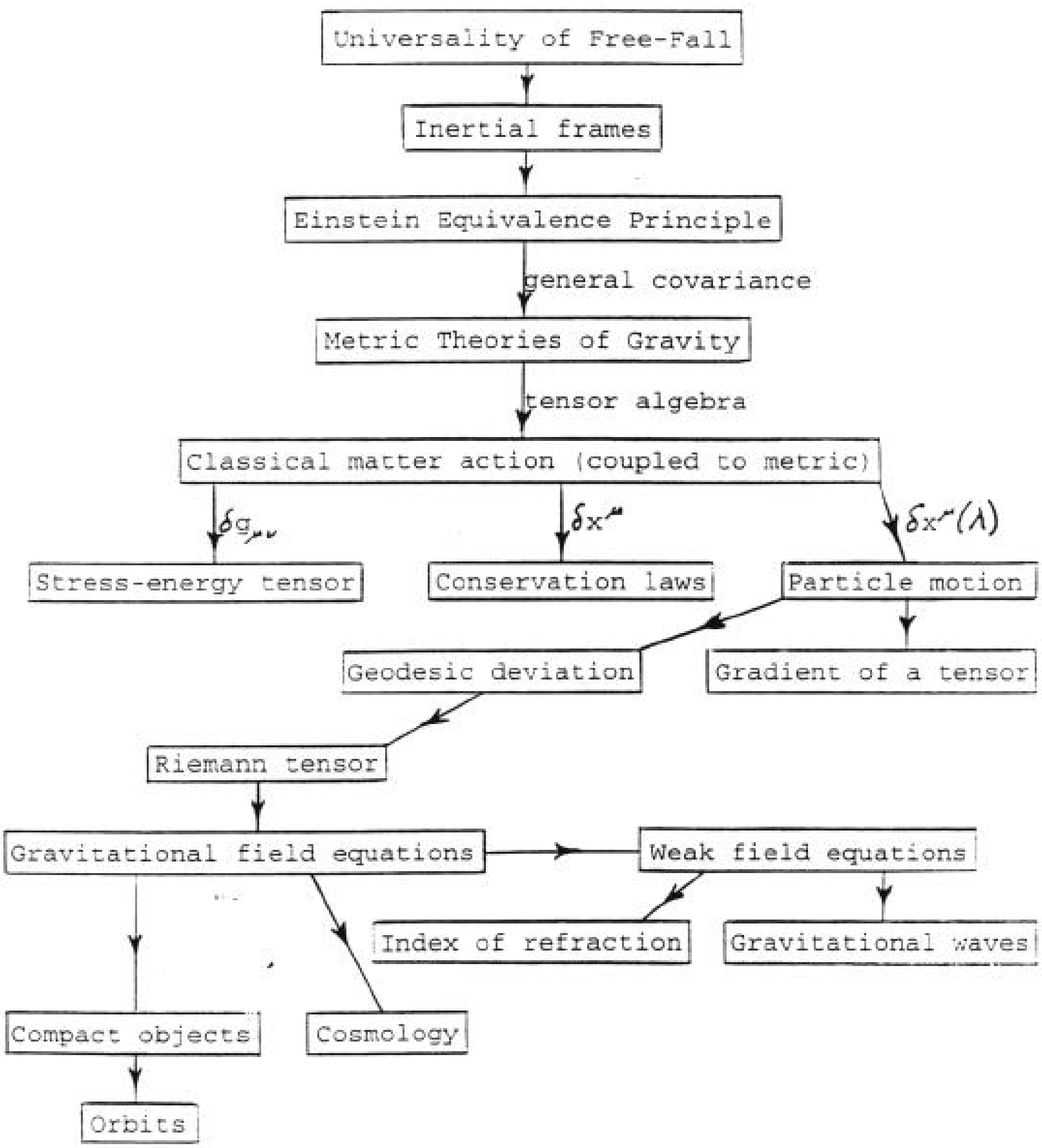

Fig. 2. The structure of these lectures.

2 Tensor Algebra in Metric Spacetimes

Tensor algebra is the study of geometrical objects (scalars, vectors, tensors, ...) at any fixed point $\mathcal{P}_0$. These objects exist independent of any coordinates, so form the proper description of physical reality.

2.1 Vector

In curved spacetime, we require a local definition of a vector. A familiar one is the tangent vector

$$\mathbf{v} = \left(\frac{\mathbf{d}\mathcal{P}}{d\lambda}\right)_{\mathcal{P}_0}$$

to some curve $\mathcal{P}(\lambda)$ at the point $\mathcal{P}_0$ where the vector exists, where $\mathbf{d}\mathcal{P}$ is the infinitesimal displacement vector. If the path parameter λ is chosen as the proper time τ (for a nonzero mass particle), the tangent vector is the four-velocity $\mathbf{U} = \mathbf{d}\mathcal{P}/d\tau$.

2.2 Tangent Space

The vectors at any point $\mathcal{P}_0$ form this abstract four dimensional vector space. All geometrical objects at this point reside in this tangent space (not in spacetime).

2.3 Basis

A basis is a set of four linearly independent vectors $\mathbf{e}_\alpha$ ($\alpha = 0, 1, 2, 3$) at a point $\mathcal{P}_0$. Any vector $\mathbf{v}$ at $\mathcal{P}_0$ can be represented by its components v^α:

$$\mathbf{v} = v^\alpha \mathbf{e}_\alpha \qquad \text{(summation convention)}\ .$$

Consider some coordinate system: four functions $x^\alpha(\mathcal{P})$. A (global) *coordinate basis* is then

$$\mathbf{e}_\alpha = \partial\mathcal{P}/\partial x^\alpha\ ,$$

indicated in figure 3. We shall only employ such bases.

Most other aspects of tensor algebra are direct generalizations from special relativity in each tangent space, as follows.

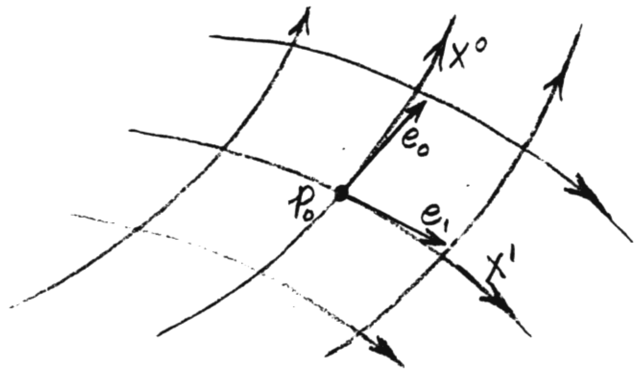

Fig. 3. Coordinate basis vectors in the tangent space at $\mathcal{P}_0$, tangent to the coordinate curves there.

2.4 Tensor

A tensor can be thought of in at least two ways:

- As a direct product: $\mathbf{T} = T^{\alpha\beta\cdots}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \cdots$.
- As a linear operator on vectors, giving a scalar (number):

$$\mathbf{T}(\mathbf{u}, \mathbf{v}, \ldots) = \mathbf{T}(\mathbf{e}_\alpha, \mathbf{e}_\beta, \ldots)u^\alpha v^\beta \cdots = T_{\alpha\beta\cdots}u^\alpha v^\beta \cdots \ .$$

The second equality defines the components of a tensor.

2.5 Metric Tensor

This generalization of the Minkowski metric of special relativity, with components $\eta_{\mu\nu} = \text{diag.}(-1, 1, 1, 1)$, produces the scalar product of vectors: $\mathbf{u} \cdot \mathbf{v} \equiv \mathbf{g}(\mathbf{u}, \mathbf{v})$. Its components then also represent scalar products:

$$\mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\beta) \equiv \mathbf{e}_\alpha \cdot \mathbf{e}_\beta = g_{\alpha\beta} = \mathbf{g}(\mathbf{e}_\beta, \mathbf{e}_\alpha) = g_{\beta\alpha} \ .$$

Other aspects are:

- The interval $ds^2 = \mathbf{g}(\mathbf{d}\mathcal{P}, \mathbf{d}\mathcal{P}) = g_{\alpha\beta}dx^\alpha dx^\beta \quad (\mathbf{d}\mathcal{P} = dx^\sigma \mathbf{e}_\sigma)$.
- Its inverse, given by $g^{\mu\sigma}g_{\sigma\nu} = \delta^\mu{}_\nu$. • 'Raising and lowering indices':

$$\begin{aligned} v_\mu &\equiv \mathbf{v} \cdot \mathbf{e}_\mu = \mathbf{g}(v^\nu \mathbf{e}_\nu, \mathbf{e}_\mu) = g_{\nu\mu}v^\nu \ , \\ v^\mu &= g^{\mu\sigma}g_{\sigma\nu}v^\nu = g^{\mu\sigma}v_\sigma \ . \end{aligned}$$

This generalizes to give

$$T_\alpha{}^\beta{}_{\gamma\cdots}{}^{\cdots} = g_{\alpha\mu}g_{\gamma\nu}T^{\mu\beta\nu\cdots} = g^{\beta\sigma}T_{\alpha\sigma\gamma\cdots}$$

and the generalized scalar product $T^{\alpha}_{\ \beta\gamma} V_\alpha N^{\beta\gamma}$, for instance.

It is important to remember that *all measured quantities are scalars.* For instance, the energy of a photon of four-momentum $\mathbf{p}$ measured by a detector of four-velocity $\mathbf{U}$ (as shown in figure 4) is $E = -\mathbf{U} \cdot \mathbf{p} = -U^\sigma p_\sigma$. This reduces to $E = p^0$ in an instantaneously comoving ($U^i = 0$) local Lorentz frame.

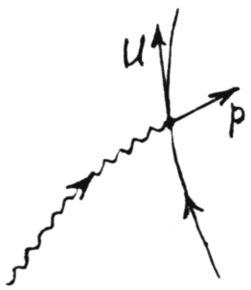

Fig. 4. The interaction of a photon of four-momentum $\mathbf{p}$ with a detector of four-velocity $\mathbf{U}$.

- Contraction produces a new tensor of rank two lower; for instance

$$Q_{\mu\nu} = M^{\sigma}{}_{\mu\sigma\nu} = g^{\sigma\tau} M_{\tau\mu\sigma\nu} \ ,$$

independent of choice of basis.

2.6 Change of Basis

This is represented by $\mathbf{e}_{\mu'} = L^{\nu}{}_{\mu'} \mathbf{e}_\nu$, generated by the tranformation matrix $L^{\nu}{}_{\mu'}(\mathcal{P}_0)$. With the inverse transformation matrix constructed from $L^{\mu}{}_{\sigma'} L^{\sigma'}{}_{\nu} = \delta^{\mu}{}_{\nu}$, one obtains the transformed components

$$T^{\alpha' \cdots}_{\ \ \beta' \cdots} = L^{\alpha'}{}_{\sigma} L^{\tau}{}_{\beta'} \cdots T^{\sigma \ \cdots}_{\ \tau \cdots}$$

of a tensor, using the above equations. Under a tranformation of coordinate bases [generated by the four functions $x^{\mu'}(x^\alpha)$], the tranformation matrices assume the form

$$L^{\alpha'}{}_{\beta} = \partial x^{\alpha'} / \partial x^{\beta} \ , \quad L^{\beta}{}_{\alpha'} = \partial x^{\beta} / \partial x^{\alpha'} \ .$$

2.7 Four-volume Element

The unique scalar which generalizes all the usual properties of a volume element is

$$dV_{(4)} = \sqrt{-g}\, dx^0 dx^1 dx^2 dx^3 \equiv \sqrt{-g}\, d^4x \ ,$$

where g is the determinant of the matrix $g_{\mu\nu}$.

2.8 Gradient of a Function

This operation on a function $f(x^\alpha)$ is represented as

$$\mathbf{d}f = f^{,\alpha}\mathbf{e}_\alpha = g^{\alpha\beta} f_{,\beta}\mathbf{e}_\alpha = g^{\alpha\beta}\frac{\partial f}{\partial x^\beta}\mathbf{e}_\alpha \ ,$$

in terms of its partial derivatives in a coordinate basis. The *directional derivative* of a function along a curve $\mathcal{P}(\lambda)$ (at $\mathcal{P}_0$) is then

$$\frac{df}{d\lambda} = \frac{dx^\alpha}{d\lambda}\frac{\partial f}{\partial x^\alpha} = v^\alpha f_{,\alpha} = \mathbf{v}\cdot\mathbf{d}f \ .$$

This scalar indicates how much the function f changes in the direction of the tangent vector $\mathbf{v}$.

3 Extremal Action Principles for Matter and Their Consequences

The fundamental special-relativistic laws of physics may obtained by extremizing the (scalar) action $\mathcal{I} = \int \mathcal{L} d^4x$, $\delta\mathcal{I} = 0$. Within metric theories, the effects of gravity *on* a classical material system may be obtained by the replacements

$$\mathcal{I} \rightarrow \int \mathcal{L}\sqrt{-g}\, d^4x \ , \ \mathcal{L} \rightarrow \mathcal{L}_M(\eta_{\mu\nu} \rightarrow g_{\mu\nu}; A_\mu, A_{[\mu,\nu]}, \text{'matter'}) \ . \qquad (1)$$

Variation with respect to A_μ gives Maxwell's equations in an arbitrary metric field. (Complete antisymmetrization and symmetrization of indices will be denoted by $[\alpha\beta\cdots]$ and $(\alpha\beta\cdots)$.) We will add the contribution of the gravitational field(s) to the action $\mathcal{I}$ later.

3.1 Stress-Energy Tensor

Under the variation $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$, the stress-energy tensor $T^{\mu\nu}$ is defined by

$$\delta\mathcal{I}_M = \tfrac{1}{2}\int T^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}\, d^4x \ . \qquad (2)$$

This definition automatically generates all of its desired properties from special relativity.

3.2 Energy-Momentum Conservation

Since the 'matter' action $\mathcal{I}_M$ is a scalar, it will be unchanged under a coordinate transformation $x^\mu \rightarrow x^{\mu'} = x^\mu + \epsilon^\mu(x^\alpha)$, here taken to be infinitesimal. It will also be unchanged by a subsequent change $x^{\mu'} \rightarrow x^\mu$ in the integration variable. This induces the net change

$$\delta g_{\mu\nu} = -(g_{\sigma\nu}\epsilon^\sigma{}_{,\mu} + g_{\mu\sigma}\epsilon^\sigma{}_{,\nu} + g_{\mu\nu,\sigma}\epsilon^\sigma) \ ,$$

the Lie derivative of $g_{\mu\nu}$. Under this variation of the metric tensor components, one obtains $\delta\mathcal{I}_M = \int T^\mu{}_{\sigma;\mu}\epsilon^\sigma\sqrt{-g}\, d^4x$, where

$$T^\mu{}_{\sigma;\mu} \equiv (1/\sqrt{-g})[\sqrt{-g}\, T^\mu{}_\sigma]_{,\mu} - \tfrac{1}{2}g_{\mu\nu,\sigma}T^{\mu\nu} \ .$$

Note that the divergence $T^{\mu\sigma}{}_{;\mu}\mathbf{e}_\sigma$ must be the components of a vector, since its contraction with the components ϵ^σ of the infinitesimal displacement vector is a scalar. Since $\delta\mathcal{I}_M = 0$ for arbitrary $\epsilon^\sigma(x^\alpha)$, we obtain the four components

$$T^\mu{}_{\sigma;\mu} = 0 \qquad (\text{or } T^{\mu\sigma}{}_{;\mu} = 0) \ , \tag{3}$$

representing conservation of energy and momentum. The latter are the equations of motion of the continuum.

3.3 Test-Particle Equation of Motion

The prescription (1) gives the action

$$\mathcal{I}_M = -\int \left(-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda$$

for a test particle, where the tangent vector to the particle's world line $x^\alpha(\lambda)$ is $\mathbf{d}\mathcal{P}/d\lambda = (dx^\mu/d\lambda)\mathbf{e}_\mu \equiv \dot{x}^\mu \mathbf{e}_\mu$. Vary the worldline, and choose λ (after the variation) so that $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -m^2$. The rest mass $m \geq 0$. The components of the particle's four-momentum are then $p^\mu = \dot{x}^\mu$. Then $\delta\mathcal{I}_M = -m^{-1}\int Q_\alpha \delta x^\alpha d\lambda = 0$, so

$$Q_\alpha \equiv dp_\alpha/d\lambda - \tfrac{1}{2} g_{\mu\nu,\alpha} p^\mu p^\nu = 0 \ . \tag{4}$$

Thus if the metric components $g_{\mu\nu}$ are independent of any coordinate x^α, the corresponding four-momentum component p_α is conserved.

We also find that the vector components

$$Q^\alpha = \frac{d^2x^\alpha}{d\lambda^2} + \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \equiv \frac{D}{D\lambda}\left(\frac{dx^\alpha}{d\lambda}\right) = \frac{Dp^\alpha}{D\lambda} = 0 \ , \tag{5}$$

where the Christoffel symbol (connection coefficient in a coordinate basis, not the components of a tensor) is

$$\left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} = \left\{ \begin{matrix} \alpha \\ \nu\mu \end{matrix} \right\} = \tfrac{1}{2} g^{\alpha\sigma}(g_{\sigma\nu,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma}) \ . \tag{6}$$

For rest masses $m > 0$, equations (4) and (5) also govern the particle's four-velocity $U^\alpha = p^\alpha/m = dx^\alpha/d\tau$, where the proper time interval $d\tau = md\lambda$.

It can be shown that in the neighborhood of a freely-falling observer, coordinates can be chosen so that $g_{\mu\nu} = \eta_{\mu\nu}$ and $g_{\mu\nu,\alpha} = 0$ along his/her worldline. It then follows from equation (5) that all test particles in that neighborhood are indeed unaccelerated ($dx^\alpha/d\tau = \text{constant} \Rightarrow dx^i/dx^0 = \text{constant}$), verifying that it is an inertial frame.

3.4 Gradient and Covariant Derivative of a Tensor

Since $Q^\alpha = Dp^\alpha/D\lambda$ are the components of a vector [given by equation (5)], it follows that for any vector field with components $V^\alpha(x^\mu)$,

$$\begin{aligned} \frac{DV^\alpha}{D\lambda} &= \frac{dV^\alpha}{d\lambda} + \left\{ {\alpha \atop \mu\nu} \right\} V^\mu \frac{dx^\nu}{d\lambda} \\ &= \left(V^\alpha{}_{,\nu} + \left\{ {\alpha \atop \mu\nu} \right\} V^\mu \right) \frac{dx^\nu}{d\lambda} \equiv V^\alpha{}_{;\nu} \frac{dx^\nu}{d\lambda} \end{aligned} \qquad (7)$$

are also the components of a vector. It thus also follows that $V^\alpha{}_{;\nu}$ must be the components of a (rank 2) tensor: the generalization of the gradient to operate on vectors. The generalization of the directional derivative is the covariant derivative, with the above components $DV^\alpha/D\lambda$.

Denoting the directional derivative operator $D/D\lambda$ along a basis vector $\mathbf{e}_\nu$ by ∇_ν, its effect on a vector $\mathbf{V}$ can also be described as

$$\nabla_\nu(V^\sigma \mathbf{e}_\sigma) = (\nabla_\nu V^\sigma)\mathbf{e}_\sigma + V^\sigma(\nabla_\nu \mathbf{e}_\sigma) \equiv V^\alpha{}_{;\nu}\mathbf{e}_\alpha \, .$$

Comparing with equation (7), we see that the connection coefficients describe how the basis vectors vary with position:

$$\nabla_\nu \mathbf{e}_\mu = \left\{ {\alpha \atop \mu\nu} \right\} \mathbf{e}_\alpha \, .$$

The application to tensors of any rank follows straightforwardly to give

$$T^{\alpha\cdots}_{\beta\cdots;\mu} = T^{\alpha\cdots}_{\beta\cdots,\mu} + \left\{ {\alpha \atop \sigma\mu} \right\} T^{\sigma\cdots}_{\beta\cdots} + \cdots - \left\{ {\sigma \atop \beta\mu} \right\} T^{\alpha\cdots}_{\sigma\cdots} - \cdots \, .$$

It then follows that the gradient of the metric tensor vanishes: $g_{\alpha\beta;\mu} = 0$.

As an interesting example, consider the Maxwell field-strength tensor, whose components $F_{\mu\nu}$ are generated from the components

$$A_{\mu;\nu} = A_{\mu,\nu} - \left\{ {\sigma \atop \mu\nu} \right\} A_\sigma$$

of the gradient of the potential via $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} \equiv 2A_{[\nu;\mu]} = 2A_{[\nu,\mu]}$. The last equality follows from the symmetry of the Christoffel symbols [equation (6)], and reflects the curious fact that only partial derivatives seem to appear in the matter Lagrangian [equation (1)].

3.5 Geodesic Deviation $\Longleftrightarrow$ Riemann Tensor

Consider two freely falling test particles with infinitesimal separation $\Delta x^\alpha \mathbf{e}_\alpha$, as indicated in figure 5. Their separation vector can be constrained to obey $\mathbf{\Delta x}\cdot\mathbf{p} = 0$. It is then purely spatial ($\Delta x_0 = 0$) in an instantaneously comoving frame ($U^i = 0$) for a nonzero mass test particle.

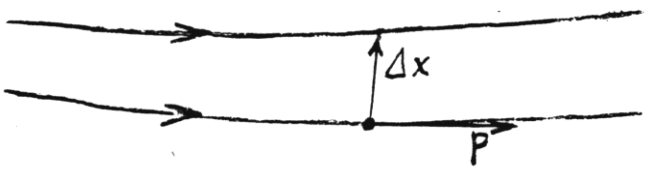

Fig. 5. The paths of two nearby test particles.

Subtracting their (geodesic) equations of motion (5) gives

$$\Delta Q^\alpha = \frac{d^2\Delta x^\alpha}{d\lambda^2} + \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\}_{,\sigma} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\Delta x^\sigma + 2\left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{d\lambda}\frac{d\Delta x^\nu}{d\lambda} = 0 \,.$$

Now since $D\Delta x^\alpha/D\lambda$ (but not $d\Delta x^\alpha/d\lambda$) are the components of a vector, so is the result of applying the operator $D/D\lambda$ [defined by equation (7)] again. Employing the above equation. this operation produces the equation of geodesic deviation

$$\frac{D^2\Delta x^\alpha}{D\lambda^2} + R^\alpha{}_{\mu\sigma\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\Delta x^\sigma = 0 \,. \tag{8}$$

Since both terms in this equation are the components of vectors, the quantities

$$R^\alpha{}_{\mu\sigma\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\}_{,\sigma} - \left\{ \begin{matrix} \alpha \\ \mu\sigma \end{matrix} \right\}_{,\nu} + \left\{ \begin{matrix} \alpha \\ \sigma\tau \end{matrix} \right\}\left\{ \begin{matrix} \tau \\ \mu\nu \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \tau\nu \end{matrix} \right\}\left\{ \begin{matrix} \tau \\ \mu\sigma \end{matrix} \right\}$$

must be the components of a (rank 4) tensor, called the Riemann (curvature) tensor. It plays a role similar to that of the electromagnetic field tensor $F_{\mu\nu}$ in the extension of the equation of motion (5) to charged test particles:

$$\frac{D}{D\tau}\left(\frac{dx^\alpha}{d\tau}\right) = \frac{q}{m}F^\alpha{}_\sigma\frac{dx^\sigma}{d\tau} \,,$$

the generalized Lorentz force equation. We see, however, that the Riemann tensor represents the physical field *gradients* (tidal forces), and only *relative* gravitational acceleration has physical meaning.

The symmetry properties of the Riemann tensor (analogous to $F_{\mu\nu} = F_{[\mu\nu]}$) are

$$R_{\alpha\beta\mu\nu} = R_{([\alpha\beta][\mu\nu])} \,, \qquad R_{[\alpha\beta\mu\nu]} = 0 \,;$$

the first giving $6 \cdot 7/2$ independent components and the second giving one less, for a total of 20 independent components. Its (unique) contractions are

$$R_{\mu\nu} = g^{\alpha\beta} R_{\beta\mu\alpha\nu} , \qquad R = g^{\mu\nu} R_{\mu\nu} ,$$

the components of the (symmetric) Ricci tensor and the Ricci scalar. It also obeys the Bianchi identities

$$R_{\alpha\beta[\mu\nu;\zeta]} = 0 , \tag{9}$$

analogous to the homogeneous Maxwell equations $F_{[\mu\nu;\zeta]} = 0$; and their (unique) double contraction

$$G^{\mu\nu}{}_{;\nu} = 0 \qquad (G^{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R) ,$$

which involves the components of the Einstein tensor **G**.

Consider the parallel transport of a vector **S** along some curve $\mathcal{P}(\lambda)$, defined by $DS^\mu / D\lambda = 0$. If the curve is the boundary of an infinitesimal area generated by the displacement vectors $\delta\mathbf{a}$ and $\delta\mathbf{b}$, the change in **S** after a complete circuit is

$$\Delta S^\mu = -R^\mu{}_{\alpha\beta\gamma} S^\alpha \delta a^\beta \delta b^\gamma . \tag{10}$$

This is another way to characterize the Riemann tensor. Now parallel transport the vector around all six faces of an infinitesimal cube, as shown in figure 6. Since the pairs of paths along each edge must cancel, the total change $\Delta S^\mu = 0$. This has been characterized as 'the boundary of a boundary equals zero'.[2] Expressing this result in terms of equation (10) then produces the Bianchi identities (9), giving them an intriguing geometrical meaning.

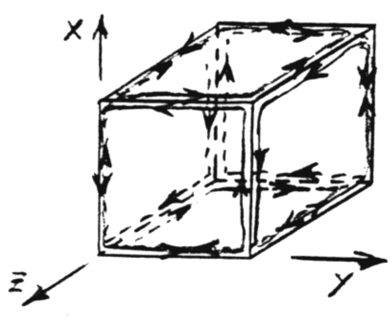

Fig. 6. The paths of vector transport which generate the Bianchi identity.

4 Gravitational Field Equations

4.1 Scalar-Tensor Theories

We now complete the implementation of the extremal action principle by adding to the matter Lagrangian density $\mathcal{L}_M$ [specified in equation (1)] a Lagrangian density $\mathcal{L}_G$ which depends solely on the gravitational field(s). Adopting the principle of simplicity that has worked so well in deriving the laws of physics, we are tempted to include nature's simplest (scalar, spin 0) field φ in addition to the metric tensor field $\mathbf{g}$. In analyzing the field equations, more insight is gained by employing the 'spin representation', in which $\hat{\mathbf{g}}$ denotes the metric tensor which corresponds to a pure spin 2 field. On the other hand, $\mathbf{g} = A^2(\varphi)\hat{\mathbf{g}}$ is the metric tensor discussed above, through which gravity couples to matter.

If we require only that the field equations be of at most second differential order, the most general Lagrangian density is then[3]

$$\mathcal{L}_G = (16\pi G)^{-1}[\hat{R} - 2\hat{g}^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - 2\Lambda(\varphi)] \ ,$$

where G is the bare gravitational constant. Thus there are two free functions in this theory[4,5]: the matter coupling function $A(\varphi)$ and the 'cosmological function' $\Lambda(\varphi)$.

Extremizing the action $\mathcal{I}_G$ with respect to variations in $\hat{g}_{\mu\nu}$ and φ then gives the field equations

$$\begin{aligned} \hat{G}_{\mu\nu} &= 8\pi G\hat{T}_{\mu\nu} - \hat{g}_{\mu\nu}\Lambda(\varphi) + 2\varphi_{,\mu}\varphi_{,\nu} - \hat{g}_{\mu\nu}\hat{g}^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} \ , &(11)\\ \hat{g}^{\mu\nu}\varphi_{,\mu;\nu} &= -4\pi G\alpha(\varphi)\hat{T} + \tfrac{1}{2}d\Lambda/d\varphi \ . &(12) \end{aligned}$$

The scalar field–matter coupling function is

$$\alpha(\varphi) \equiv d\ln A/d\varphi = a_1 + a_2(\varphi - \varphi_0) + \cdots \ , \qquad (13)$$

where the expansion is about the present cosmological value φ_0 of the scalar field. Expanding the self interaction $\Lambda(\varphi)$ in the same way shows that the effective range of the scalar field is of order $(d^2\Lambda/d\varphi^2)_0^{-1/2}$.

The stress-energy tensor $\hat{\mathbf{T}}$ is defined, as in equation (2), with respect to variations of $\hat{\mathbf{g}}$. It obeys the modified conservation laws [compare with equation (3)]

$$\hat{T}_{\mu\ ;\nu}^{\ \nu} = \alpha(\varphi)\hat{T}\varphi_{,\mu} \ , \qquad (14)$$

indicating the separate effects of the spin 2 and spin 0 fields.

If the coupling function $A(\varphi)$ has a minimum, Damour & Nordtvedt[6] and Santiago, Kalligas, & Wagoner[7] have shown that in most cases the theory is attracted toward that minimum during the expansion of the universe, thus approaching general relativity : $\varphi =$ constant, $A(\varphi) =$ constant, $\alpha(\varphi) = 0$. This is in accord with the small experimental limit[8] $a_1^2 < 10^{-3}$. [The Brans-Dicke theory[8] is the special case $\alpha(\varphi) =$ constant, $\Lambda(\varphi) = 0$.]

Although there is local interest in this broad class of theories, for the most part we shall concentrate on general relativity for the remainder of these lectures. The two exceptions will be gravitational waves and cosmology, where a scalar field introduces qualitatively new effects.

In addition, we will employ the fact that most matter in the universe is well approximated as a perfect fluid, described by the stress-energy tensor (obtained from the EEP)

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \ , \tag{15}$$

where ρ is the mass-energy density, p is the pressure, and $\mathbf{U}$ is the four-velocity of the fluid. Such a fluid flows adiabatically (conserving specific entropy).

4.2 Weak-Field Equations

Throughout almost the entirety of all regions much smaller than that of the observable universe, gravitational fields can be considered weak. This means that (except near black holes and neutron stars) one can choose coordinates such that the metric assumes the nearly Minkowski form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x^\alpha) \ , \qquad |h_{\mu\nu}| \ll 1 \ .$$

For instance, within the solar system, $|h_{\mu\nu}| \lesssim GM_\odot/R_\odot c^2 = 2.12 \times 10^{-6}$. We shall consider isolated sources $T_{\mu\nu}$, and can neglect the cosmological constant $\Lambda(\varphi_0)$ within such regions.

We work to first order in $h_{\mu\nu}$, and utilize our freedom of general infinitesimal coordinate transformations $x^{\alpha'}(\mathcal{P}) = x^\alpha(\mathcal{P}) + \xi^\alpha(\mathcal{P})$, which produces the change

$$h_{\mu'\nu'} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \tag{16}$$

in the metric perturbation. This is directly analogous to the gauge transformation $A_{\mu'} = A_\mu + \chi_{,\mu}$ in electrodynamics; and leaves the Riemann tensor components

$R_{\alpha\beta\mu\nu}$, like the Maxwell field strength tensor components $F_{\alpha\beta}$, invariant to this order. We can then use our freedom in choosing the four functions $\xi^\alpha(\mathcal{P})$ to impose the coordinate condition

$$\bar{h}^{\mu\sigma}{}_{,\sigma} = 0 \ , \quad \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h \quad (h = \eta^{\alpha\beta}h_{\alpha\beta}) \ ,$$

analogous to the Lorentz gauge condition $A^\sigma{}_{,\sigma} = 0$ in electrodynamics.

The Einstein field equations (11) (with $\varphi = \varphi_0 =$ constant) then produce the weak field equations

$$\eta^{\alpha\beta}\bar{h}_{\mu\nu,\alpha\beta} \equiv \Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \ , \tag{17}$$

identical in structure to those in electrodynamics. (With the Lorentz gauge condition, equation (17) is consistent with the conservation laws $T^{\mu\nu}{}_{,\nu} = 0$; analogous to $J^\mu{}_{,\mu} = 0$ in electrodynamics.) Thus the solution is of the same form:

$$\bar{h}_{\mu\nu}(x^\alpha) = 4G \int T_{\mu\nu}(x^0 - |x^i - x^i_*|, x^i_*)|x^i - x^i_*|^{-1} d^3x^i_* \ . \tag{18}$$

A Newtonian system is one in which all (macroscopic and microscopic) velocities are nonrelativistic (and thus the retardation in equation (18) is negligible), in addition to having a weak field. In such systems the dominant component of the stress-energy tensor (15) is seen to be $T_{00} \cong \rho$. Therefore we see from equation (18) that the dominant component of the (trace-reversed) metric perturbation is $\bar{h}_{00} = -4\Phi$, where Φ is the Newtonian gravitational potential. (This also identifies the meaning of the coupling constant G.) Then the spacetime interval becomes

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \cong -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2 + dy^2 + dz^2) \ . \tag{19}$$

Incidentally, the result $h_{00} = -2\Phi$ can be obtained more generally by comparing Newton's second law with the geodesic equation of motion (5) for slowly-moving test particles. In the same limit (in which $dx^\alpha/d\tau \cong \delta^\alpha{}_0$), the spatial components of the equation of geodesic deviation (8) become

$$\frac{d^2\Delta x^i}{d\tau^2} \cong -R^i{}_{0j0}\Delta x^j \cong -\frac{\partial^2\Phi}{\partial x^i \partial x^j}\Delta x^j \ ,$$

showing how the tidal gravitational forces affect the separation of nearby particles.

5 Weak-Field Applications

5.1 Index of Refraction of Gravity

Consider a photon passing through a Newtonian gravitational potential (for which $|\partial\Phi/\partial t| \ll |\nabla\Phi|$) produced by some localized distribution of mass. This is a good approximation for all observed systems. In the geometrical optics limit, we follow a photon initially travelling in the direction $\mathbf{e}_x$ far from the masses, so subsequently $p^x = dx/d\lambda \cong p^t \equiv p$. As shown in figure 7, it will be deflected by a very small angle $\vec{\alpha}$, with components $\alpha^N \cong p^N/p$, where the index $N = y, z$.

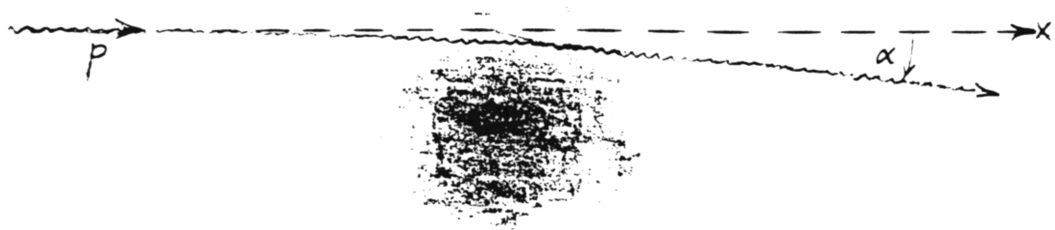

Fig. 7. Photon path through a localized weak field.

The equation of motion (5) then gives

$$\frac{dp^N}{d\lambda} \cong -\left[\left\{ {N \atop tt} \right\} + 2\left\{ {N \atop tx} \right\} + \left\{ {N \atop xx} \right\}\right] p^2 \cong -2\Phi_{,N}p^2$$

when the weak-field metric (19) is inserted in the Christoffel symbols. With $d\ell \equiv dx \cong p d\lambda$ and $d\alpha^N/dx \cong p^{-1}dp^N/dx$, we obtain

$$\vec{\alpha} = -2\int \nabla_\perp \Phi \, d\ell = \int \nabla_\perp n \, d\ell \, , \qquad (20)$$

where we have identified the effective *gravitational index of refraction* $n(x^i) \equiv 1 - 2\Phi$ (≥ 1).

Using the fact that $0 = ds^2 \cong g_{tt}dt^2 + g_{xx}dx^2$ for the photon, we obtain its coordinate velocity

$$\frac{dx}{dt} \cong \left(\frac{-g_{tt}}{g_{xx}}\right)^{1/2} \cong 1 + 2\Phi \cong 1/n \, ,$$

as expected. The time delay, relative to a photon traveling between the same initial and final values of x in the absence of any mass, is then

$$\Delta t = -2\int \Phi\, d\ell = 2\int |\Phi| d\ell = \int (n-1) d\ell \,. \tag{21}$$

However, when the emitter and receiver of the photon are very far from the mass coordinate time t equals proper time τ at both locations. So this is the observed time delay measured by such pairs of observers.

Thus in both respects (bending and delay), empty space acts as if it had this index of refraction.

5.2 Gravitational Waves

Unfortunately, all gravitational waves will be weak when they reach our detectors. The vacuum weak-field equations (17) $\Box \bar{h}_{\mu\nu} = 0$ and Lorentz gauge condition $\bar{h}_{\mu\ ,\sigma}^{\ \sigma} = 0$ allow a representation in terms of plane transverse monochromatic waves

$$\bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\alpha x^\alpha}\ ; \qquad A_{\mu\sigma} k^\sigma = 0\ , \quad k_\sigma k^\sigma = 0\ ,$$

with wave vector $\mathbf{k}$. However, both the field equations and Lorentz gauge condition are preserved under another infinitesimal coordinate (gauge) transformation (16) if the generator also satisfies the wave equation $\Box \xi_\mu = 0$. One can then use the four additional degrees of freedom to set $\bar{h}_{0i} = 0$ and $\bar{h} = 0$ (so now $\bar{h}_{\mu\nu} = h_{\mu\nu}$). In summary, we have constructed the transverse-traceless (TT) gauge, in which the eight independent conditions

$$h_{\mu 0} = h_{jk,k} = h^k{}_k = 0$$

leave two independent polarization states, again in direct analogy with electrodynamics.

We can also include the possibility of a weak scalar wave $\varphi_1 \equiv \varphi - \varphi_0$, since the tensor field equations (11) are unaffected through first order in φ_1 (except that $h_{\mu\nu} \rightarrow \hat{h}_{\mu\nu} \cong h_{\mu\nu} - 2a_1\varphi_1\eta_{\mu\nu}$ in the above equations). The scalar field equation (12) (with Λ constant) becomes $\Box\varphi_1 = 0$, giving the same plane-wave representation.

To understand the response of a gravitational-wave detector, consider slowly moving free test particles whose separation is much less than the gravitational wavelengths involved. Now employ a local Lorentz frame (a different choice of gauge), in which physical (e. g., radar) and coordinate distances are equal through first order in the particle separation $\mathbf{\Delta x}$. The equation (8) of geodesic deviation becomes $d^2\Delta x^i/d\tau^2 \cong -R^i{}_{0j0}\Delta x^j$ as before, involving only the matter coupling metric $\mathbf{g} = A^2(\varphi)\hat{\mathbf{g}}$. In terms of our spin representation,

$$R_{i0j0} = \hat{R}_{i0j0} + a_1(\varphi_{,ij} - \delta_{ij}\varphi_{,00})\ .$$

In the previous TT gauge, one obtains $\hat{R}^{TT}_{i0j0} = -\frac{1}{2}\hat{h}^{TT}_{ij,00}$. However, the gauge (coordinate) invariance of the weak-field Riemann tensor allows us to use this expression in the above equation of geodesic deviation, giving

$$d^2\Delta x^i/d\tau^2 \cong [\tfrac{1}{2}\hat{h}^{TT}_{ij,00} + a_1(\delta_{ij}\varphi_{,00} - \varphi_{,ij})]\Delta x^j\ . \tag{22}$$

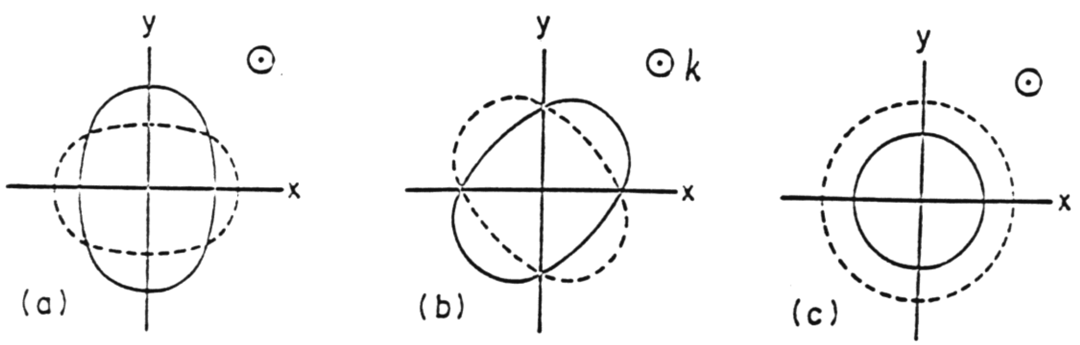

Fig. 8. Wave–induced distortions of a ring of test particles.

In figure 8 we show the resulting positions of an initially circular ring of test particles (at phases $\pi/2$ and $3\pi/2$) for each polarization state: (a) $\hat{h}^{TT}_{xx} = -\hat{h}^{TT}_{yy}$, (b) $\hat{h}^{TT}_{xy} = \hat{h}^{TT}_{yx}$, (c) φ_1. They remain in the plane transverse to the propagation vector $\vec{k}$ shown.

For separations $\vec{\Delta x}$ in the same direction as $\vec{k}$, equation (22) also shows that there is no response to any of the three wave components.

6 Strong-Field Applications

6.1 Compact Objects

Throughout any spherically symmetric spacetime, we can choose Schwarzschild coordinates, in which the interval assumes the form

$$ds^2 = -e^{2\Phi(r,t)}dt^2 + e^{2\lambda(r,t)}dr^2 + r^2[d\theta^2 + \sin^2\theta d\phi^2] \ , \tag{23}$$

so that the proper area (measured by a set of observers at fixed r and t) of any spherical surface is $4\pi r^2$. We consider here isolated bodies, so the metric potentials $\Phi, \lambda \to 0$ as $r \to \infty$. In addition, we shall consider static ($U^i = 0, \ \partial/\partial t = 0$) bodies.

Then the only non-trivial momentum conservation equation $T_r{}^\sigma{}_{;\sigma} = 0$ gives hydrostatic equilibrium:

$$\frac{dp}{dr} = -(\rho + p)\frac{d\Phi}{dr} \ , \tag{24}$$

also indicating that $\Phi(r)$ is the generalized Newtonian potential. The only structural difference from the Newtonian equation is the addition of the pressure to the 'inertial mass-energy density'.

The $\{tt\}$ component of the Einstein field equation gives, after an integration,

$$e^{2\lambda} = \left[1 - \frac{2Gm(r)}{r}\right]^{-1} \ , \qquad m(r) = 4\pi\int_0^r \rho(r_*)r_*^2 dr_* \ . \tag{25}$$

Let $r = R$ be the radius of the star, defined by $p(R) = 0$. Then $m(r \geq R) = M$, the total gravitational mass (defined by applying Kepler's third law to distant orbits).

The $\{rr\}$ component of the Einstein field equation gives the generalization of the Newtonian field equation:

$$\frac{d\Phi}{dr} = G\frac{(m + 4\pi r^3 p)}{r(r - 2Gm)} \ . \tag{26}$$

We see that pressure also contributes to the 'gravitational mass-energy density', and strong fields ($2Gm/r \to 1$) also steepen the potential gradient (and therefore, from equation (24), the pressure gradient). The $\{\theta\theta\}$ and $\{\phi\phi\}$ components of the field equation are redundant.

Finally, if the equation of state assumes the form

$$p = p(\rho, s(\rho)) \quad \text{(specific entropy } s \text{ determined separately)} \ , \tag{27}$$

we have four equations (24, 25, 26, 27) to determine Φ, λ(and m), p, and ρ. For the major applications, white dwarfs and neutron stars, the entropy effectively vanishes since the particles providing the pressure support (electrons or neutrons, respectively) are degenerate.

Continuity of $\Phi(r)$ and $\lambda(r)$ at the stellar surface allows matching to the exterior (or black hole) Schwarzschild solution

$$e^{2\Phi} = e^{-2\lambda} = 1 - 2GM/r \ ,$$

also obtained from the above equations. With the additional boundary condition $m \propto r^3$ as $r \to 0$, a set of stellar models is then a one-parameter [e. g., $\rho_c \equiv \rho(0)$] family. Such equilibrium models are stable when $dM/d\rho_c \geq 0$.

The maximum mass of white dwarfs and neutron stars are of the same order, $M_P^3/m_n^2 = 1.86 M_\odot$, where the Planck mass $M_P \equiv \sqrt{\hbar c/G} = 2.1 \times 10^{-5}$ g and m_n is the nucleon mass. An extensive survey of this subject is provided by Shapiro and Teukolsky.[9]

6.2 Orbits

One of the most powerful probes of the strong gravitational fields near neutron stars and black holes is analysis of the orbits of test particles (or those comprising gaseous accretion disks), here taken to have rest mass $m > 0$. Within the Schwarzschild geometry, we can define the orbital plane as $\theta = \pi/2$. From equation (4), with $p^\mu = m dx^\mu/d\tau$, we then see that both the energy and angular momentum (per unit mass)

$$\tilde{E} = -\frac{p_t}{m} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \ , \qquad \tilde{L} = \frac{p_\phi}{m} = r^2 \frac{d\phi}{d\tau} \ ,$$

are conserved along each orbit. The relation $g_{\mu\nu}p^\mu p^\nu = -m^2$ then gives us the remaining (energy) equation

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \tilde{V}^2(\tilde{L}, r) \ , \qquad \tilde{V}^2 = \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right) \ .$$

As in Newtonian theory, we can understand the orbits via plots of the effective potential $\tilde{V}$, shown in figure 9 for $r > 2GM$ (nonrotating black hole horizon).

For instance, radial turning points occur where $\tilde{V} = \tilde{E}$, and stable (and unstable) circular orbits correspond to the minima (and maxima) of $\tilde{V}$. However, we notice two new (relativistic) features: (a) A particle is captured by the black hole if its specific angular momentum is low enough ($\tilde{L} < 2\sqrt{3}GM$) or its specific energy is high enough ($\tilde{E} > \tilde{V}_{max}$); (b) Stable circular orbits only exist for $\tilde{L} > 2\sqrt{3}GM$, at $r > 6GM$.

For rotating (Kerr) black holes, the metric tensor is much more complicated, but effective potentials can still be obtained[2] for orbits with angular momentum parallel or anti-parallel to that of the black hole. The 'dragging of inertial frames' by the rotation of the mass produces many interesting new effects, such as a region (the 'ergosphere') ouside the horizon where all observers must rotate in the same direction as the hole.

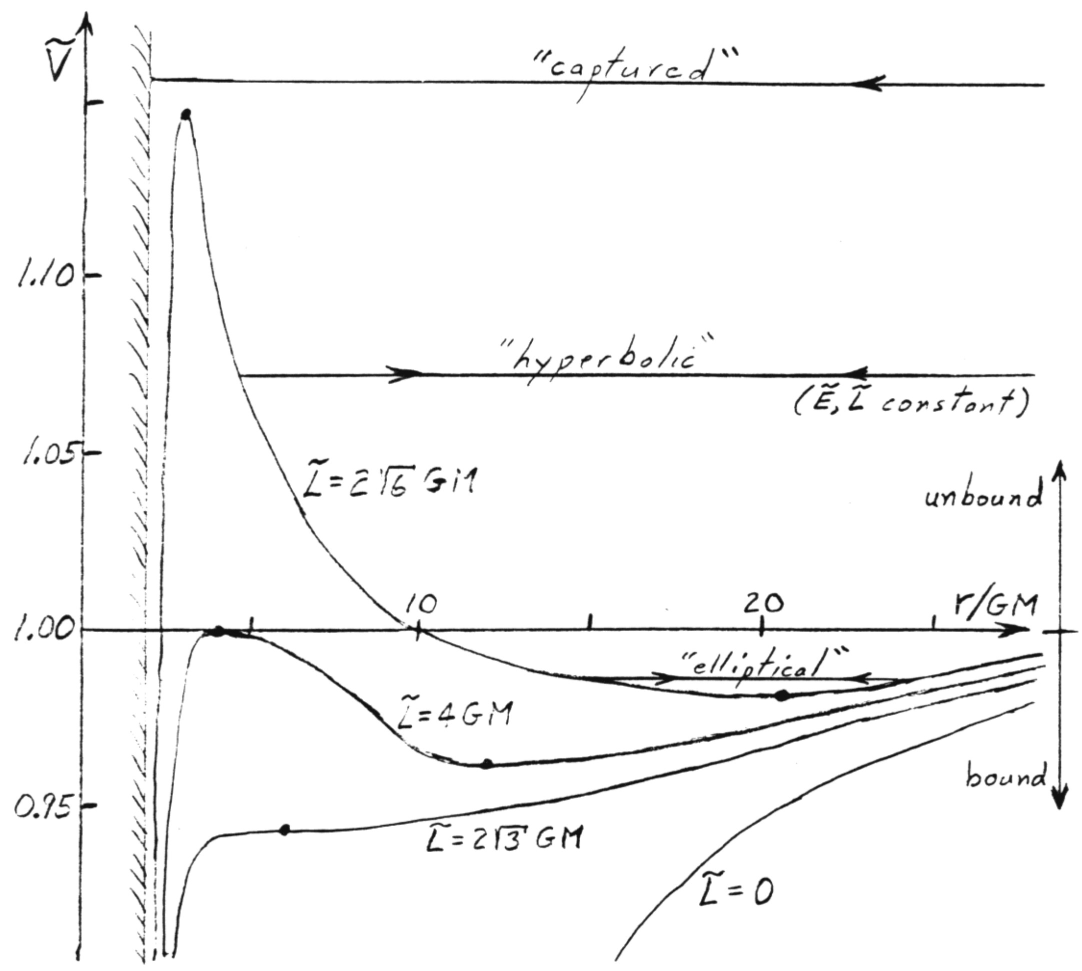

Fig. 9. The effective potential $\tilde{V}$ is plotted for various choices of $\tilde{L}$. Also shown are the three classes of orbits (constant $\tilde{E}, \tilde{L}$).

6.3 Cosmology

6.3.1 Kinematical Analysis

We shall first outline some general features of cosmology within the framework of any metric theory of gravity (incorporating the EEP). The analysis of the large-scale properties and evolution of the universe[10] has at its foundation one key observation, *isotropy*, and one key assumption, the *Copernican Principle.* (By large scale we shall mean distances $D \gtrsim 3 \times 10^8$ light years, roughly 1/30 of the radius of the visible universe.)

We first define a preferred set of observers: those who view the cosmic microwave background (CMB) radiation as isotropic on the corresponding angular scales ($\gtrsim$ 2 degrees). The CMB is isotropic to an accuracy of 10^{-5} (after removing the effect of our motion), the size of the observed intensity variations imprinted by the primordial density fluctuations which formed galaxies, etc. Galaxies (such as ours) have random velocities with respect to this frame such that $\langle v^2/c^2 \rangle^{1/2} \sim 10^{-3}$.

We shall then attach our spatial coordinates to such observers, so they become comoving with respect to the average motion of the matter. We also take the time coordinate to be their proper time, so the averaged (over volumes $\Delta V \sim D^3$) four-velocity of the matter has components $U^\alpha = \delta^\alpha_0$.

The Copernican Principle (CP) asserts that our position in the universe is not special, but typical. The basis for this assumption is the similarity of our galaxy and local group of galaxies to other such systems observed throughout the universe. One then obtains the following consequence[11]:

$$\text{CP} + \text{Observed isotropy} \Rightarrow \text{Universal isotropy} \Rightarrow \text{Homogeneity.}$$

'Universal' means with respect to all comoving observers. The property of large-scale homogeneity has been dubbed the Cosmological Principle. It means that all large-scale properties of the universe depend only on the proper time t of the comoving observers. However, we see from the above that it is not an independent principle; it follows from the CP (so the same acronym is appropriate).

Another consequence of assuming the CP is the form of the metric, which corresponds to the fact that the curvature of the three-spaces $t =$ constant must also be uniform. We adopt the representation in which the interval is

$$ds^s = -dt^2 + a^2(t)[d\chi^2 + \Sigma^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)] \,, \tag{28}$$

where χ is a dimensionless comoving radial coordinate, the curvature $K = k/a^2(t)$, and

$$\Sigma = \begin{cases} \sin\chi\,, & k = +1 \\ \chi\,, & 0 \\ \sinh\chi\,, & -1 \end{cases} .$$

Of course, a flat space ($k = 0$) is also the limit of the other two choices as the radius of curvature $a(t) \to \infty$.

Essentially all of our information about the universe comes to us via (radial) photons. So we integrate equation (28) for $ds = 0 = d\theta = d\phi$ from a comoving emitter (at time t and radial coordinate χ) to us (at time t_0 and $\chi = 0$) to give

$$\chi = \int_t^{t_0} a^{-1}(t')\,dt' \,. \tag{29}$$

Applying equation (29) to two photons separated by one period of oscillation, the fact that χ remains fixed gives the fundamental redshift (Z) formula

$$1 + Z \equiv \lambda_0/\lambda = a(t_0)/a(t) \,. \tag{30}$$

Thus we see that the scale factor $a(t)$ directly stretches wavelengths, as well as determining physical (radar) distances such as the radial one $dl = a(t)d\chi$.

A final consequence of the symmetry induced by the CP plus observed isotropy is the fact that the matter stress-energy tensor $T_{\mu\nu}$ must assume the same form as that for a perfect fluid, given by equation (15). We shall now denote the corresponding effective density and pressure of the matter by ρ_m and p_m.

6.3.2 Dynamical Analysis

In order to understand what governs the dynamics of the evolution of the universe, we take the metric theory of gravity to be within the class of scalar-tensor theories presented above. We again employ the spin representation, in which the spin–2 metric $\hat{\mathbf{g}} = A^{-2}(\varphi)\mathbf{g}$, where $\mathbf{g}$ is the (physical representation) metric that couples directly to matter. However, we can retain the same form of the metric [given by the interval (28)] if we instead employ the time $d\hat{t} = A^{-1}(\varphi)dt$ and scale factor $\hat{a}(\hat{t}) = A^{-1}(\varphi)a(t)$. But we must remember that physical distances and times are given by the interval $ds^2 = A^2(\varphi)d\hat{s}^2$. Correspondingly, physical matter densities and pressures are obtained from $\rho_m = A^{-4}(\varphi)\hat{\rho}_m$ and $p_m = A^{-4}(\varphi)\hat{p}_m$.

The field equations (11) can then be put in the form

$$(3/\hat{a})d^2\hat{a}/d\hat{t}^2 = -4\pi G[\hat{\rho}_m + \hat{\rho}_\varphi + 3(\hat{p}_m + \hat{p}_\varphi)] \ , \quad (31)$$

$$(3/\hat{a}^2)[(d\hat{a}/d\hat{t})^2 + k] = 8\pi G(\hat{\rho}_m + \hat{\rho}_\varphi) \ , \quad (32)$$

while the scalar field equation (12) assumes the form

$$d^2\varphi/d\hat{t}^2 + (3/\hat{a})(d\hat{a}/d\hat{t})d\varphi/d\hat{t} + \tfrac{1}{2}d\Lambda/d\varphi = -4\pi G\alpha(\varphi)[\hat{\rho}_m - 3\hat{p}_m] \ . \quad (33)$$

We have introduced the effective scalar field density and pressure

$$\hat{\rho}_\varphi \equiv (8\pi G)^{-1}[(d\varphi/d\hat{t})^2 + \Lambda(\varphi)] \ , \quad \hat{p}_\varphi \equiv (8\pi G)^{-1}[(d\varphi/d\hat{t})^2 - \Lambda(\varphi)] \ . \quad (34)$$

In addition we have 'microscopic conservation of energy', in the form

$$d[(\hat{\rho}_m + \hat{\rho}_\varphi)\hat{a}^3]/d\hat{t} + (\hat{p}_m + \hat{p}_\varphi)d\hat{a}^3/d\hat{t} = 0 \ . \quad (35)$$

Using equation (35) renders equation (31) redundant, while equation (32) can be put in the form of 'macroscopic energy conservation':

$$\tfrac{1}{2}(d\hat{a}/d\hat{t})^2 + \mathcal{U}(\hat{a}) = -k/2 \ , \quad (36)$$

where the effective gravitational potential $\mathcal{U} = -(4\pi G/3)\hat{a}^2(\hat{\rho}_m + \hat{\rho}_\varphi)$. For a small comoving sphere of radius $R_* = \chi\hat{a}(\hat{t})$ and mass $M_* = (4\pi R_*^3/3)(\hat{\rho}_m + \hat{\rho}_\varphi)$, one finds that $\chi^2\mathcal{U} = -GM_*/R_*$.

Let us consider the present epoch (when $\varphi = \varphi_0$), noting that we can set $A(\varphi_0) \equiv 1$. In addition, we use the fact that its derivative [see equation (13)] is small: $\alpha^2(\varphi_0) \equiv \alpha_0^2 \leq 10^{-3}$. Therefore the spin and matter representations are approximately the same today. The measured Hubble constant is then

$$H_0 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)_0 = \hat{H}_0 + \alpha_0\left(\frac{d\varphi}{dt}\right)_0 \ .$$

The fundamental dynamical equation (36) then assumes the form

$$\Omega_0 + \Omega_K + \Omega_\Lambda + \Omega_{\dot{\varphi}} = [1 - \alpha_0(d\varphi/d\ln a)_0]^2 \ , \quad (37)$$

where we adopt the standard notation

$$\Omega_0 \equiv \frac{8\pi G(\rho_m)_0}{3H_0^2} \ , \quad \Omega_K \equiv -\frac{k}{(a_0H_0)^2} \ , \quad \Omega_\Lambda \equiv \frac{\Lambda(\varphi_0)}{3H_0^2} \ , \quad \Omega_{\dot{\varphi}} \equiv \frac{(d\varphi/dt)_0^2}{3H_0^2} \ .$$

Note that equation (37) differs from the standard result[10] because of the contribution of a possible time dependence of the scalar field via $\Omega_{\dot{\varphi}}$ and the term which makes the right hand side no longer equal to unity. We point this out to illustrate one way in which a dynamical scalar field can impact the comparison of cosmological models with observations.

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