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E-144: The Weizsäcker-Williams Approximation to Trident Production in Electron-Photon Collisions

Abstract

To help judge the validity of the Weizsäcker-Williams approximation to trident production I first compare with an 'exact' calculation for the weak-field case. The agreement is within a factor of 1.5 except close the threshold where the cross section is low. This study also suggests a more invariant formalism than we have been using thus far; a version for the strong-field case is stated.

1 The Weak-Field Case

1.1 Introduction

As a test of the applicability of the Weizsäcker-Williams method we first consider the weak-field case in which only a single initial photon is involved:

$$\omega_0 + e \to e' + e^+ e^-. \tag{1}$$

A complete calculation for this process for unpolarized electrons and photons is available [V. Vortruba, Phys. Rev. **73**, 1468 (1948)] but apparently the analytic form is too complex to be enlightening. A useful summary is given in sec. 11-4 of the book *The Theory of Photons and Electrons* by J.M. Jauch and F. Rohrlich, 2nd ed., (Springer-Verlag, New York, 1976).

Most discussions of reaction (1) use the frame in which the initial electron is at rest. In E-144 the electron is ultrarelativistic in the lab frame. To be able to discuss the process in either frame it is useful to emphasize the relativistic invariants of the problem. In particular, we use

$$s = (\omega_0 + e)^2. \tag{2}$$

where s is the square of the center of mass energy of reaction (1) and in eq. (2) ω_0 and e represent the 4-momenta of the initial electron and photon. The threshold for reaction (1) is $s_{min} = 9m^2$, corresponding to the case when all final-state particles are at rest in the c.m. frame.

Just above threshold the cross section for the trident process (1) varies as

$$\sigma_T = 9.2 \times 10^{-4} \alpha r_0^2 \left(\frac{s - 9m^2}{m^2} \right)^2, \qquad (s - 9m^2 \ll m^2), \tag{3}$$

where $\alpha = e^2/\hbar c$ is the fine structure constant, $r_0 = e^2/mc^2$ is the classical electron radius, m is the electron rest mass and c is the speed of light. Far above threshold the cross section varies as

$$\sigma_T = \alpha r_0^2 \left(\frac{28}{9} \ln \frac{s}{m^2} - \frac{100}{9} \right), \qquad (s \gg 9m^2), \tag{4}$$

In E-144 we are near threshold for trident production, but for us $s - s_{min}$ extends up to $\mathcal{O}(m^2)$. Hence we are between the regions of applicability of the asymptotic relations (3) and (4). A numerical tabulation of the trident cross section has been given by K.J. Mork, Phys. Rev. **160**, 1065 (1967) based on the analytic calculation of Vortruba. Figure 1 (taken from Mork's paper) compares theory and experiment for reaction (1).

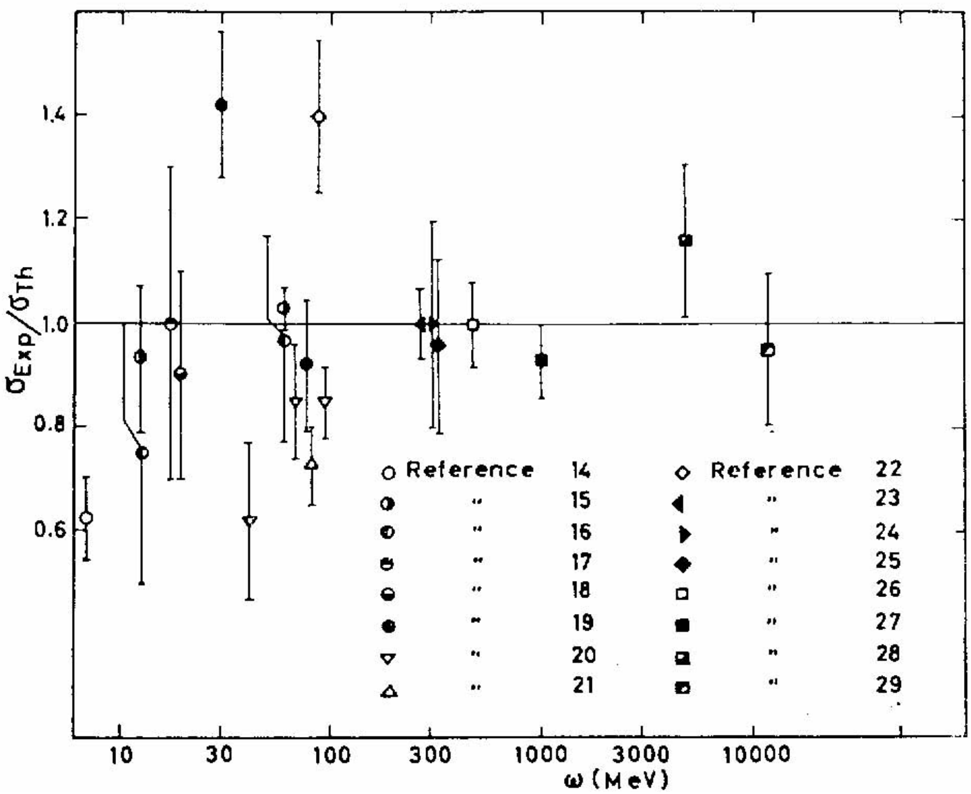

Figure 1: Comparison of theory and experiment for reaction (1). From Mork (1967), which includes references to the experiments.

We will compare the Weizsäcker-Williams approximation to the result of Mork below.

Mork also reported numerical results from a simplified calculation of trident production in which diagrams (b) and (d) of Fig. 2 are neglected [A. Borsellino, Nuovo Cim. **4**, 112 (1947)]. These diagrams are referred to as γ-e or as Compton diagrams in that the initial-state photon couples directly to the initial electron rather than the e^+e^- pair. Well above threshold the Compton diagrams contribute little to the cross section, while near threshold they interfere to reduce the cross section, as summarized in Fig. 3 below. Thus the neglect of the Compton diagrams results in an overestimate of the cross section.

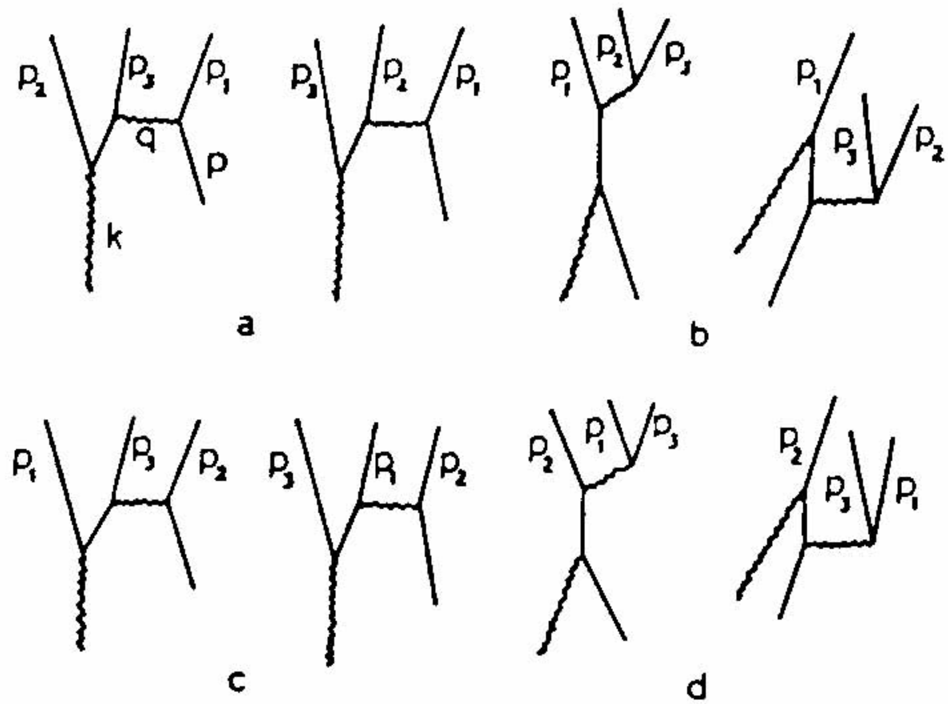

Figure 2: Feynman diagrams for reaction (1). Diagrams (b) and (d) are called γ-e or Compton diagrams.

1.2 The Weizsäcker-Williams Approximation

In a frame in which the initial electron is ultrarelativistic its electric and magnetic fields are nearly transverse and of nearly equal magnitude. That is, the fields appear to be almost identical to those of a packet of real photons, which I will label by ω_1. A Fourier transform of the time dependence of the electron's field integrated over observers at impact parameters $b > b_{min}$ to the electron's trajectory yields the photon number spectrum

$$N(\omega_1) \approx \frac{2\alpha}{\pi\omega_1} \ln \frac{\gamma}{\omega_1 b_{min}}, \tag{5}$$

where γ is the Lorentz factor for the initial electron. See, for example, sec. 15.4 of *Classical Electrodynamics* by J.D. Jackson, 2nd ed., (Wiley, New York, 1975). We then suppose that a (virtual) photon ω_1 from this spectrum combines with the incident photon ω_0 to produce an e^+e^- pair via the Breit-Wheeler process:

$$\omega_0 + \omega_1 \to e^+e^-. \tag{6}$$

The Weizsäcker-Williams approximation to the trident cross section is then

$$\sigma_T = \frac{2\alpha}{\pi} \int_{\omega_{1,min}}^{\omega_{1,max}} \frac{d\omega_1}{\omega_1} \ln \frac{\gamma}{\omega_1 b_{min}} \sigma_{BW}(\omega_0, \omega_1), \tag{7}$$

where σ_{BW} is the Breit-Wheeler cross section.

The Breit-Wheeler cross section can be expressed in terms of s', the square of the center of mass energy of the photon-photon system. For a frame in which the two photons collide head on, $s' = 4\omega_0\omega_1$, where in this expression ω stands for the photon energy. Then

$$\sigma_{BW} = 4\pi r_0^2 \frac{m^2}{s'} \left[\frac{3-\beta^4}{2} \ln \frac{1+\beta}{1-\beta} - \beta(2-\beta^2) \right], \tag{8}$$

where

$$\beta = \sqrt{1 - \frac{4m^2}{s'}} \tag{9}$$

is v/c of the positron (or partner electron) in the pair rest frame. The threshold condition is, of course, $s'_{min} = 4m^2$. The asymptotic forms are

$$\sigma_{BW} \approx \pi r_0^2 \beta, \qquad (\beta \ll 1 \Leftrightarrow s' - 4m^2 \ll m^2), \tag{10}$$

and

$$\sigma_{BW} \approx 2\pi r_0^2 \frac{m^2}{s'} \left(\ln \frac{s'}{m^2} - 1 \right), \qquad (s' \gg 4m^2). \tag{11}$$

See, for example, sec. 13-3 of Jauch and Rohrlich.

Before inserting eq. (8) into (7) the latter should be put into a form that is more manifestly covariant (as Schwinger would say). My approach is to replace ω_1 in (7) by $s' = 4\omega_0\omega_1$. Immediately $d\omega_1/\omega_1 = ds'/s'$.

Then the lower limit of integration, originally $\omega_{1,min}$, becomes $s'_{min} = 4m^2$.

The upper limit of integration becomes s'_{max} for the Breit-Wheeler process embedded in the trident reaction (1). Another interpretation of s' is as the square of the invariant mass of the e^+e^- pair: $s' = m^2_{e^+e^-}$. Then s'_{max} occurs when as much energy as possible goes into the mass of the e^+e^- pair. This occurs when both the pair and the scattered electron e' are at rest in the c.m. frame of reaction (1). In this case

$$s = (m + m_{e^+e^-})^2 = (m + \sqrt{s'_{max}})^2, \qquad \text{and hence} \qquad s'_{max} = (\sqrt{s} - m)^2, \tag{12}$$

where s is the square of the c.m. energy of reaction (1).

Finally we need to reinterpret the argument of the logarithm in eq. (7). It should be an invariant, should be greater than 1, and should have ω_1 in the denominator. The simplest form is then s'_{max}/s'. This could be multiplied by a number of order 1, which is the usual ambiguity of the Weizsäcker-Williams method.

Altogether, the proposed invariant combination of (7) and (8) is

$$\begin{aligned} \sigma_T &= \frac{2\alpha}{\pi} \int_{4m^2}^{s'_{max}} \frac{ds'}{s'} \ln \frac{s'_{max}}{s'} \sigma_{BW}(s') \\ &= 8\alpha r_0^2 \int_{4m^2}^{s'_{max}} \frac{m^2 ds'}{s'^2} \ln \frac{s'_{max}}{s'} \left[\frac{3-\beta^4}{2} \ln \frac{1+\beta}{1-\beta} - \beta(2-\beta^2) \right], \end{aligned} \tag{13}$$

where β and s'_{max} are given by eqs. (9) and (12). This form can be evaluated in a frame in which the initial electron is at rest even though it is unclear that the field of the electron is equivalent to a collection of real photons in this frame.

Figures 3-5 show results of numerical calculations of eq. (13) along with the 'exact' cross section as tabulated by Mork and the cross sections calculated by Borsellino by ignoring diagrams (b) and (d) of Fig. 2. The agreement of the Weizsäcker-Williams approximation and the exact calculation is fairly good, although worst near threshold where the rate is very low.

The results are plotted as a function of the invariant $(s - s_{min})/2m^2$ which is a measure of how far the reaction is above threshold. For the initial electron at rest this invariant is E_{gamma}/m where E_{gamma} is the energy of the initial photon.

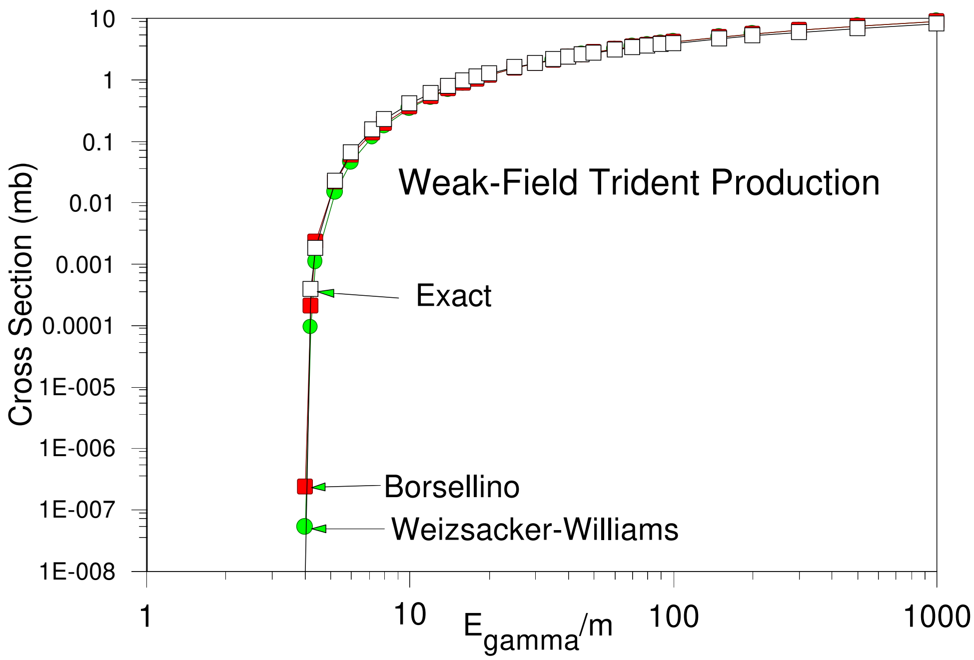

Figure 3: Calculated cross sections for the trident process (1) for the initial electron at rest and initial photon of energy E_{gamma}. Then $E_{gamma}/m = (s - s_{min})/2m^2$, the invariant measure of the energy of the interaction above threshold.

In the Weizsäcker-Williams approximation the initial photon ω_0 interacts only with the e^+e^- pair, not directly with the initial electron. Thus the approximation neglects diagrams (b) and (d) of Fig. 2. This feature is shared with the approximation of Borsellino, and indeed Figs. 3-5 show that the Weizsäcker-Williams approximation tracks Borsellino's results more closely than the 'exact' results.

The Weizsäcker-Williams approximation corresponds to the use of transverse but otherwise unpolarized virtual photons. This feature is shared with the exact calculation of diagrams (a) and (c) of Fig. 2. Only for the neglected diagrams (b) and (d) could there be polarization of the virtual photons in case the initial photon is polarized. For the Compton diagrams (b) and (d) the virtual photons would take on the polarization of the initial photon for energies of the virtual photon near the maximum.

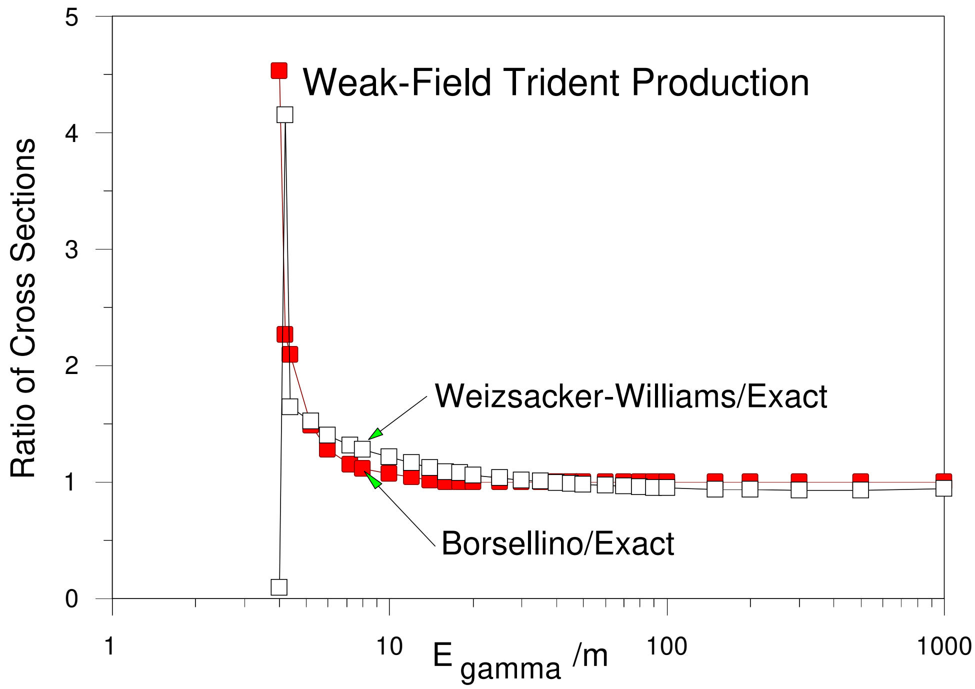

Figure 4: Ratio of calculated cross sections of the trident process (1).

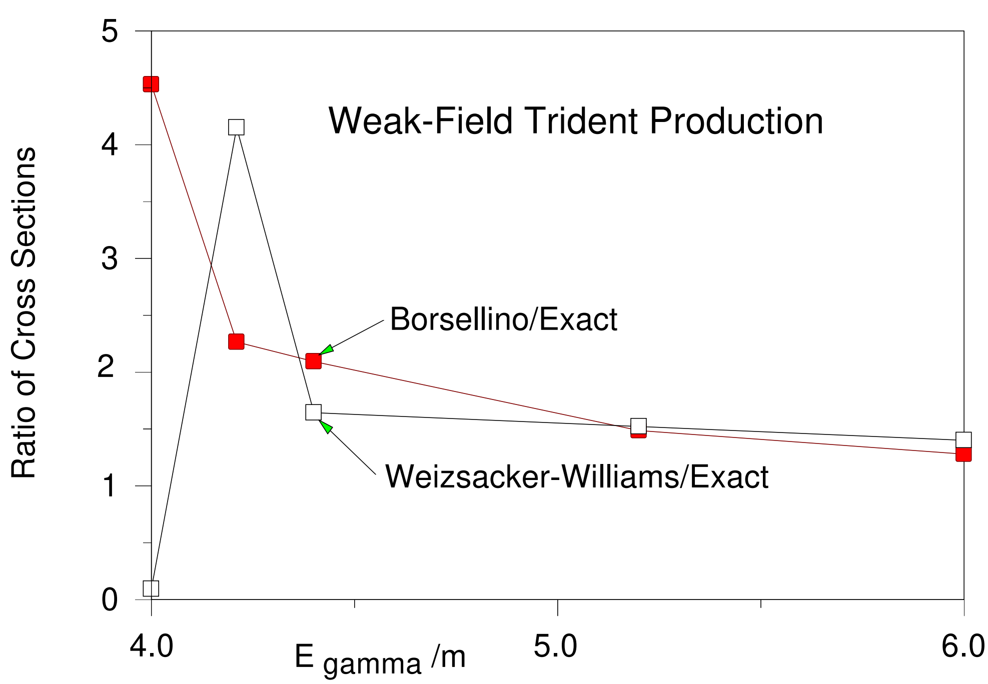

Figure 5: Ratio of calculated cross sections of the trident process (1).

2 The Strong-Field Case

Multiphoton trident production can occur in a strong field of initial-state photons:

$$e + n\omega_0 \to e' + e^+e^-. \tag{14}$$

Extrapolating from eq. (13) I propose the Weizsäcker-Williams approximation for reaction (14) be formulated as

$$Rate_T = \frac{2\alpha}{\pi}\sum_n \int_{4\overline{m}^2}^{s'_{n,max}} \frac{ds'_n}{s'_n} \ln\frac{s'_{n,max}}{s'_n} Rate_{BW}(s'_n, \eta), \tag{15}$$

where $\overline{m} = m\sqrt{1+\eta^2}$ is the shifted mass of the electron in the strong field and $\eta = e\mathcal{E}_{rms}/m\omega c$ is the field-strength parameter. In this we calculate a rate rather than a cross section, using the results of Nikishov and Ritus. For n initial-state (laser) photons the sub-process is the multiphoton Breit-Wheeler reaction

$$n\omega_0 + \omega_1 \to e^+e^-, \tag{16}$$

for which the square of the c.m. energy is

$$s'_n = (n\omega_0 + \omega_1)^2, \tag{17}$$

where in this expression ω stands for the 4-momentum of a photon. Equation (12) becomes

$$s'_{n,max} = (\sqrt{s_n} - \overline{m})^2, \qquad \text{with} \qquad s_n = (e + n\omega_0)^2 = \overline{m}^2 + 2n(e\cdot\omega_0), \tag{18}$$

where e and ω_0 are the initial-state 4-momenta (including mass-shift effects, hence $e^2 = \overline{m}^2$).

If $s'_{n,max} < 4\overline{m}^2$ there is no contribution at order n. This condition can be stated another way. The threshold condition is that the final-state electron and the e^+e^- pair are both at rest in the c.m. frame of the $e + n\omega_0$ system, and that $m_{e^+e^-} = 2\overline{m}$. That is,

$$s_{n,min} = (e + n\omega_0)^2 \geq (\overline{m} + 2\overline{m})^2 = 9\overline{m}^2. \tag{19}$$

For a head-on collision between a relativistic electron and the initial-state photons this becomes

$$\overline{m}^2 + 4nE_0\omega_0 \geq 9\overline{m}^2, \qquad \text{or} \qquad n \geq \frac{2\overline{m}^2}{E_0\omega_0}, \tag{20}$$

where E_0 is the energy of the initial electron. Strictly speaking, $E_0 = q_0$, the quasienergy of the initial electron which is related by

$$q = e + \frac{\eta^2 m^2}{2(e\cdot\omega_0)}\omega_0, \tag{21}$$

where e, q and ω_0 are 4-momenta in this expression. For our example, $q_0 \approx E_0 + \eta^2 m^2/4E_0 \approx E_0$.

For $E_0 = 46.6$ GeV and $\omega_0 = 2.3$ eV we must have $n \geq 5$. As noted in sec. 1.2, this corresponds to all initial-state photons coupling only to the e^+e^- pair, and ignores the strong-field generalizations of diagrams (b) and (d) of Fig. 2.

In the Weizsäcker-Williams approximation the virtual photon ω_1 is unpolarized even if the initial-state photon ω_0 is polarized. Hence the Breit-Wheeler rate used in eq. (15) should be for unpolarized ω_1 but with whatever polarization holds for the initial-state photons ω_0.

2.1 Numerical Results

The above procedures have been implemented in the E-144 simulation.

The requirements (18) and (19) that energy be conserved during pair creation has a striking effect on the calculated rate. First, the minimum number of laser photons is $n = 5$ (and it turns out that there is no significant rate unless $n > 5$). Second, the maximum energy of the virtual photon, $\omega_{1,max}$, that can contribute is much less than the electron beam energy E_0. The latter point can be anticipated by the approximation of head-on collisions for which eq. (18) tells us

$$\omega_{1,max} = E_0 - \frac{\overline{m}^2}{2n\omega_0}\left(\sqrt{1 + \frac{4nE_0\omega_0}{\overline{m}^2}} - 1\right). \tag{22}$$

Some representative values are given in the Table.

Table 1: Maximum energy $\omega_{1,max}$ of a virtual photon that can contribute to the trident process for $E_0 = 46.6$ GeV, $\omega_0 = 2.35$ eV and $\overline{m} = m$ as a function of the number n of laser photons, assuming head-on collisions.

n	6	8	10	12	20
$\omega_{1,max}$ (GeV)	25.1	27.2	28.7	30.0	33.1

Figure 6 shows results of the trident-rate calculation for various numbers of laser photons. The solid curves are the proper results while the dashed curves show the effect of setting $\omega_{1,max}$ to E_0.

Figure 7 shows the contribution to the rate as a function of the invariant measure of energy above threshold. Only the case $n = 6$ is close enough to threshold that the Weizsäcker-Williams approximation is significantly in error, and the sign of the error is to overestimate the rate.

Finally, Figure 8 shows the total rate of trident production (16) as a function of laser intensity parameter η^2 for typical conditions of E-144. Also shown is the calculated rate for pair creation by the two-step process

$$e + n\omega_0 \to e' + \omega_1, \qquad \text{followed by} \qquad \omega_1 + m\omega_0 \to e^+e^-. \tag{23}$$

The trident process is only a 1% correction to the two-step production process.

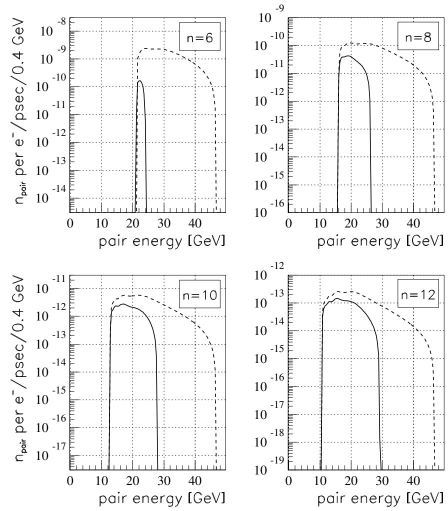

Figure 6: Calculated rates for the trident process (16) in typical conditions of E-144. Solid curves: $\omega_{1,max}$ taken from eq. (18); dashed curves: $\omega_{1,max} = E_0$.

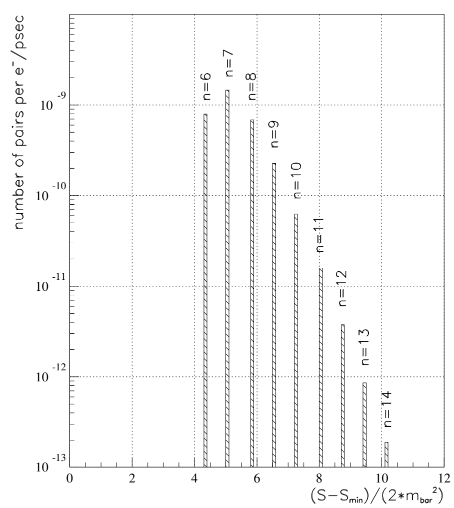

Figure 7: Contributions to the rate of trident production as a function of the invariant measure $(s - s_{min})/2\overline{m}^2$ of energy above threshold.

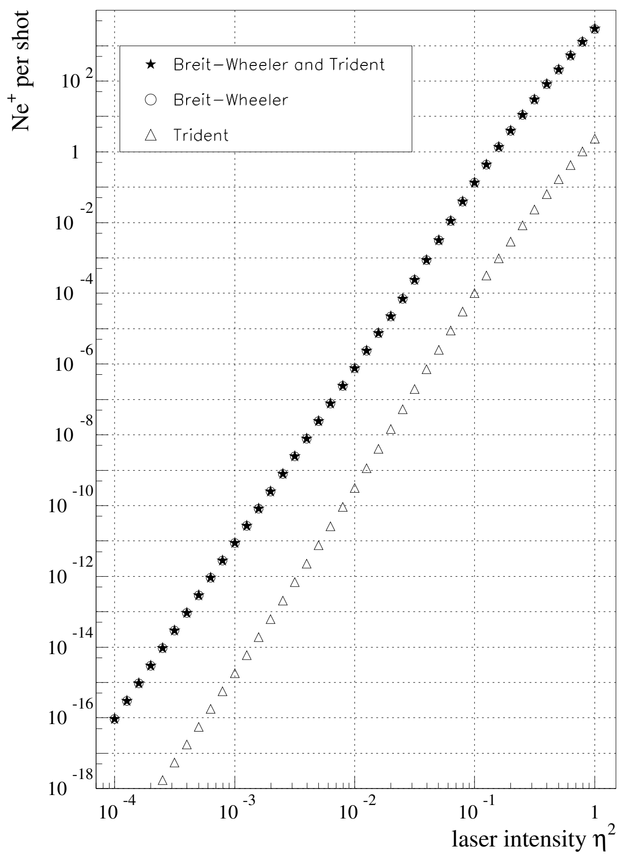

Figure 8: Calculated rates for trident production (16) and two-step pair creation (23) for typical conditions of E-144.