Various calculations for linear polarization and/or tridents

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0.1 Compton scattering off a linearly polarized laser

We are given the 4-vectors k, q, p, and a, and the invariants

$$u = \frac{(kk')}{(kq')}, \quad \lambda = \frac{2(kp)}{m^2}$$

and the value of ϕ, which is defined as the angle between the planes $(\vec{k}, \vec{q'})$ and $(\vec{k}, \vec{a})$ in the center-of-mass system (CMS). The CMS is defined by $n\vec{k} + \vec{q} = 0$, where n is the number of photons (also given). In addition, we have the relations $nk + q = k' + q'$, $q^2 = q'^2 = m_*^2$, and $k^2 = k'^2 = 0$. In terms of the given 4-vectors, invariants, and ϕ, what are

$$\beta = \frac{e^2 a^2}{8} \left(\frac{1}{(kp)} - \frac{1}{(kp')} \right)$$

and

$$\alpha = e \left(\frac{(ap)}{(kp)} - \frac{(ap')}{(kp')} \right),$$

and u_n, the maximum value of u for a given number of photons n?

(You can tell what metric I'm using by the fact that $q^2 = m_*^2 > 0$. Thus, $a^2 < 0$ for a wave, and $\eta^2 = -e^2 a^2 / m^2$. The value of η is a given too, of course.)

For the Compton scattering off a linearly polarized laser, I get

$$u_n = 2n \frac{(kq)}{m_*^2} = \frac{n\lambda}{1 + \eta^2},$$

$$\beta = \frac{\eta^2}{2\lambda} u$$

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beta = eta*eta * u / (2.*lambda);
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and

$$\alpha = \sqrt{8} \frac{\eta}{\sqrt{1+\eta^2}} n \cos\phi \sqrt{\frac{u}{u_n}(1 - \frac{u}{u_n})} = (\sqrt{2}\cos\phi) z$$

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alpha= sqrt(8.*u/un*(1.-u/un)*eta*eta/(1.+eta*eta))*cos(phi)*n;
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where $z \equiv 2n \frac{\eta}{\sqrt{1+\eta^2}} \sqrt{\frac{u}{u_n}(1 - \frac{u}{u_n})}$.

The maximum values are

$$\beta_{max} = \frac{\eta^2}{2(1+\eta^2)} n$$

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betamax = n*eta*eta/(1.+eta*eta)/2.;
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and

$$\alpha_{max} = \sqrt{\frac{2\eta^2}{1+\eta^2}}\, n$$

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alphamax = n*sqrt(2.*eta*eta/(1.+eta*eta));
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0.2 Pair creation by a photon in the field of a linearly polarized laser

This process is related to Compton scattering by the replacement $p \to -p$, $k' \to -l$, where p is now the final state positron's 4-momentum and l is the incident photon which creates the pair. In addition, the definition of the kinematic parameter u will change to be symmetric under exchange of p and p'. Because we may wish to use virtual photons of a fast electron's field to calculate trident rates, I will allow $l^2 \neq 0$.

Thus, we are given the 4-vectors k, l, and a, and the invariants

$$u = \frac{(kl)^2}{4(kq)(kq')}, \quad \lambda = \frac{2(kl)}{m^2}$$

and the value of ϕ, which is defined as the angle between the planes $(\vec{k}, \vec{q}')$ and $(\vec{k}, \vec{a})$ in the center-of-mass system (CMS). The CMS is defined by $n\vec{k} + \vec{l} = 0$. In addition, we have the relations $nk + l = q + q'$, $q^2 = q'^2 = m_*^2$, and $k^2 = 0$. In terms of the given 4-vectors, invariants, and ϕ, what are

$$\beta = \frac{-e^2 a^2}{8}\left(\frac{1}{(kp)} + \frac{1}{(kp')}\right)$$

and

$$\alpha = e\left(\frac{(ap)}{(kp)} - \frac{(ap')}{(kp')}\right),$$

and u_n and n_{min}, the maximum value of u for a given number of photons n and the minimum value of n required to make a pair, respectively?

Note that the definitions of α and β above reflect the replacement $p \to -p$.

My answers:

$$u_n = \frac{n\lambda + l^2/m^2}{4(1+\eta^2)},$$

$$n_{min} = \left\lceil \frac{4(1+\eta^2) - l^2/m^2}{\lambda} \right\rceil,$$

$$\beta = \frac{2\eta^2}{\lambda} u$$

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beta = 2.0*eta*eta * u / lambda;
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and

$$\alpha = \sqrt{8}\frac{\eta}{\sqrt{1+\eta^2}}\left(n + \frac{l^2}{m^2\lambda}\right)\cos\phi\sqrt{\frac{u}{u_n}\left(1 - \frac{u}{u_n}\right)} = \left(1 + \frac{l^2}{nm^2\lambda}\right)(\sqrt{2}\cos\phi)z$$

```
alpha=  sqrt(8.*u/un*(1.-u/un)*eta*eta/(1.+eta*eta))
        *(n+l2_over_m2/lambda)*cos(phi);
```

where $z \equiv 2n\frac{\eta}{\sqrt{1+\eta^2}}\sqrt{\frac{u}{u_n}(1 - \frac{u}{u_n})}$ as for Compton scattering.

The maximum values are

$$\beta_{max} = \frac{\eta^2}{2(1+\eta^2)}\left(n + \frac{l^2}{m^2\lambda}\right)$$

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betamax = (n+l2_over_m2/lambda)*eta*eta/(1.+eta*eta)/2.;
```

and

$$\alpha_{max} = \sqrt{\frac{2\eta^2}{1+\eta^2}}\left(n + \frac{l^2}{m^2\lambda}\right)$$

```
alphamax = (n+l2_over_m2/lambda)*sqrt(2.*eta*eta/(1.+eta*eta));
```

I have written out the equations for β and α in both mathematical symbols and in "computerese" for maximum clarity. In the "computerese" equations, I define `l2_over_m2` $\equiv l^2/m^2$, which is a measure of how far off-shell the incident virtual photon is as compared to the electron mass.

Note that the equations for β in the two processes are identical except for a factor of 4, which is due to the "historical" factor of 4 in the definition of u combined with my choice of how to define λ. The equations for α are identical except for the adjustment $n \rightarrow n + \frac{l^2}{m^2\lambda}$, which allows for the possibility of off-shell incident photons. For $l^2 \ll m^2$, the effect on α, u_n, and n_{min} is small.

0.3 Final states in the lab frame

Compton:

$$E_f = E_i/(1+u)$$

$$p_\perp = m_* \sqrt{u(u_n - u)}/(1+u)$$

Pair production:

$$E_f = E_i(1 \pm \sqrt{1 - 1/u})/2$$

$$p_\perp = m_* \sqrt{u(u_n - u)}/u$$

0.4 The value of $l^2 = (ll)$ for virtual photons of the fast electron's field

This is partly an attempt to address the question of whether it is okay to use the method of virtual quanta to estimate trident pair production rates. The basic idea is to require the original electron to stay on-shell as its virtual photon creates a pair. Is $(ll)/m^2$ always small?[1] Do we need it to be? Is (ll) always reasonable and appropriate?

Consider the case of trident pair production by the field of an electron which does not undergo hard scattering, i.e., no photons are absorbed from the laser by the original electron. Let the original electron's initial and final 4-momentum be given by q and q', respectively, and require that $q^2 = q'^2 = m_*^2$. A virtual photon with 4-momentum $l = q - q'$ creates the pair.

The method of virtual quanta requires us to identify one particle as the source of the field and the other(s) as the "struck" particle or system. Clearly, the electron is the source of the field; apparently, the laser photons or perhaps the sea of potential e+e- pairs itself is the struck system. Unfortunately, the method usually requires us to work in the rest frame of the struck system; the Lorentz-boosted electric field of the source particle (which should be ultra-relativistic in the struck system's rest frame) appears as an E-M pulse traveling parallel to the source particle, and this is the crucial idea behind the method of virtual photons. Obviously, it is unreasonable to work in the rest frame of the laser photons. Probably the best choice would be to work in the rest frame of the created pair, but this rest frame depends on the momentum of the virtual photon, which is a complication not dealt with in the standard formalism. For the moment, let us *assume* that it is okay to work in any reference frame in which the source particle is ultra-relativistic before and after the creation of the pair. Let us call such a reference frame the "lab frame."

(Another complication is that we need to pick a minimum impact parameter, b_{min}, closer than which the process in question either does not occur or is calculated in some other way. I can't think of any *a priori* reason why trident production should be suppressed closer than some distance, although the math below does suggest a possible cutoff due to (ll) acquiring a large negative value

[1] This section will use (ll) instead of l^2 in order to avoid superscript confusion later.

as the transferred momentum increases. For now, we'll ignore the problem of b_{min}, as it doesn't relate to the problem of finding (ll).)

In the lab frame, let's take q, q', l to be all along the e^1 axis. Then

$$q'^2 = (q-l)^2 = {q^0}^2 - {q^1}^2 - 2q^0l^0 + 2q^1l^1 + {l^0}^2 - {l^1}^2$$

$$l^0(l^0 - 2q^0) = l^1(l^1 - 2q^1)$$

$$l^0 = q^0 - \sqrt{{q^0}^2 + l^1(l^1 - 2q^1)}$$

$$(ll) = {l^0}^2 - {l^1}^2 = 2q^0l^0 - 2q^1l^1 = 2\left({q^0}^2 - q^1l^1 - q^0\sqrt{{q^0}^2 + l^1(l^1 - 2q^1)}\right)$$

(Note it is not possible to eliminate both the energy l^0 and the momentum l^1 from the expression for (ll). How far off the mass-shell the virtual photon is will depend on the energy l^0 or forward momentum transferred to the pair l^1. At most one of these can be eliminated.)

The zero-derivative points of (ll) are at $l^1 = 0$ and $l^1 = 2q^1$. At $l^1 = 0$, we have $(ll) = l^0 = 0$. At $l^1 = 2q^1$, we have $(ll) = -4{q^1}^2$ and $l^0 = 0$. The point $l^1 = q^1$ is not an extremum of (ll); in fact, $(ll) \to -\infty$ and $l^0 \to -\infty$ as $l^1 \to \pm\infty$. The most positive value of l^0 is obtained at $l^1 = q^1$, where $l^0 = q^0 - m_*$ and $(ll) = -2m_*(q^0 - m_*)$. In short, (ll) is always negative, while l^0 is positive in the range $0 < l^1 < 2q^1$ and negative for $l^1 > 2q^1$.

Thus, we have an expression for (ll), and we know that the virtual photon is always space-like, but there is no particular reason in the kinematics why the magnitude of (ll) should be less than m_*^2. A suitable choice of b_{min} might provide that condition from the cutoff of the Weizsäcker-Williams spectrum. The fact that (ll) is negative indicates that pair has a deeper potential well out of which to climb, and must absorb more photons from the laser as (ll) increases. This will suppress the pair production at high transferred momentum, which may provide justification for a corresponding minimum impact parameter b_{min}. If this is in fact the case, then the trident rate should be unchanged by reducing b_{min} by any factor, assuming the trident simulation code is working properly.

For $l^1 \ll q^0, q^1$, and $q^0, q^1 \gg m_*$, I find

$$l^0 = l^1\left(1 - \frac{l^1 m_*^2}{{q^1}^3} - \frac{m_*^2}{2{q^1}^2} + \mathcal{O}\left((\frac{l^1 m_*}{q^2})^2\right) + \mathcal{O}\left((\frac{m_*}{q^1})^4\right)\right)$$

$$(ll) = \frac{{l^1}^2 m_*^2}{{q^1}^2}\left(-1 + \frac{m_*^2}{{q^1}^2} - \mathcal{O}\left(\frac{l^1}{q^1}\right) + \mathcal{O}\left(\frac{l^1 m_*^2}{{q^1}^3}\right)\right)$$

Unfortunately, this doesn't give us a very good idea where $|(ll)| \approx m_*^2$. The above expression tells us that $|(ll)| \ll m_*^2$ for $l^1 \ll q^1$, and we know that $|(ll)| \approx 2m_*q^1 \gg m_*^2$ for $l^1 \approx q^1$. However, if we define $l^1 = (1 - 1/r)q^1$, and expand (ll) in powers of m_*/q^1, it turns out that

$$(ll) = \frac{(r-1)^2}{r}m_*^2 + \mathcal{O}(\frac{m_*^4}{{q^1}^2}),$$

so $r \approx 6$ gives $(ll) \approx -4m_*^2$, increasing n_{min} by a factor of 2. So $l^1 \approx \frac{5}{6}q^1$ might be a reasonable cutoff, with corresponding $b_{min} \approx 6\hbar/(5q^1)$. This cutoff does not obey $l^1 \ll q^0$, so the more general (and complicated) expressions for l^0 and (ll) will have to be used in the trident simulation.

The major part of the Weizsäcker-Williams spectrum cuts off at photon energies of $\hbar\omega_{max} = \hbar\gamma/b_{min}$, where $\gamma \equiv q^0/m_*$ is the Lorentz boost of the incident electron in the lab frame. With the value for b_{min} suggested above, this would give $\hbar\omega_{max} = \gamma q^1 \gg q^1$. It seems wrong to have the virtual photon energy spectrum extending up to 10^5 times the source particle's energy, especially when the kinematics indicate that the maximum allowable energy for the virtual photon is $q^0 - m_*$. Since no virtual photon could have an energy higher thant $q^0 - m_*$ and still allow its source particle to stay on-shell, what should be done with that part of the spectrum? Simply discard it? The choice $b_{min} = \hbar/m_*$ would give $\hbar\omega_{max} = q^0$, but would correspond to a transferred momentum $l^1 = m_*/c$, too small for any cut-off effect from the kinematics. I suspect that these problems require us to rederive the method of virtual quanta from the ground up in some way that handles them gracefully.

All in all, I'm not too sure about applying the method of virtual quanta to this process, but hopefully calculating (ll) and incorporating non-zero (ll) into the kinematics of the pair production will help make it a little more defensible. Above all, it will be important to include provisions for b_{min} to be decreased by a factor of 2 or more to confirm the cutoff-independence of the result (note this will put the cutoff for l^1 above q^1, with all that entails!) and to verify the assumption that using the lab frame gives the same pair rates as any other reference frame in which q^0 and q^1 are much bigger than m_*. Even if the simulation results do show frame-invariance and independance of b_{min} below some value, I'll still feel uncomfortable if the spectrum has to be chopped abruptly at $\hbar\omega = q_0$. Perhaps someone can think of a better approach to use for the pair production at impact parameter $b < \hbar/m_*$? The quantity $\hbar/m_*$ is just the Compton wavelength of the (mass-shifted) electron, which is 137 times larger than the electron radius; the potential at that impact parameter is just 1/137 times the electron mass, so that doesn't really help. Hmmm....