Classical "Dressing" of a Free Electron in a Plane Electromagnetic Wave

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The energy and momentum densities of the fields of a free electron in a plane electromagnetic wave include interference terms that are the classical version of the "dressing" of the electron the arises in a quantum analysis. The transverse mechanical momentum of the oscillating electron is balanced by the field momentum resulting from the interference between the driving wave and the static part of the electron's field. The interference between the wave and the oscillating part of the electron's field leads to a longitudinal field momentum and a negative field energy that are difficult to interpret in detail.

I. INTRODUCTION

The behavior of a free electron in a electromagnetic wave is one of the most commonly discussed topics in classical electromagnetism. Yet, several basic issues remain to be clarified. These relate to the question: to what extent can net energy be transferred from an electromagnetic pulse (such as that of a laser) in vacuum to a free electron?

These issues are made more complex by quantum considerations, including the role of the "quasimomentum" of an electron that is "dressed" by an electromagnetic wave [1].

As a small step towards understanding of the larger issues, we consider a simpler question here. The response of a free electron to a plane electromagnetic wave is oscillatory motion in the plane perpendicular to the direction of the wave, in the first approximation. Thus, the electron has momentum transverse to the direction of the wave. However, the wave contains momentum only in its direction, and the radiated wave contains no net momentum (in the nonrelativistic limit). How is momentum conserved in this process?

The general sense of the answer has been given by Poynting [2], who noted that an electromagnetic field can be said to contain a flux of energy (energy per unit area per unit time) given by

$$\mathbf{S} = \frac{c\mathbf{E}\times\mathbf{B}}{4\pi}, \tag{1}$$

in Gaussian units, where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field (taken to be in vacuum throughout this paper) and c is the speed of light.

Poincaré [3] noted that this flow of energy can also be associated with a momentum density given by

$$\mathbf{P}_{\rm field} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}, \tag{2}$$

Hence, in the problem of a free electron in a plane electromagnetic wave we are led to seek an electromagnetic field momentum that is equal and opposite to the mechanical momentum of the electron.

In this paper we demonstrate that indeed the mechanical momentum of the oscillating electron is balanced by the field momentum in the interference term between the incident wave and the static field of the electron. We are left with some subtleties when we consider the interference between the incident wave and the oscillating field of the electron.

II. GENERALITIES

We consider a plane electromagnetic wave that propagates in the $+z$ direction of a rectangular coordinate system. A fairly general form of this wave is

$$\begin{aligned}\mathbf{E}_{\rm wave} &= \hat{\mathbf{x}}E_x\cos(kz-\omega t) - \hat{\mathbf{y}}E_y\sin(kz-\omega t),\\ \mathbf{B}_{\rm wave} &= \hat{\mathbf{x}}E_y\sin(kz-\omega t) + \hat{\mathbf{y}}E_x\cos(kz-\omega t),\end{aligned} \tag{3}$$

where $\omega = kc$ is the angular frequency of the wave, $k = 2\pi/\lambda$ is the wave number and $\hat{\mathbf{x}}$ is a unit vector in the x direction, *etc.* When either E_x or E_y is zero we have a linearly polarized wave, while for $E_x = \pm E_y$ we have circular polarization.

A free electron of mass m oscillates in this field such that its average position is at the origin. This simple statement hides the subtlety that our frame of reference is not the lab frame of an electron that is initially at rest but which is overtaken by a wave [4–6]. If the velocity of the oscillating electron is small, we can ignore the $\mathbf{v}/c\times\mathbf{B}$ force and take the motion to be entirely in the plane $z = 0$. Then, (also ignoring radiation damping) the equation of motion of the electron is

$$m\ddot{\mathbf{x}} = e\mathbf{E}_{\rm wave}(0,t) = e(\hat{\mathbf{x}}E_x\omega t + \hat{\mathbf{y}}E_y\omega t). \tag{4}$$

Using eq. (3) we find the position of the electron to be

$$\mathbf{x} = -\frac{e}{m\omega^2}(\hat{\mathbf{x}}E_x\cos\omega t + \hat{\mathbf{y}}E_y\sin\omega t), \tag{5}$$

and the mechanical momentum of the electron is

$$\mathbf{p}_{\rm mech} = m\dot{\mathbf{x}} = \frac{e}{\omega}(\hat{\mathbf{x}}E_x\sin\omega t - \hat{\mathbf{y}}E_y\cos\omega t). \tag{6}$$

The root-mean-square (rms) velocity of the electron is

$$\dot{x}_{\rm rms} = \sqrt{\langle \dot{x}^2\rangle} = \frac{e}{m\omega}\sqrt{\frac{E_x^2+E_y^2}{2}} = \frac{eE_{\rm rms}}{m\omega c}c. \tag{7}$$

The condition that the $\mathbf{v}/c \times \mathbf{B}$ force can be neglected is then

$$\eta \equiv \frac{eE_{\rm rms}}{m\omega c} \ll 1, \tag{8}$$

where the dimensionless measure of field strength, η, is a Lorentz invariant. Similarly, the rms departure of the electron from the origin is

$$x_{\rm rms} = \frac{eE_{\rm rms}}{m\omega^2} = \frac{\eta\lambda}{2\pi}. \tag{9}$$

Thus, condition (8) also insures that the extent of the motion of the electron is small compared to a wavelength, and so we may use the dipole approximation when considering the fields of the oscillating electron.

The fields associated with the electron can be regarded as the superposition of those of an electron at rest at the origin plus those of a dipole consisting of the actual oscillating electron and a positron at rest at the origin. Thus, we can write the electric field of the electron as $\mathbf{E}_{\rm static} + \mathbf{E}_{\rm osc}$ and the magnetic field as $\mathbf{B}_{\rm osc}$, where the oscillating fields have the pure frequency ω in the low-velocity limit.

The entire electromagnetic momentum density can then be written

$$\mathbf{P}_{\rm field} = \frac{(\mathbf{E}_{\rm wave} + \mathbf{E}_{\rm static} + \mathbf{E}_{\rm osc}) \times (\mathbf{B}_{\rm wave} + \mathbf{B}_{\rm osc})}{4\pi c}. \tag{10}$$

However, in seeking the field momentum that opposes the mechanical momentum of the electron, we should not include either of the self-momenta $\mathbf{E}_{\rm wave} \times \mathbf{B}_{\rm wave}$ or $(\mathbf{E}_{\rm static} + \mathbf{E}_{\rm osc}) \times \mathbf{B}_{\rm osc}$. The former is independent of the electron, while the latter can be considered as a part of the mechanical momentum of the electron according to the concept of "renormalization".

We therefore restrict our attention to the interaction field momentum

$$\mathbf{P}_{\rm int} = \mathbf{P}_{\rm wave,static} + \mathbf{P}_{\rm wave,osc}, \tag{11}$$

where

$$\mathbf{P}_{\rm wave,static} = \frac{\mathbf{E}_{\rm static} \times \mathbf{B}_{\rm wave}}{4\pi c}. \tag{12}$$

and

$$\mathbf{P}_{\rm wave,osc} = \frac{\mathbf{E}_{\rm wave} \times \mathbf{B}_{\rm osc} + \mathbf{E}_{\rm osc} \times \mathbf{B}_{\rm wave}}{4\pi c}. \tag{13}$$

We recall from eq. (6) that the mechanical momentum of the oscillating electron has pure frequency ω. Since the wave and the oscillating part of the electron's field each have frequency ω, the term $\mathbf{P}_{\rm wave,osc}$ contains harmonic functions of ω^2, which can be resolved into a static term plus ones in frequency 2ω. Hence we should not expect this term to cancel the mechanical momentum. Rather, we look to the term $\mathbf{P}_{\rm wave,static}$, since this has pure frequency ω.

III. THE MOMENTUM DENSITY $\mathbf{P}_{\rm WAVE,STATIC}$

The static field of the electron at the origin is, in rectangular coordinates,

$$\mathbf{E}_{\rm static} = \frac{e}{r^3}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}), \tag{14}$$

where r is the distance from the origin to the point of observation. Combing this with eq. (3) we have

$$\begin{aligned}\mathbf{P}_{\rm wave,static} = \frac{e}{4\pi c r^3}\{&-\hat{\mathbf{x}} z E_x \cos(kz - \omega t) \\ &+\hat{\mathbf{y}} z E_y \sin(kz - \omega t) \\ &+\hat{\mathbf{z}}[x E_x \cos(kz - \omega t) - y E_y \cos(kz - \omega t)]\}.\end{aligned} \tag{15}$$

When we integrate this over all space to find the total field momentum, the terms in $\hat{\mathbf{z}}$ vanish as they are odd in either x or y. Likewise, after expanding the cosine and sine of $kz - \omega t$, the terms proportional to $z \cos kz$ vanish on integration. The remaining terms are thus

$$\begin{aligned}\mathbf{p}_{\rm wave,static} &= \int_V \mathbf{P}_{\rm wave,static} \\ &= \frac{e}{4\pi c}(-\hat{\mathbf{x}} E_x \sin\omega t + \hat{\mathbf{y}} E_y \cos\omega t) \int_V \frac{z \sin kz}{r^3} \\ &= \frac{e}{\omega}(-\hat{\mathbf{x}} E_x \sin\omega t + \hat{\mathbf{y}} E_y \cos\omega t) = -\mathbf{p}_{\rm mech},\end{aligned} \tag{16}$$

after an elementary volume integration.

It is noteworthy that the integration is independent of any hypothesis as to the size of a classical electron. Indeed, the integrand of (16) can be expressed as $\cos\theta \sin(kr\cos\theta)/r^2$ via the substitution $z = r\cos\theta$. Hence, the integral over a spherical shell is independent of r for $kr \ll r$, and significant contributions to the integral occur for radii up to one wavelength of the electromagnetic wave. This contrasts with the self-momentum density of the electron which is formally divergent; if the integration is cut off at a minimum radius (the classical electron radius), the dominant contribution occurs within twice that radius.

Thus, we have demonstrated the principal result of this paper.

IV. THE MOMENTUM DENSITY $\mathbf{P}_{\rm WAVE,OSC}$

Several subtleties in the argument appear when we consider the other interference term in the momentum density (11). For this we must first display the electromagnetic fields of an oscillating electron.

A. The Fields $\mathbf{E}_{\rm osc}$ and $\mathbf{B}_{\rm osc}$

Since we restrict our attention to an electron that oscillates with amplitude much less than a wavelength of the driving wave, and the electron attains velocities that are much less than the speed of light, it is sufficient to use the dipole approximation to the fields of the electron. While these fields are well known, they are typically presented in imaginary notation, of which only the real part has physical significance. This notation is very useful for discussions in which only time-averaged behavior is of interest. However, we wish to consider the details of momentum balance at an arbitrary moment, and it is preferable to use purely real notation.

We begin by noting that the retarded vector potential of the oscillating electron at a point $\mathbf{r}$ at time t can be written

$$\mathbf{A}_{\rm osc}(\mathbf{r},t) = \frac{e}{c}\frac{\dot{\mathbf{x}}(t'=t-r/c)}{r} \qquad (17)$$
$$= -\frac{e^2}{m\omega c r}[\hat{\mathbf{x}}E_x\sin(kr-\omega t)+\hat{\mathbf{y}}E_y\cos(kr-\omega t)],$$

using eq. (5) for the motion $\mathbf{x}$ of the electron. The oscillating part of the scalar potential is obtained by integration of the Lorentz gauge condition:

$$\nabla\cdot\mathbf{A}_{\rm osc}+\frac{1}{c}\frac{\partial\phi_{\rm osc}}{\partial t}=0. \qquad (18)$$

We find

$$\phi_{\rm osc} = -\frac{e^2}{m\omega^2}\left\{E_x\left[\frac{kx}{r^2}\sin(kr-\omega t)+\frac{x}{r^3}\cos(kr-\omega t)\right]\right.$$
$$\left.+E_y\left[\frac{ky}{r^2}\cos(kr-\omega t)-\frac{y}{r^3}\sin(kr-\omega t)\right]\right\}. \qquad (19)$$

The constant static potential is omitted in the above.

The electric and magnetic fields are, of course, found from the potentials via

$$\mathbf{B}=\nabla\times\mathbf{A} \qquad \text{and} \qquad \mathbf{E}=-\nabla\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}. \qquad (20)$$

The lengthy expressions for the rectangular components of the fields are

$$B_{{\rm osc},x} = -\frac{e^2E_y}{m\omega^2}\left[\frac{k^2z}{r^2}\sin(kr-\omega t)+\frac{kz}{r^3}\cos(kr-\omega t)\right],$$
$$B_{{\rm osc},y} = -\frac{e^2E_x}{m\omega^2}\left[\frac{k^2z}{r^2}\cos(kr-\omega t)-\frac{kz}{r^3}\sin(kr-\omega t)\right],$$
$$B_{{\rm osc},z} = \frac{e^2E_x}{m\omega^2}\left[\frac{k^2y}{r^2}\cos(kr-\omega t)-\frac{ky}{r^3}\sin(kr-\omega t)\right] \qquad (21)$$
$$+\frac{e^2E_y}{m\omega^2}\left[\frac{k^2x}{r^2}\sin(kr-\omega t)+\frac{kx}{r^3}\cos(kr-\omega t)\right],$$

and

$$E_{{\rm osc},x} = -\frac{e^2E_x}{m\omega^2}\left[\left(\frac{3kx^2}{r^4}-\frac{k}{r^2}\right)\sin(kr-\omega t)\right.$$
$$\left.+\left(\frac{k^2}{r}-\frac{k^2x^2}{r^3}+\frac{3x^2}{r^5}-\frac{1}{r^3}\right)\cos(kr-\omega t)\right]$$
$$-\frac{e^2E_y}{m\omega^2}\left[\frac{3kxy}{r^4}\cos(kr-\omega t)\right.$$
$$\left.+\left(\frac{k^2xy}{r^3}-\frac{3xy}{r^5}\right)\sin(kr-\omega t)\right],$$
$$E_{\rm osc,y} = -\frac{e^2E_x}{m\omega^2}\left[\frac{3kxy}{r^4}\sin(kr-\omega t)\right.$$
$$\left.-\left(\frac{k^2xy}{r^3}-\frac{3xy}{r^5}\right)\cos(kr-\omega t)\right]$$
$$-\frac{e^2E_y}{m\omega^2}\left[\left(\frac{3ky^2}{r^4}-\frac{k}{r^2}\right)\cos(kr-\omega t)\right. \qquad (22)$$
$$\left.-\left(\frac{k^2}{r}-\frac{k^2y^2}{r^3}+\frac{3y^2}{r^5}-\frac{1}{r^3}\right)\sin(kr-\omega t)\right],$$
$$E_{\rm osc,z} = -\frac{e^2E_x}{m\omega^2}\left[\frac{3kxz}{r^4}\sin(kr-\omega t)\right.$$
$$\left.-\left(\frac{k^2xz}{r^3}-\frac{3xz}{r^5}\right)\cos(kr-\omega t)\right]$$
$$-\frac{e^2E_y}{m\omega^2}\left[\frac{3kyz}{r^4}\cos(kr-\omega t)\right.$$
$$\left.+\left(\frac{k^2yz}{r^3}-\frac{3yz}{r^5}\right)\sin(kr-\omega t)\right].$$

B. Components of $\mathbf{P}_{\rm wave,osc}$

Since the wave fields have no z component, the x component of $\mathbf{P}_{\rm wave,osc}$ is given by

$$P_{{\rm wave,osc},x} = \frac{E_{{\rm wave},y}B_{{\rm osc},z}-E_{{\rm osc},z}B_{{\rm wave},y}}{4\pi c}. \qquad (23)$$

From eqs. (21) and (22) we see that both $B_{{\rm osc},z}$ and $E_{{\rm osc},z}$ are odd in either x or y. Therefore, the volume integral of $P_{{\rm wave,osc},x}$ vanishes, and we do not consider it further. Likewise, $P_{{\rm wave,osc},y}$ vanishes on integration. This confirms the claim made at the end of sec. II that the interference term $\mathbf{P}_{\rm wave,osc}$ is not relevant to the balance of transverse momentum between the electron and the fields.

However, the z component of $\mathbf{P}_{\rm wave,osc}$ does not vanish on integration, and requires further discussion. As the details include some surprises (to the author) I present them at length.

$$P_{{\rm wave,osc},z} =$$
$$\frac{E_{{\rm w},x}B_{{\rm o},y}-E_{{\rm w},y}B_{{\rm o},x}+E_{{\rm o},x}B_{{\rm w},y}-E_{{\rm o},y}B_{{\rm w},x}}{4\pi c} =$$
$$-\frac{e^2E_x^2\cos(kz-wt)}{4\pi m\omega^2c}\left[\frac{k^2z}{r^2}\cos(kr-\omega t)-\frac{kz}{r^3}\sin(kr-\omega t)\right]$$
$$-\frac{e^2E_y^2\sin(kz-wt)}{4\pi m\omega^2c}\left[\frac{k^2z}{r^2}\sin(kr-\omega t)+\frac{kz}{r^3}\cos(kr-\omega t)\right]$$
$$-\frac{e^2E_x^2\cos(kz-wt)}{4\pi m\omega^2c}\left[\left(\frac{3kx^2}{r^4}-\frac{k}{r^2}\right)\sin(kr-\omega t)\right.$$

$$
\begin{aligned}
&+\left(\frac{k^2}{r}-\frac{k^2x^2}{r^3}+\frac{3x^2}{r^5}-\frac{1}{r^3}\right)\cos(kr-\omega t)\Bigg] \\
&-\frac{e^2E_xE_y\cos(kz-wt)}{4\pi m\omega^2 c}\left[\frac{3kxy}{r^4}\cos(kr-\omega t)\right. \\
&\left.+\left(\frac{k^2xy}{r^3}-\frac{3xy}{r^5}\right)\sin(kr-\omega t)\right] \qquad (24) \\
&-\frac{e^2E_xE_y\sin(kz-wt)}{4\pi m\omega^2 c}\left[-\frac{3kxy}{r^4}\sin(kr-\omega t)\right. \\
&\left.+\left(\frac{k^2xy}{r^3}-\frac{3xy}{r^5}\right)\cos(kr-\omega t)\right] \\
&+\frac{e^2E_y^2\sin(kz-wt)}{4\pi m\omega^2 c}\left[\left(\frac{3ky^2}{r^4}-\frac{k}{r^2}\right)\cos(kr-\omega t)\right. \\
&\left.-\left(\frac{k^2}{r}-\frac{k^2y^2}{r^3}+\frac{3y^2}{r^5}-\frac{1}{r^3}\right)\sin(kr-\omega t)\right].
\end{aligned}
$$

The terms of $P_{\text{wave,osc},z}$ that are proportional to E_yE_y are odd on both x and y, and so will vanish on integration.

We now consider the implications of eq. (24) separately for waves of circular and linear polarization.

C. Circular Polarization

For a circularly polarized wave, we have $E_x^2 = E_y^2$. Consequently the dimensionless measure of field strength is $\eta = eE_x/m\omega c = eE_y/m\omega c$, according to (8). The prefactors $e^2E_x^2/4\pi m\omega^2 c$ and $e^2E_y^2/4\pi m\omega^2 c$ can therefore both be written $\eta^2 mc/4\pi$, and have dimensions of momentum.

The terms of eq. (24) in E_x^2 and E_y^2 can be combined in pairs via the identities

$$
\begin{aligned}
&\cos(kz-\omega t)\cos(kr-\omega t)+\sin(kz-\omega t)\sin(kr-\omega t) \\
&=\cos kz\cos kr+\sin kz\sin kr, \qquad (25)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sin(kz-\omega t)\cos(kr-\omega t)-\cos(kz-\omega t)\sin(kr-\omega t) \\
&=\sin kz\cos kr-\cos kz\sin kr. \qquad (26)
\end{aligned}
$$

A detail: the second term of eq. (24) in E_x^2 contains factors of x^2, while second term of in E_y^2 contains factors of y^2. But during integration, we can replace y^2 by x^2, after which the terms can be combined via (25-26).

We see already that the volume integral of $P_{\text{wave,osc},z}$ will contain no time dependence!

On integration, terms such as $f(x,r)\sin kz$ and $g(x,r)z\cos kz$ that are odd in z will vanish. The integrated field momentum is thus,

$$
p_{\text{wave,osc},z}=\int_V P_{\text{wave,osc},z}=-\frac{\eta^2 mc}{4\pi}I_1=-\frac{4}{3}\eta^2 mc, \quad (27)
$$

where I_1 is the volume integral whose integrand is

$$
\begin{aligned}
&\frac{k^2z}{r^2}\sin kz\ \sin kr+\frac{kz}{r^3}\sin kz\ \cos kr \\
&+\left(\frac{3kx^2}{r^4}-\frac{k}{r^2}\right)\cos kr\ \sin kr \\
&+\left(\frac{k^2}{r}-\frac{k^2y^2}{r^3}+\frac{3y^2}{r^5}-\frac{1}{r^3}\right)\cos kr\ \cos kr. \qquad (28)
\end{aligned}
$$

We return to the significance of eq. (27) after describing the evaluation of integral I_1.

As seen from eq. (27), the integral I_1 must be dimensionless, although it is apparently a function of the wave number k. However, the form of (28) indicates that I_1 is actually independent of the length scale, so we can set $k = 1$ during integration.

To perform the integration we consider a volume element $r^2dr\ d\cos\theta\ d\phi$ in a spherical coordinate system with angle θ defined relative to the z axis. It is more convenient to keep $z = r\cos\theta$ as a variable of integration, using $dz = rd\cos\theta$. Then the volume integration has the form

$$
\int_V=\int_0^\infty rdr\int_{-r}^{r}dz\int_0^{2\pi}d\phi. \qquad (29)
$$

Most terms of (28) are independent of ϕ, so their ϕ integral is just 2π. For the terms in x^2, we have

$$
\in_0^{2\pi} tx^2\ d\phi=\int r^2\sin^2\theta\cos^2\phi\ d\phi=\pi(r^2-z^2). \qquad (30)
$$

While each of the four main terms of (28) diverges on integration, it turns out that the two terms in $\cos z$ taken together are finite (and likewise for the two terms in $\sin z$). We find that

$$
I_1=I_A+I_B=\frac{16\pi}{3}, \qquad (31)
$$

where

$$
\begin{aligned}
I_A&=2\pi\int_0^\infty dr\ \frac{\sin r}{r}\int_{-r}^{r}dz\ z\sin z \\
&\quad+2\pi\int dr\ \frac{\cos r}{r^2}\int dz\ z\sin z \\
&=4\pi, \qquad (32)
\end{aligned}
$$

and

$$
\begin{aligned}
I_B&=\pi\int dr\ \frac{\sin r}{r}\int dz\ \cos z \\
&\quad-3\pi\int dr\ \frac{\sin r}{r^3}\int dz\ z^2\cos z \\
&\quad+\pi\int dr\ \cos r\left(1+\frac{1}{r^2}\right)\int dz\ \cos z \\
&\quad+\pi\int dr\ \frac{\cos r}{r^2}\left(1-\frac{3}{r^2}\right)\int dz\ z^2\cos z \\
&=\frac{4\pi}{3}. \qquad (33)
\end{aligned}
$$

We are left with the result (27) that the integral of the interference term in the field momentum density has a constant longitudinal term for an electron oscillating in a circularly polarized wave.

This is quite odd! Even for circularly polarized waves that are strong (large η), there always exists a frame in which a free electron's motion is purely transverse to the direction of the wave vector [7]. Our analysis has been performed in this frame.

It appears that the present example is another of the type in which integrals of the Poynting vector lead to a constant momentum that has no physical significance [8,9].

D. Linear Polarization

Consider now the case of a linearly polarized wave with electric field along the x axis. Then $E_{\rm rms} = E_x/\sqrt{2}$, and the prefactors in (24) can be written as $\eta^2 mc/2\pi$.

The remaining terms in the momentum density $P_{\rm wave,osc,\it z}$ have time dependences that can be expressed as sums of pure frequencies via the identities

$$\begin{aligned} &2\cos(kz-\omega t)\cos(kr-\omega t) \\ &= \cos kr\cos kr + \sin kr\sin kr \\ &\quad +(\cos kz\cos kr - \sin kz\sin kr)\cos 2\omega t \\ &\quad +(\cos kr\sin kr + \sin kz\sin kr)\sin 2\omega t, \end{aligned} \tag{34}$$

and

$$\begin{aligned} &2\cos(kz-\omega t)\sin(kr-\omega t) \\ &= \cos kr\sin kr - \sin kr\cos kr \\ &\quad +(\cos kz\sin kr + \sin kz\cos kr)\cos 2\omega t \\ &\quad +(\sin kr\sin kr - \cos kz\cos kr)\sin 2\omega t, \end{aligned} \tag{35}$$

Inserting these into eq. (24) and keeping only those terms that are even in z, we find the integrated field momentum to be

$$\begin{aligned} p_{\rm wave,osc,\it z} &= \int_V P_{\rm wave,osc,\it z} \\ &= -\frac{\eta^2 mc}{4\pi}(I_1 + I_2\cos 2\omega t + I_3\sin 2\omega t), \end{aligned} \tag{36}$$

where integral $I_1 = 16\pi/3$ has been discussed in (28-33),

$$I_2 = -I_A + I_B = -\frac{8\pi}{3}, \tag{37}$$

and integral I_3 has the integrand,

$$\begin{aligned} &\frac{k^2 z}{r^2}\sin kz\ \sin kr - \frac{kz}{r^3}\sin kz\ \cos kr \\ &-\left(\frac{3kx^2}{r^4} - \frac{k}{r^2}\right)\cos kr\ \sin kr \\ &+\left(\frac{k^2}{r} - \frac{k^2y^2}{r^3} + \frac{3y^2}{r^5} - \frac{1}{r^3}\right)\cos kr\ \cos kr. \end{aligned} \tag{38}$$

On evaluation, $I_3 = 0$.

Hence, the longitudinal component of the interference field momentum of a free electron in a linearly polarized wave is

$$p_{\rm wave,osc,\it z} = -\frac{4}{3}\eta^2 mc + \frac{2}{3}\eta mc\cos 2\omega t. \tag{39}$$

While we can dismiss the constant term as having no physical significance as in the case of circular polarization, the term in $\cos 2\omega t$ bears further scrutiny.

It is well known that the motion of a free electron in a linearly polarized wave of frequency ω is not purely transverse, but has a longitudinal component at frequency 2ω [5,7]. The general motion (in the frame in which the electron is at rest on average) is a "figure 8" in the plane containing the electric field vector and the wave vector, the x-z plane in our example. For weak fields as considered here, the motion is

$$x = -\frac{eE_x}{m\omega}\cos\omega t, \qquad z = -\frac{e^2E_x^2}{8m^2\omega^2 c}\sin 2\omega t, \tag{40}$$

so that the z component of the mechanical momentum of the electron is

$$p_{\rm mech,\it z} = m\dot{z} = -\frac{\eta^2 mc}{2}\cos 2\omega t. \tag{41}$$

The term in $p_{\rm wave,osc,\it z}$ at frequency 2ω is $-4/3$ of the longitudinal component of the mechanical momentum associated with the "figure 8" motion of the electron. Thus, we have not been completely successful in accounting for momentum conservation when the small, oscillatory longitudinal momentum is considered. It is possible that the dipole approximation is not accurate enough in this case; but since the discrepancy remains a factor of 4/3 for arbitrarily small η, the difficulty is more likely eleswhere.

The present factor of 4/3 is presumably not the same as the famous factor of 4/3 that often arises in analysis of the electromagnetic energy and momentum of the self fields of an electron [10,11]. A further appearance of a factor of 4/3 in the present example occurs when we consider the field energy of the interference terms.

V. THE INTERFERENCE FIELD ENERGY

It is also interesting to examine the electromagnetic field energy of an electron in a plane wave. As for the momentum density (10), we can write

$$U_{\rm total} = \frac{(\mathbf{E}_{\rm wave} + \mathbf{E}_{\rm static} + \mathbf{E}_{\rm osc})^2 + (\mathbf{B}_{\rm wave} + \mathbf{B}_{\rm osc})^2}{8\pi}, \tag{42}$$

for the field energy density. Again, we no not consider the divergent energies of the self fields, but only the interference terms,

$$U_{\rm int} = U_{\rm wave,static} + U_{\rm wave,osc}, \tag{43}$$

where

$$U_{\rm wave,static} = \frac{\mathbf{E}_{\rm wave} \cdot \mathbf{E}_{\rm static}}{8\pi}. \tag{44}$$

and

$$U_{\rm wave,osc} = \frac{\mathbf{E}_{\rm wave} \cdot \mathbf{E}_{\rm osc} + \mathbf{B}_{\rm wave} \cdot \mathbf{B}_{\rm osc}}{8\pi}. \tag{45}$$

In general, the interference field energy density is oscillating. Here, we look for terms that are nonzero after averaging over time. We see at once that

$$\langle U_{\rm wave,static} \rangle = 0, \tag{46}$$

since all terms have time dependence of $\cos \omega t$ or $\sin \omega t$. In contrast, $\langle U_{\rm wave,osc} \rangle$ will be nonzero as its terms are products of sines and cosines:

$$\begin{aligned}
&U_{\rm wave,osc} = \\
&- \frac{e^2 E_x^2 \cos(kz - wt)}{8\pi m\omega^2} \left[\left(\frac{3kx^2}{r^4} - \frac{k}{r^2} \right) \sin(kr - \omega t) \right. \\
&\qquad \left. + \left(\frac{k^2}{r} - \frac{k^2 x^2}{r^3} + \frac{3x^2}{r^5} - \frac{1}{r^3} \right) \cos(kr - \omega t) \right] \\
&- \frac{e^2 E_x E_y \cos(kz - wt)}{8\pi m\omega^2} \left[\frac{3kxy}{r^4} \cos(kr - \omega t) \right. \\
&\qquad \left. + \left(\frac{k^2 xy}{r^3} - \frac{3xy}{r^5} \right) \sin(kr - \omega t) \right], \\
&+ \frac{e^2 E_x E_y \sin(kz - wt)}{8\pi m\omega^2} \left[\frac{3kxy}{r^4} \sin(kr - \omega t) \right. \\
&\qquad \left. - \left(\frac{k^2 xy}{r^3} - \frac{3xy}{r^5} \right) \cos(kr - \omega t) \right] \\
&+ \frac{e^2 E_y^2 \sin(kz - wt)}{8\pi m\omega^2} \left[\left(\frac{3ky^2}{r^4} - \frac{k}{r^2} \right) \cos(kr - \omega t) \right. \\
&\qquad \left. - \left(\frac{k^2}{r} - \frac{k^2 y^2}{r^3} + \frac{3y^2}{r^5} - \frac{1}{r^3} \right) \sin(kr - \omega t) \right] \\
&- \frac{e^2 E_y^2 \sin(kz - wt)}{8\pi m\omega^2} \left[\frac{k^2 z}{r^2} \sin(kr - \omega t) + \frac{kz}{r^3} \cos(kr - \omega t) \right] \\
&- \frac{e^2 E_x^2 \cos(kz - wt)}{8\pi m\omega^2} \left[\frac{k^2 z}{r^2} \cos(kr - \omega t) - \frac{kz}{r^3} \sin(kr - \omega t) \right].
\end{aligned} \tag{47}$$

The terms in $E_x E_y$ will vanish on integration over volume. The various time averages are

$$\begin{aligned}
&\langle 2\cos(kz - \omega t)\cos(kr - \omega t) \rangle \\
&\qquad = \cos kz \cos kr + \sin kz \sin kr, \\
&\langle 2\sin(kz - \omega t)\cos(kr - \omega t) \rangle \\
&\qquad = \sin kz \cos kr - \cos kz \sin kr, \\
&\langle 2\cos(kz - \omega t)\sin(kr - \omega t) \rangle \\
&\qquad = \cos kz \sin kr - \sin kz \cos kr, \\
&\langle 2\sin(kz - \omega t)\sin(kr - \omega t) \rangle \\
&\qquad = \cos kz \cos kr + \sin kz \sin kr.
\end{aligned} \tag{48}$$

After performing the time average on eq. (47), we keep only terms that are even in z. These terms have the form (28), and so we find that

$$u_{\rm int} = \int_V \langle U_{\rm wave,osc} \rangle = -\frac{e^2 (E_x^2 + E_y^2)}{16\pi m\omega^2} I_1 = -\frac{2}{3}\eta^2 mc^2, \tag{49}$$

for waves of either linear or circular polarization. As with the case of the interference field momentum, this interference field energy is distributed over a volume of order a cubic wavelength around the electron. Being an interference term, its sign can be negative.

We can interpret the quantity,

$$\frac{u_{\rm int}}{c^2} = -\frac{2}{3}\eta^2 m, \tag{50}$$

as a correction to the mass of the electron. This correction depends on η^2, which equals $v_{\rm rms}^2/c^2$ (for small η, recall eq. (7)). Thus, this mass correction is of the same order as the relativistic mass increase. Indeed, a general result for the motion of an electron in a plane wave of arbitrary strength η is that its rms relativistic mass, often called its effective mass, is [4,7]

$$m_{\rm eff} = m\sqrt{1 + \eta^2}. \tag{51}$$

For small η, the increase in mass is

$$\Delta m \approx \frac{1}{2}\eta^2 m. \tag{52}$$

Thus, the mass decrease that we found by considering the energy of the interference terms between the electromagnetic fields of the wave and electron is

$$\frac{u_{\rm int}}{c^2} = -\frac{4}{3}\Delta m. \tag{53}$$

Experience suggests that the mass increase deduced by a relativistic argument is the correct description. Then we infer that the mass decrease, $u_{\rm int}/c^2$, is a component of the electromagnetic mass associated with the electron, which leads to the physical mass by a nonclassical process of "renormalization",

[1] K.T. McDonald, "Comment on "Experimental Observation of Electrons Accelerated in Vacuum to Relativistic Energies by a High-Energy Laser" by G. Malka, E. Lefebvre and J.L. Miquel, Phys. Rev. Lett. **78**, 3314 (1997)", accepted for publication in Phys. Rev. Lett.

[2] J.H. Poynting, "On the Transfer of Energy in the Electromagnetic Field" Phil. Trans. **175**, 343-361 (1884); also, pp. 174-193, *Collected Scientific Papers* (Cambridge U. Press, 1920).

[3] H. Poincaré, "Théorie de Lorentz et le Principe de la Réaction", Arch. Neérl. **5**, 252-278 (1900).

[4] T.W.B. Kibble, "Frequency Shift in High-Intensity Compton Scattering", Phys. Rev. **138**, B740-753, (1965); "Radiative Corrections to Thomson Scattering from Laser Beams", Phys. Lett. **20**, 627-628 (1966); "Refraction of Electron Beams by Intense Electromagnetic Waves", Phys. Rev. Lett. **16**, 1054-1056 (1966); 'Mutual Refraction of Electrons and Photons", Phys. Rev. **150**, 1060-1069 (1966); "Some Applications of Coherent States", *Cargèse Lectures in Physics*, Vol. 2, ed. by M. Lévy (Gordon and Breach, New York, 1968), pp. 299-345.

[5] E.S. Sarachik and G.T.Schappert, "Classical Theory of the Scattering of Intense Laser Radiation by Free Electrons", Phys. Rev. D **1**, 2738-2753 (1970).

[6] K.T. McDonald and K. Shmakov, "Temporary Acceleration of Electrons While Inside an Intense Electromagnetic Pulse", submitted to Phys. Rev. Lett.

[7] L. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon Press, Oxford, 1975), prob. 2, § 47 and prob. 2, § 49; p. 112 of the 1941 Russian edition.

[8] R.H. Romer, "Question #26. Electromagnetic Field Momentum", Am. J. Phys. **63**, 777-779 (1995).

[9] K.T. McDonald, "Answer to Question #26 ["Electronmagnetic field momentum," Robert H. Romer, Am. J. Phys. **63**(9), 777-779 (1995)]", Am. J. Phys. **64**, 15-16 (1995).

[10] J. Schwinger, "Electromagnetic Mass Revisited", Found. Phys. **13**, 373-383 (1983).

[11] P. Moylan, "An Elementary Account of the Factor of 4/3 in the Electromagnetic Mass", Am. J. Phys. **63**, 818-820 (1995).