

Lattice Calculations and Hadron Physics

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1 Introduction

Lattice QCD aims to understand the strong interaction of hadrons from the first principles of QCD for quarks and gluons with the aid of numerical simulations. The physical quantities calculated with the method are many, ranging from the spectrum of light hadrons to a variety of electroweak matrix elements. Practical limitations such as finite lattice volume and spacing, and the use of the quenched approximation of ignoring sea quarks are gradually being lifted due to development of computer power, particularly those of dedicated parallel computers. In this talk, we review progress in lattice QCD, focusing on efforts to calculate weak matrix elements relevant for the determination of the Cabibbo-Kobayashi-Maskawa (CKM) matrix.

We start the review with discussion of the latest results for the light hadron spectrum and quark masses in Section 2. In addition to being a topic basic to all of lattice QCD, the recent progress in the quenched and full QCD calculations has yielded an interesting result, relevant for the CKM determination as well, that dynamical sea quarks have a significant effect on the value of the strange quark mass.

The CP violation in K meson decays continues to be an active topic with recent experimental reports on ε'/ε [1, 2]. In Section 3 we discuss the present status of the lattice calculation of the K meson mixing parameter B_K and the decay amplitudes including ε'/ε , for which there has been recent renewed attention.

Understanding physics of B mesons is very important for the determination of the CKM matrix in the Standard Model and to detect the physics beyond it. It will be even more so as detailed data from the B factories, BaBar [3] and Belle [4], become available. In Section 4 we present the lattice effort for calculating the weak matrix elements of heavy-light mesons. After a brief discussion on the theoretical issues of heavy quark calculations on a lattice, we present recent results on the B meson decay constant f_B , the mixing parameter B_B , and the form factors

of semi-leptonic decays. For the decay constant, serious attempts exist already to detect sea quark effects in its value. The other, more complicated quantities, are still calculated in the quenched approximation. For technical details and recent developments of heavy quark physics on the lattice, we refer to recent reviews at the lattice conferences [5, 6].

In Section 5 we sketch to what extent the lattice results help constrain the CKM matrix determination. We conclude with a brief summary in Section 6.

2 Light hadron spectrum and quark masses

2.1 Light hadron spectrum

Calculation of the light hadron spectrum has been pursued since 1981 because an agreement of the calculated spectrum with experiment provides a fundamental confirmation for the validity of QCD in the non-perturbative low-energy domain, and also gives a measure of the reliability of the calculational techniques.

A recent precision result of the CP-PACS Collaboration [7] for the light hadron spectra in the continuum limit within the quenched approximation is shown in Fig. 1. The up and down quarks are assumed to be degenerate ($m_u = m_d \equiv m_l$) in this calculation, and the experimental ρ and neutral π meson masses are used to fix the scale (lattice spacing a) and the light quark mass (m_l). For the strange quark mass, two choices are compared, one employing the K meson mass (filled symbols; K -input) and other with the ϕ meson mass (open symbols; ϕ -input). Experimental values are shown by horizontal lines.

This result shows an overall agreement of the light hadron spectrum in the quenched lattice QCD at a 5–10% level, as previously demonstrated by the GF11 Collaboration [8]. However, it is also clear that there is systematic deviation between the quenched spectrum and experiment beyond the calculational error of 2–3%. In particular, the hyperfine splitting between the ϕ , K^* meson masses, and the K meson mass is smaller than the experimental one.

As is shown in Fig. 2, however, the K^* and ϕ meson masses from K input agree much better in the 2 flavor full QCD [9] than in the quenched QCD with the experimental values after taking the continuum limit. The effect of dynamical sea quarks is really important for reproducing the correct spectra.

2.2 Light quark masses

The masses of quarks are fundamental parameters of the Standard Model. Because of quark confinement, they can only be determined indirectly from a comparison of experimental hadron masses and their theoretical prediction in terms

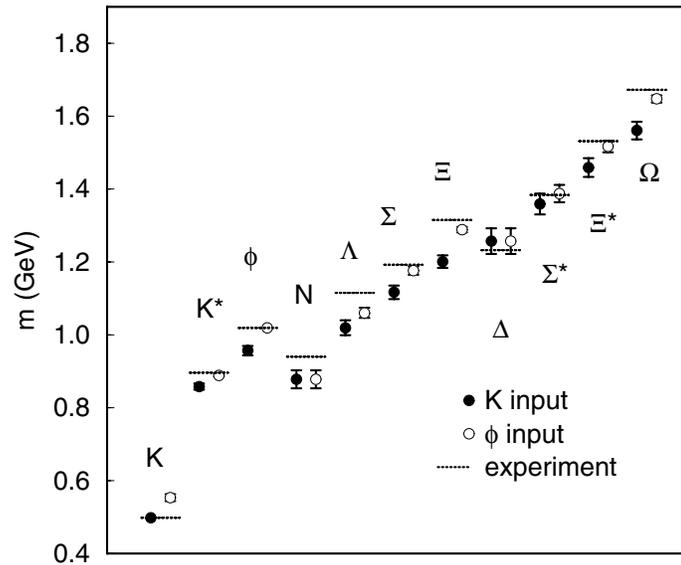
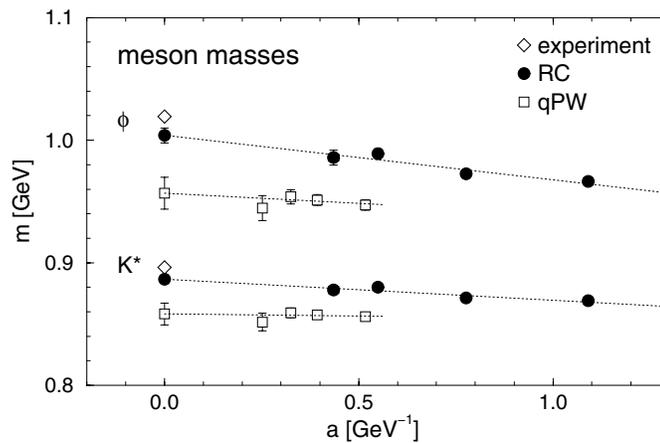


Figure 1: Light hadron spectra in quenched QCD.

Figure 2: ϕ and K^* meson masses from K input as a function of a in full QCD (filled symbols), together with the quenched results (open symbols).

of quark mass parameters. For these reasons, much effort in lattice calculations have been devoted for their extraction. In this section, we discuss the determination of strange quark mass m_s which, as we shall see below, has an important impact on the ε'/ε in K meson decays.

A summary of results for $m_s^{\overline{MS}}(2 \text{ GeV})$ in quenched QCD with the Wilson quark

action [10], compiled by Bhattacharya and Gupta in 1997 [11], is shown in Fig. 3 as a function of a . Also shown are linear continuum extrapolations as explained in the caption.

From this analysis for results with the Wilson quark action, and similar ones for the other types of quark actions, they concluded that their best estimate of the \overline{MS} mass at 2 GeV in quenched QCD was $m_s^{\overline{MS}}(2\text{GeV}) = 110(20)(11)\text{MeV}$, where the first error is estimated from the difference in the results among Wilson, Clover [12], and KS [13] quark actions, and the second one is the uncertainty of the scale ($1/a$) determination. This value is already close to the lower bound from QCD sum rules [14], $m_s^{\overline{MS}}(2\text{ GeV}) \geq 90 - 100\text{ MeV}$.

Two problems were manifest in their analysis. First, as can be seen from the figure, the large spread of data at finite a makes reliable continuum extrapolation difficult. This difficulty can be overcome by high precision calculations. Indeed, the CP-PACS data in the figure is precise enough for a reasonable linear extrapolation.

A more difficult problem, made quite clear by the CP-PACS precision data, is that the value of strange quark mass from K -input disagrees with the one from ϕ -input, even in the continuum limit. This is one of the manifestations of the quenching error.

The full QCD result with 2 flavors of dynamical quarks from the CP-PACS collaboration this year [9] is shown in Fig. 4. Here, the strange quark mass $m_s^{\overline{MS}}(2\text{GeV})$, defined through the axial-vector Ward-Takahashi identity, is plotted as a function of a . The filled symbols in the figure are full QCD results [9] obtained with the Clover quark action and renormalization group (RG) improved gauge action, while the open ones are the previous quenched results with the Wilson quark action and the ordinary plaquette gauge action [7]. Results from both K -input (circles) and ϕ -input (squares) are given as in the previous figure. Numerically, we obtain in the continuum limit that $m_s^{\overline{MS}}(2\text{ GeV}) = 89.3(6.6)\text{ MeV}$ from K -input and $93.7(8.8)\text{ MeV}$ from ϕ -input in the 2 flavor full QCD, while $115(2)\text{ MeV}$ and $143(6)\text{ MeV}$ are the values for the K - and ϕ -input, respectively, in quenched QCD.

It is very encouraging to observe that strange quark masses from two different inputs agree with each other within errors in the continuum limit. This reflects the fact, discussed already, that the hyperfine splitting among strange mesons agrees well with experiment in the 2 flavor full QCD.

It is surprising, however, that the quenching errors on the strange quark mass are larger than expected: 20% for K -input and 40% for ϕ input. Moreover, the strange quark mass in full QCD is much smaller than the previous quenched one, and is very close to the lower bound from QCD sum rules [14].

We quote the averaged value from the two inputs,

$$m_s^{\overline{MS}}(2\text{GeV}) = 91(13)\text{MeV}, \quad (1)$$

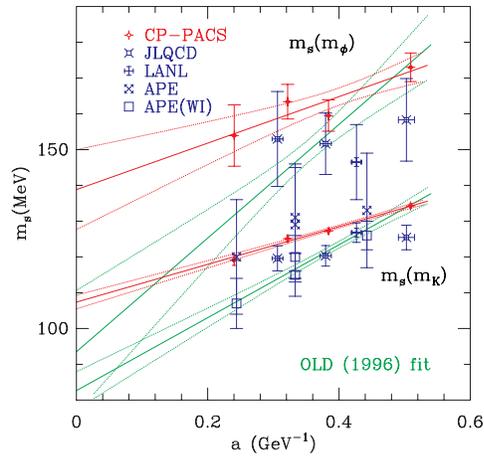


Figure 3: Strange quark mass in quenched QCD with Wilson quark actions. Two continuum extrapolations are shown for each input: one is the fit using all but the CP-PACS data, the other is the fit using only the CP-PACS data.

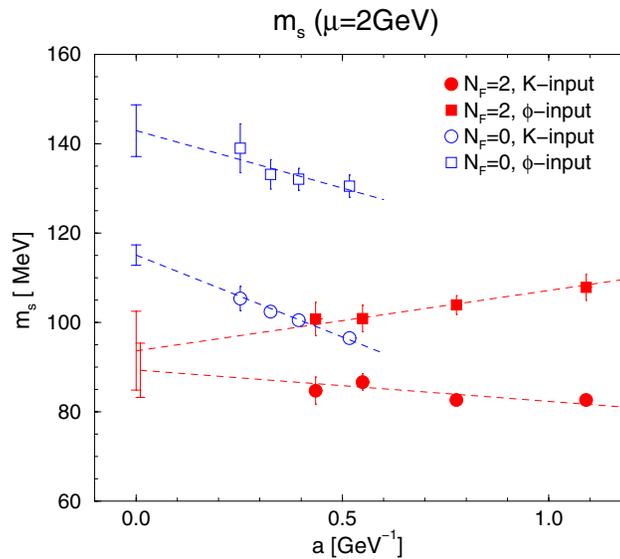


Figure 4: Strange quark mass in full QCD with Clover quark action (filled symbols) as well as the one in quenched QCD with Wilson quark action (open symbols), from K -input (circles) and ϕ -input (squares).

as the CP-PACS result for the strange quark mass at 2 GeV in 2 flavor full QCD, which is equivalent to $m_s^{\overline{MS}}(m_c) = 100(15)$ MeV at the scale of the charm quark mass.

Let us comment on the systematic uncertainties in this result: (i) The 2 flavor QCD simulation still neglects the dynamical effect of the strange quark itself. Although this effect is expected to be smaller than that from zero to 2 dynamical flavors, the value of the strange quark mass may be further reduced. (ii) The lightest dynamical sea quark masses used in the simulation corresponds to a π to ρ mass ratio $m_\pi/m_\rho = 0.55$. One has to extrapolate the result obtained at these heavier quark masses to the physical light quark mass, where $m_\pi/m_\rho = 0.18$. There may be systematic uncertainties associated with opening of decay channels such as $\rho \rightarrow 2\pi$ which are difficult to assess for the current range of heavy sea quarks. (iii) For the renormalization factor to convert the bare quark mass on the lattice into the renormalized \overline{MS} quark mass in the continuum, the perturbative value calculated to one-loop order is used. It is certainly desirable to use the renormalization factor calculated non-perturbatively [15]. For the present analysis, effects of higher order terms that have been neglected are partly estimated by comparison of the quark mass evaluated through the axial-vector Ward-Takahashi identity and those from the mass parameter in the action. The influence on $m_s^{\overline{MS}}(2 \text{ GeV})$ is found to be 2-3 MeV, which has already been included in the estimate of the central value and the error. (iv) There exists also an uncertainty in the continuum extrapolation. Variations of the strange quark mass by a change of the ansatz in the continuum extrapolation, from linear in a to, for example, linear plus quadratic in a or $a \cdot \alpha_{\overline{MS}}(1/a)$, are found to be 3-6 MeV. This has already been included in the error estimation.

3 CP-violation in K meson decay

3.1 K_0 - \overline{K}_0 mixing parameter B_K

The indirect CP-violation in K meson decays is parameterized by ε , which is related to the CKM matrix elements by

$$\eta [(1 - \rho)A + B] \hat{B}_K = \frac{|\varepsilon|}{C}, \quad (2)$$

where ρ and η are the Wolfenstein parameters, and A , B , and C are numerical constants. \hat{B}_K is the renormalization group invariant form of B_K , the K_0 - \overline{K}_0 mixing parameter, which is defined as the K meson matrix element of the $\Delta S = 2$ four

quark operator:

$$B_K = \frac{\langle \bar{K}_0 | \bar{s} \gamma_\mu (1 - \gamma_5) d \cdot \bar{s} \gamma_\mu (1 - \gamma_5) d | K_0 \rangle}{\frac{8}{3} \langle \bar{K}_0 | \bar{s} \gamma_\mu (1 - \gamma_5) d | 0 \rangle \langle 0 | \bar{s} \gamma_\mu (1 - \gamma_5) d | K_0 \rangle}. \quad (3)$$

Thus, the lattice calculation of B_K provides an important constraint on the CKM matrix.

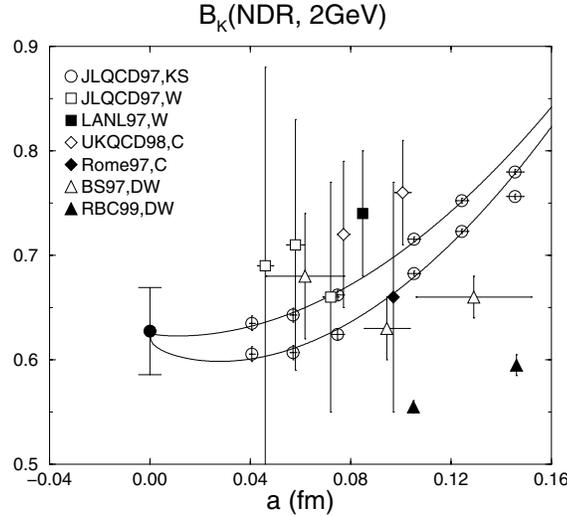


Figure 5: $B_K(\text{NDR}, 2 \text{ GeV})$ in quenched QCD with KS, Wilson/Clover and Domain-wall quark actions, as a function of the lattice spacing a . Two solid lines represent the continuum extrapolation by simultaneously fitting B_K from two different operators of the KS quark, with the constraint that two agree in the continuum limit.

In Fig. 5 we summarize the quenched lattice results for $B_K(\text{NDR}, 2 \text{ GeV})$, renormalized at 2 GeV in the \overline{MS} scheme with the naive dimensional regularization (NDR), as a function of lattice spacing. These results have been obtained with a variety of lattice quark actions. The Wilson quark action [10, 16, 17] breaks chiral symmetry explicitly, while the improved Wilson quark action (Clover or SW quark action) [12, 18, 19] reduces the magnitude of the chiral symmetry breaking. On the other hand, the Kogut-Susskind (KS) quark action [13] retains a $U(1)$ subgroup of chiral symmetry, which is sufficient to guarantee the correct chiral behavior of the B_K parameter. Finally, the domain-wall (DW) quark action [20], recently developed and employed in lattice QCD calculations [21, 22], maintains both chiral and flavor symmetries at the expense of introducing the extra dimension. After taking the continuum limit, all formulations should give the same answer.

As seen from the figure, the result from the KS fermion approach of the JLQCD Collaboration [23] is the most extensive and statistically accurate. They employ two different lattice operators for B_K , and matching to the continuum operator is made with the one-loop renormalization factor. In order to take into account the 2-loop ambiguity, in addition to the a^2 scaling violation of the KS quark action, they fit the results of B_K from two operators simultaneously to the form

$$B_K(a) = B_K + b a^2 + c \alpha_{\overline{MS}}(1/a)^2, \quad (4)$$

with a unique value of B_K in the continuum limit. In the figure, the fit is represented by solid lines. By this procedure, the JLQCD collaboration obtain

$$B_K(\text{NDR}, 2\text{GeV}) = 0.628(42), \quad (5)$$

which corresponds to $\hat{B}_K = 0.87(6)$. The error in the continuum limit is much larger than those of individual data at non-zero a , due to the two-loop ambiguity α^2 in Eq. (4). More accurate results will require precision determination of the renormalization factor in some non-perturbative way.

So far, the degenerate quark masses, $m_s = m_d$, have to be used to avoid a quenched pathology that B_K diverges in the limit that $m_s \neq m_d \rightarrow 0$. A 4 to 8% increase of B_K is predicted by chiral perturbation theory (χ PT) for non-degenerate, physical quark masses [24]. As far as the quenching error is concerned, the value of B_K in 3 flavor full QCD is found to increase by 5% [25]. However, this should be considered as a preliminary estimate, because no chiral and continuum extrapolations have been made in the calculation, and the degenerate quark masses are still used.

3.2 Direct CP-violation

The direct CP-violation in K meson decays parameterized by ε' is important because a non-zero value of ε' strongly supports that the complex phase of the CKM matrix is the source of CP-violation in the Standard Model. Recent experiments consistently suggest that ε'/ε is non-zero, and the world average of experimental values is now $21.3(2.8)\times 10^{-4}$ [1].

In order to see whether this value is consistent with the standard model prediction or suggests new physics, one has to theoretically calculate ε'/ε within the framework of the Standard Model. An approximate formula takes the form [26]

$$\begin{aligned} \varepsilon'/\varepsilon &\simeq \text{Im}(\lambda_t) \times 13 \cdot \left[\frac{110\text{MeV}}{m_s(2\text{GeV})} \right]^2 \\ &\times \left[B_6^{1/2}(1 - \Omega_{\eta+\eta'}) - 0.4 \cdot B_8^{3/2} \left(\frac{m_t}{165\text{GeV}} \right)^{2.5} \right] \left(\frac{\Lambda_{\overline{MS}}^{(4)}}{340\text{MeV}} \right), \quad (6) \end{aligned}$$

where $B_{6(8)}^{1/2(3/2)}$ is the matrix element of the QCD (electroweak) penguin operator in the $\Delta I = 1/2(3/2)$ K meson decay, normalized by the vacuum insertion estimate, $\lambda_t = V_{td}V_{ts}^*$, and $\Omega_{\eta+\eta'}$ is the suppression factor by the isospin breaking effect in the quark masses. Because this formula indicates that the ε'/ε increases if the strange quark mass decreases, it is interesting to calculate its value using the lattice result of m_s obtained in full QCD, which is smaller than the previous estimate.

We have evaluated ε'/ε employing a more precise formula [26] and taking the standard estimate of parameters that $B_6^{1/2} = 1.0(3)$, $B_8^{3/2} = 0.8(2)$, $\Lambda_{\overline{MS}}^{(4)} = 340(50)$ MeV, $\text{Im}\lambda_t = 1.33(14) \times 10^{-4}$ and $\Omega_{\eta+\eta'} = 0.25(8)$. For comparison, we use two values of the strange quark mass, $m_s^{\overline{MS}}(m_c) = 100(20)$ MeV from the lattice result and 130(25) MeV often employed in phenomenology. Here we conservatively add 5 MeV to the error of lattice result.

We list the result in Table 1 varying the strange quark mass within the estimated error band; the other parameters are set to be their central values. Taking ambiguities in other parameters into account, the estimation in the Standard Model for ε'/ε , for a small strange quark mass indicated by the recent 2 flavor lattice QCD calculations, is still consistent with the experimental value.

$m_s(m_c) = 100(20)$ MeV: this talk		$m_s(m_c) = 130(25)$ MeV: Ref. [26]	
$m_s(m_c)$	ε'/ε	$m_s(m_c)$	ε'/ε
80 MeV	21.2×10^{-4}	105 MeV	11.5×10^{-4}
100 MeV	13.0×10^{-4}	130 MeV	7.0×10^{-4}
120 MeV	8.5×10^{-4}	155 MeV	4.4×10^{-4}

Table 1: The strange quark mass and ε'/ε

It should be noted that only the strange quark mass is varied here while other parameters are kept fixed. This has inherent uncertainties because other parameters such as $B_6^{1/2}$ and $B_8^{3/2}$ also depend on the strange quark mass. (See [27] for a quark mass independent parameterization of these matrix elements.) For a complete prediction for ε'/ε from the Standard Model, we need to evaluate $B_6^{1/2}$ and $B_8^{3/2}$ also by lattice calculations. This possibility will be considered next.

The direct calculation of the $K \rightarrow \pi\pi$ amplitude, in particular for the $\Delta I = 1/2$ case, is notoriously difficult, mainly due to the euclidean nature of space-time on the lattice [28]. Instead of the direct calculation, one may evaluate $K \rightarrow \pi$ amplitudes, which are related to the $K \rightarrow \pi\pi$ amplitude at the lowest order of χ PT as follows [29]

$$\langle \pi^+ \pi^- | O_i | K^0 \rangle = \langle \pi^+ | O_i - \alpha_i O_{sub} | K^+ \rangle \cdot \frac{m_K^2 - m_\pi^2}{m_M^2 f_M}, \quad (7)$$

where O_i is a $\Delta S = 1$ four quark operator, and α_i is determined by

$$0 = \langle 0 | O_i - \alpha_i O_{sub} | K^0 \rangle \quad (8)$$

and $O_{sub} \equiv (m_d + m_s)\bar{s}d + (m_d - m_s)\bar{s}\gamma_5 d$, which can mix with O_i under operator renormalizations. Here m_K and m_π are physical, the K and π meson masses, while m_M is the unphysical degenerate K and π meson mass used in the lattice calculation of the $K \rightarrow \pi$ matrix element. Therefore, the left-hand side of Eq. (7) should be independent on m_M at the lowest order of χ PT. For this method to work, chiral symmetry of the lattice quark action is crucially important.

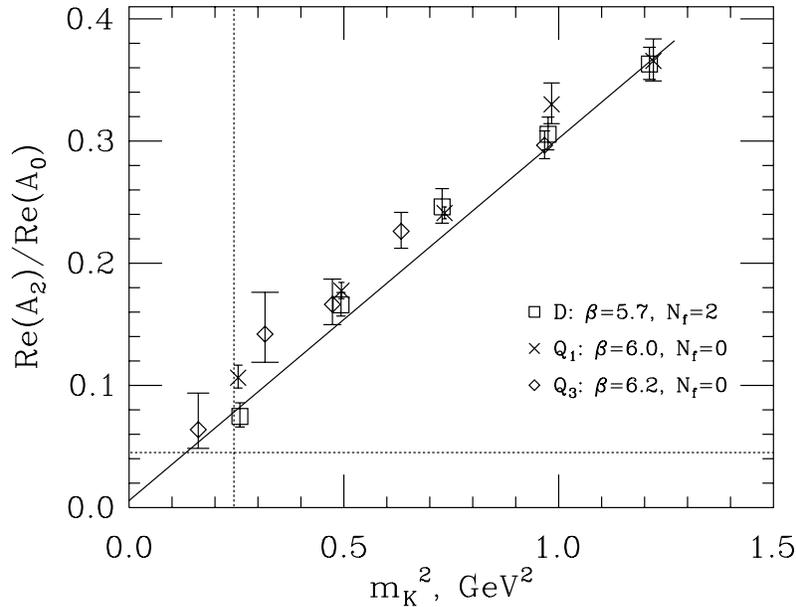


Figure 6: $\text{Re}(A_2)/\text{Re}(A_0)$ in quenched QCD with KS quark action at $\beta = 6.0$ (crosses) and $\beta = 6.2$ (diamonds), as a function of the (unphysical) K meson mass squared, together with the one in full QCD at $\beta = 5.7$ (squares). A horizontal dotted line represents the experimental value of the ratio, while the vertical one is the experimental value of K meson mass squared.

Recently, applying this method with the KS quark action [30] or the domain-wall quark action [22, 31], both of which have good chiral properties, statistically meaningful results have been obtained for $K \rightarrow \pi\pi$ amplitudes, relevant for the $\Delta = 1/2$ rule. In Fig. 6, $\text{Re}(A_2)/\text{Re}(A_0)$, where A_I is the $K \rightarrow \pi\pi$ amplitude for the final $\pi\text{-}\pi$ state with the isospin I , is plotted as a function of m_M^2 (denoted as m_K^2 in the figure), in both quenched and full QCD with the KS quark action [30]. Although the ratio is still larger than the experimental value (so that the $\Delta I = 1/2$

enhancement is still smaller than the experimental one) the strong m_M^2 dependence, mainly caused by $\text{Re}(A_2)$, is observed, the result suggests that the method is useful. Calculations with better control over systematic errors as well as statistical errors will be the next step. A similar result for $\text{Re} A_0$ with small m_M^2 dependence has recently been reported; this result employs domain-wall quarks in the quenched approximation [22, 31].

Although results for ε'/ε have also been obtained by the same method with the KS quark action, it has been found that effects associated with 1-loop renormalization of weak operators are too large to obtain a reliable estimate of ε'/ε [30]. Partially employing non-perturbative renormalization factors, a negative value of ε'/ε is obtained [30]. Recently, a large negative value of ε'/ε was also reported from the calculation with domain-wall quarks with non-perturbative renormalization factors [22, 31]. The value of $B_6^{1/2}$ is also large and negative. However, further confirmation of these results, which include a systematic study of scaling violations for the continuum extrapolation and the validity of the reduction method via χ PT, will be needed for establishing a lattice estimate of ε'/ε .

For the $\Delta I = 3/2$ decay, on the other hand, the estimation by the lowest order of the χ PT gives reasonable results. For example, the quenched results of $B_7^{3/2}$ and $B_8^{3/2}$ are given in Table 2. Furthermore, the direct calculation of the $\Delta I = 3/2$ $K \rightarrow \pi\pi$ decay amplitude successfully gives $\langle \pi^+\pi^0 | Q_+ | K^+ \rangle = (8.9-10.6) \times 10^{-3} \text{ GeV}^3$ in the quenched approximation [33], in good agreement with the experimental value $10.4 \times 10^{-3} \text{ GeV}^3$.

References	action	a	Renormalization	$B_7^{3/2}(2 \text{ GeV})$	$B_8^{3/2}(2 \text{ GeV})$
GBS96 [17]	Wilson	$\neq 0$	Perturbative	0.58(2) $^{(7)}_{(3)}$	0.81(3) $^{(3)}_{(2)}$
KGS97 [32]	KS	$\rightarrow 0$	Perturbative	0.62(3)(6)	0.77(4)(4)
Rome97 [19]	Clover	$\neq 0$	Non-Perturbative	0.72(5)	1.03(3)
Rome97 [19]	Clover	$\neq 0$	Perturbative	0.58(2)	0.83(2)
UKQCD98 [18]	Clover	$\neq 0$	Perturbative	0.58 $^{(5)}_{(4)}$ $^{(2)}_{(8)}$	0.80(8) $^{(1)}_{(4)}$

Table 2: Lattice estimates for $B_7^{3/2}$ and $B_8^{3/2}$ in the quenched approximation.

4 Weak matrix elements of heavy-light mesons

4.1 Lattice action for heavy quarks

A practical problem for lattice study of heavy quark physics is that the cut-off $1/a$ in the current simulations, which is typically $1 \sim 4 \text{ GeV}$, is smaller than the b quark mass, $m_b \simeq 4 \text{ GeV}$. Therefore, the discretization error of order $m_b a$ is

expected to be large. If the Wilson/Clover action is used for the heavy quark, the numerical simulation has to be restricted to the charm quark mass region to avoid the large discretization errors, and the results has to be extrapolated to the b quark mass with the help of results in the static limit. We refer to this method as the extrapolation method below.

It has been pointed out [34], however, that direct simulations at the b quark mass is possible if the large discretization errors are removed by a careful interpretation of the results. We call this the direct method.

Although the continuum limit can be taken in both methods, the a dependence of the results is complicated and non-linear in the latter.

If the non-relativistic formulation of quark is used instead, one can directly simulate the b quark more easily and accurately. This method, called lattice NRQCD [35], is an effective theory defined by an inverse expansion in the heavy quark mass. Taking the continuum limit is not possible with this method, and the systematic error increases for the charm quark.

Because there is no best method so far, one always has to check that different methods give consistent results within estimated errors.

4.2 Leptonic decay constants

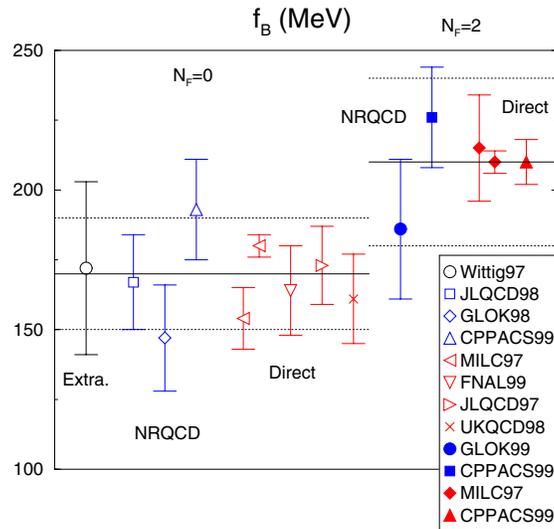


Figure 7: f_B , from different methods in quenched (open) and full (filled) QCD.

One of the simplest quantities among weak matrix elements is the leptonic decay constant of a heavy-light meson, which is important for determining V_{td}

and V_{ts} from neutral B meson mass differences:

$$\Delta M_q = \frac{G_F M_W^2}{6\pi^2} \eta_{B_q} S(m_t/M_W) f_{B_q}^2 \hat{B}_{B_q} |V_{tq}|^2, \quad (9)$$

where f_{B_q} denotes the decay constant of the B_q meson. In particular, a significant effort among lattice groups has been devoted to the calculation of $f_B \equiv f_{B_d}$, which has not been experimentally measured yet.

After many years of calculations, results for f_B from several groups with different methods are gradually converging. In Fig. 7, the latest results for f_B from different methods, in both quenched ($N_f = 0$) and full ($N_f = 2$) QCD, are summarized.

Results in quenched QCD, represented by open symbols, from the extrapolation method [36], NRQCD [37, 38, 39], and the direct method [40, 41, 42], are consistent within 10 % errors, showing that the lattice methods currently employed indeed work for b quark. Recent calculations in full QCD with 2 flavors of dynamical quarks ($N_f = 2$), denoted by filled symbols, from NRQCD [43, 39] and the direct method [40, 44, 45], indicate that f_B increases by 10-20 %.

Quantity	$N_f = 0$	$N_f = 2$	Experiments
f_B	170(20) MeV	210(30) MeV	
f_{B_s}	195(20) MeV	245(30) MeV	
f_D	200(20) MeV		≤ 310 MeV
f_{D_s}	220(20) MeV		241(21)(30) MeV
f_{B_s}/f_B	1.15(4)	1.16(4)	
f_{D_s}/f_D	1.10(6)		

Table 3: Lattice estimates for leptonic decay constants in $N_f = 0$ and 2.

A summary of leptonic decay constants is given in Table 3. The present best estimate of f_B in 2 flavor QCD is 210(30) MeV, which is larger than the previous quenched result. The ratio, f_{B_s}/f_B , which has less systematic uncertainty, on the other hand, remains unchanged from quenched QCD to full QCD. This quantity will become important if both ΔM_d and ΔM_s are experimentally measured.

4.3 B meson mixing parameter B_B

As is seen in Eq. (9), the mixing parameters of neutral B_q mesons B_{B_q} ($q = d, s$), defined by

$$B_{B_q}(\mu) = \frac{\langle \bar{B}_q | \bar{b} \gamma_\mu (1 - \gamma_5) q \cdot \bar{b} \gamma_\mu (1 - \gamma_5) q | B_q \rangle}{\frac{8}{3} \langle \bar{B}_q | \bar{b} \gamma_\mu (1 - \gamma_5) q | 0 \rangle \langle 0 | \bar{b} \gamma_\mu (1 - \gamma_5) q | B_q \rangle}, \quad (10)$$

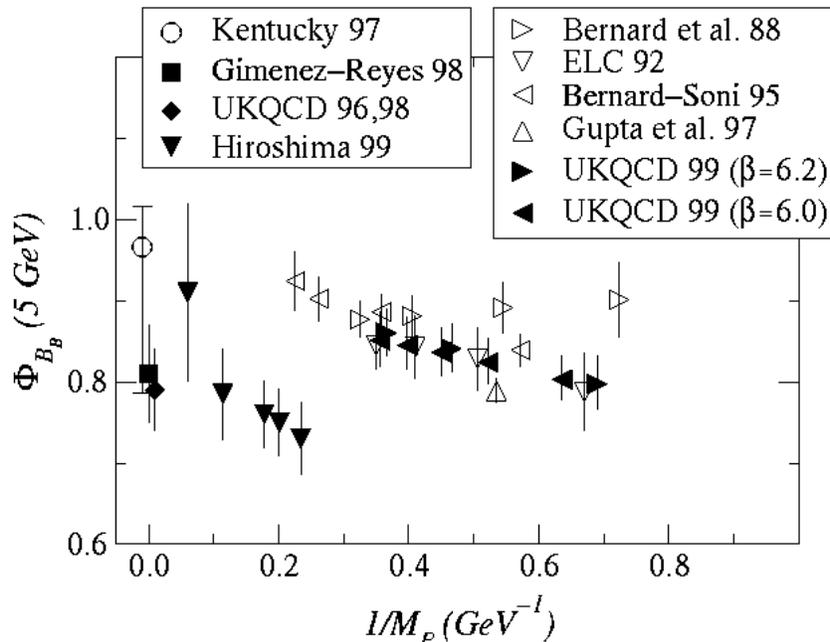


Figure 8: A compilation of quenched results for $\Phi_{B_B}(5 \text{ GeV})$ as a function of a heavy meson mass inverse, $1/M_P$.

are important to determine V_{td} and V_{ts} from ΔM_q , together with decay constants f_{B_q} .

Lattice results so far are obtained within the quenched approximation at fixed lattice spacings. A compilation of lattice results for $\Phi_{B_B}(\mu) \equiv (\alpha_s(M_P)/\alpha_s(M_B))^{2/\beta_0} B_B(\mu)$ at $\mu = 5 \text{ GeV}$ is shown in Fig. 8, as a function of a heavy meson mass inverse, $1/M_P$. The coupling factor is introduced to cancel the $\ln(M_P)$ dependence in the B parameter [46].

Data in the figure are classified into 3 distinct groups. The first is the calculation from the static-light approximation, where the heavy meson mass is fixed to be infinite. The results are consistent among the Kentucky group [47], Gimenez-Reyes [48, 49], and UKQCD [50], though the error of the Kentucky group is rather large due to the ambiguity of 1-loop renormalization in Wilson quark action. Therefore, we take an average over results of the latter two groups, who employ the Clover quark action, as the current lattice estimate, giving $B_{B_d}(m_b) = 0.80(5)$ ($\hat{B}_{B_d} = 1.28(8)$).

The second result comes from a NRQCD-light method by the Hiroshima group [51].

They observe a heavy meson mass dependence: B_B decreases as the B meson mass decreases from the static limit. As is seen from the figure, their result in the static limit is reasonably consistent with the previous result from static-light calculations.

There are several calculations using the extrapolation method. They are rather old results from Wilson quark action [52, 53, 54, 17], except UKQCD's this year with the Clover action [55]. Although the results are internally consistent, they seem to differ from the static result in the static limit and the NRQCD result at the B meson mass, as is seen in the figure. More work is needed to reduce this uncertainty and to obtain the reliable estimate for B_B .

We take the static result as our tentative estimate for B_B , because systematic errors are best controlled in this method so far:

$$B_{B_d}(m_b) = 0.80(5), \quad (\hat{B}_{B_d} = 1.28(8)), \quad \xi_{sd} \equiv \frac{f_{B_s} \sqrt{B_{B_s}}}{f_{B_d} \sqrt{B_{B_d}}} = 1.17(6). \quad (11)$$

Combining this value with the previous estimate of f_B , one obtains

$$\sqrt{\hat{B}_{B_d}} f_{B_d} = 240(36) \text{MeV}. \quad (12)$$

4.4 Form factors in semi-leptonic decays

In this section form factors in semi-leptonic decays of B or D mesons are considered. For pseudo-scalar to pseudo-scalar decays there are two form factors, $f^+(q^2)$ and $f^0(q^2)$, which are defined by

$$\langle P(k) | V^\mu | H(p) \rangle = f^+(q^2) \left[(p+k)^\mu - \frac{m_H^2 - m_P^2}{q^2} q^\mu \right] + f^0(q^2) \frac{m_H^2 - m_P^2}{q^2} q^\mu, \quad (13)$$

here $q = p - k$ is the momentum transfer, $H = B$ or D , and $P = D, K, \eta$, or π . For pseudo-scalar to vector decays, there are four form-factors $V(q^2)$, $A_{1,2}(q^2)$, and $A(q^2)$, defined by

$$\begin{aligned} \langle V(k, \varepsilon) | V^\mu | H(p) \rangle &= \frac{2V(q^2)}{m_H + m_V} \varepsilon^{\mu\nu\alpha\beta} p_\nu k_\alpha \varepsilon_\beta^* \\ \langle V(k, \varepsilon) | A^\mu | H(p) \rangle &= i(m_H + m_V) A_1(q^2) \varepsilon^{*\mu} - i \frac{A_2(q^2)}{m_H + m_V} \varepsilon^* \cdot p (p+k)^\mu \\ &\quad + i \frac{A(q^2)}{q^2} 2m_V \varepsilon^* \cdot p q^\mu \end{aligned}$$

where $V = D^*, K^*, \phi$ or ρ .

$D \rightarrow K^{(*)}\ell\nu, \pi(\rho)\ell\nu$

We first consider D meson decays to light mesons, $K, K^*, \pi,$ or ρ . The purpose of this calculation is two-fold: Assuming that $V_{cs} = 1 - \lambda^2/2$, one can establish the validity of lattice calculations for semi-leptonic form factors. Conversely, one may extract V_{cs} or V_{cb} directly from experimental data, using the lattice estimate of the form factors.

The calculation of these form factors is easier than others, because direct simulations at m_c or m_s can avoid the subtleties of the quark mass extrapolation. In addition, the fact that all physical as well as unphysical q^2 regions are covered makes interpolation to $q^2 = 0$ possible and easy.

As an example, Fig. 9 shows a lattice result [56], presented already in 1995, for the form factor $f^+(q^2)$ of the $D \rightarrow K\ell\nu$ decay as a function of q^2 in quenched QCD. As this figure shows, lattice methods work well for these decays. For a summary of quenched estimates for these form factors at $q^2 = 0$, see [57].

$B \rightarrow D^{(*)}\ell\nu$

The large branching ratio of $B \rightarrow D^{(*)}\ell\nu$ decays allows a very precise determination of $|V_{cb}|$. Differential decay rates are related to $|V_{cb}|^2$ and the form factor $\mathcal{F}_{B \rightarrow D}$ by

$$\frac{d\Gamma}{d\omega}(B \rightarrow D^{(*)}\ell\nu) = (\text{known factor}) \times |V_{cb}|^2 |\mathcal{F}_{B \rightarrow D^{(*)}}(\omega)| \quad (14)$$

where $\omega = \mathbf{v}_B \cdot \mathbf{v}_{D^{(*)}}$ is the velocity transfer. For example, the very accurate result for $|\mathcal{F}_{B \rightarrow D^*}(\omega)V_{cb}|$ from the DELPHI collaboration is shown in Fig. 10. Taking the $\omega \rightarrow 1$ limit, one obtains $|\mathcal{F}_{B \rightarrow D^*}(1)V_{cb}| = (37.95 \pm 1.34 \pm 1.59) \times 10^{-3}$ [58].

Now the most important task of lattice QCD is the determination of $|\mathcal{F}_{B \rightarrow D^{(*)}}(1)|$, in order to extract $|V_{cb}|$ from experimental results. In the limit $\omega \rightarrow 1$,

$$\mathcal{F}_{B \rightarrow D}(1) = h_+(1) - \frac{m_B - m_D}{m_B + m_D} h_-(1) \quad (15)$$

$$\mathcal{F}_{B \rightarrow D^*}(1) = h_{A_1}(1), \quad (16)$$

in terms of new form factors, defined by

$$\frac{\langle D(\mathbf{v}') | \bar{c} \gamma_\mu b | B(\mathbf{v}) \rangle}{\sqrt{m_B m_D}} = [h_+(\omega)(\mathbf{v} + \mathbf{v}')_\mu + h_-(\omega)(\mathbf{v} - \mathbf{v}')_\mu] \quad (17)$$

$$\frac{\langle D^*(\mathbf{v}') | \bar{c} \gamma_\mu \gamma_5 b | B(\mathbf{v}) \rangle}{\sqrt{m_B m_D^*}} = (\omega + 1) \varepsilon_\mu^* h_{A_1}(\omega). \quad (18)$$

Because the heavy quark symmetry implies $\mathcal{F}_{B \rightarrow D}(1) = \mathcal{F}_{B \rightarrow D^*}(1) = 1$ in the $m_b, m_c \rightarrow \infty$ limit, the heavy quark mass dependence has to be determined precisely.

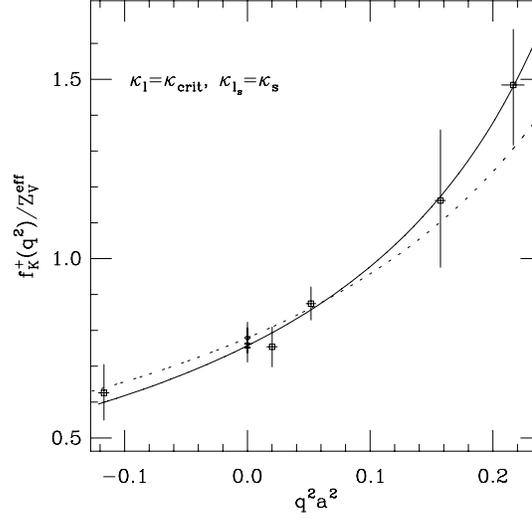


Figure 9: The form factor $f^+(q^2)$ for $D \rightarrow K \ell \nu$ decay, as a function of q^2 .

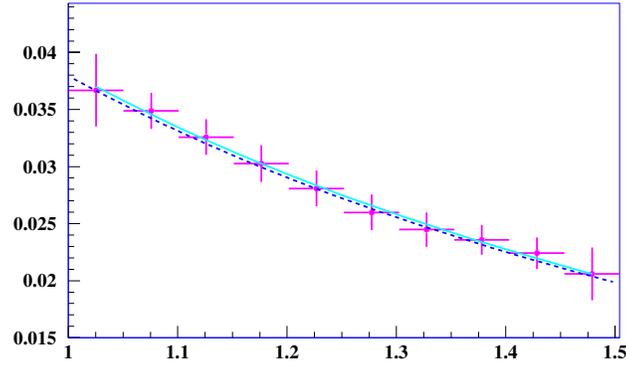


Figure 10: $|\mathcal{F}_{B \rightarrow D^*}(\omega) V_{cb}|$ as a function of ω from DELPHI collaboration.

These form factors have been evaluated recently by the Fermilab group [59], using the direct method with Clover quark action, in quenched approximation at $a^{-1} \sim 1$ GeV. In order to cancel systematic as well as statistical errors of lattice matrix elements, they employ the ratio method, through which $|h_+(1)|^2$, for example, is obtained as

$$|h_+^{B \rightarrow D}(1)|^2 = \frac{\langle D | V_0^{cb} | B \rangle \langle B | V_0^{bc} | D \rangle}{\langle D | V_0^{cc} | D \rangle \langle B | V_0^{bb} | B \rangle}. \quad (19)$$

	Lattice	QCD sum rule
$\mathcal{F}_{B \rightarrow D}(1)$	1.058 (16) $^{(+14)}_{(-6)}$	0.98(7)
$\mathcal{F}_{B \rightarrow D^*}(1)$	0.935 (22) $^{(+23)}_{(-24)}$	0.91(6)/0.92(4)

Table 4: Lattice results for $|\mathcal{F}_{B \rightarrow D^*}(1)|$, together with those from QCD sum rule.

The precision of the result makes it possible to determine the heavy quark mass dependence of h 's up to $1/m_{b,c}^3$ for $h_+(1)$ and $1/m_{b,c}^2$ for $h_{-,A_1}(1)$.

A summary of their results, together with results from the QCD sum rule approach [60], is given in Table 4. Already errors of lattice results are comparable to or even smaller than those of QCD sum rule results. Because the deviation from 1, not the value itself, is important for $F_{B \rightarrow D^{(*)}}(1)$, the systematic as well as the statistical errors should be reduced further. Improved calculations with smaller lattice spacings for the continuum extrapolation and/or the inclusion of the full QCD effect should be performed as the next step. Results from such calculations, together with experimental data, will reduce the error in $|V_{cb}|$ significantly.

As far as the shape of $F_{B \rightarrow D^{(*)}}(\omega)$ is concerned, the lattice calculation has a longer history than that for $F_{B \rightarrow D^{(*)}}(1)$. It is important to calculate the shape of form factors, as a check of theoretical method and for a reduction of errors in the extrapolation of experimental data.

Heavy quark symmetry implies that these form factors are related to one universal, mass-independent function, $\xi(\omega)$, the famous Isgur-Wise function:

$$h_{+,A_1}(\omega) = (1 + \gamma_{+,A_1}(\omega) + \beta_{+,A_1}(\omega))\xi(\omega), \quad (20)$$

$$h_-(\omega) = (\gamma_-(\omega) + \beta_-(\omega))\xi(\omega), \quad (21)$$

where β_{\pm,A_1} represent radiative corrections, while $\gamma_{\pm,A_1}(\omega)$ represent heavy quark mass dependences. The Isgur-Wise function $\xi(\omega)$ has already been calculated on the lattice. For example, the recent result for $h_+(\omega)/(1 + \beta_+(\omega))$ from the UKQCD collaboration [61] with the Clover quark action is given in Fig. 11. They employ four different heavy quark masses, and find the mass-independence of $h_+(\omega)/(1 + \beta_+(\omega))$, which suggests that the correction to the heavy quark mass limit is indeed small. Therefore, within errors, $h_+(\omega)/(1 + \beta_+(\omega))$ can be identified with the Isgur-Wise function, $\xi(\omega)$. In future investigations, it will be important to extract the mass dependent terms $\gamma_{\pm,A_1}(\omega)$, in order to construct the shape of $F_{B \rightarrow D^{(*)}}(\omega)$ precisely.

$B \rightarrow \pi \ell \nu, \rho \ell \nu$

The last quantities of our concern are the form factors of $B \rightarrow \pi, \rho$ decays, the rare decays that can be used to determine $|V_{ub}|$.

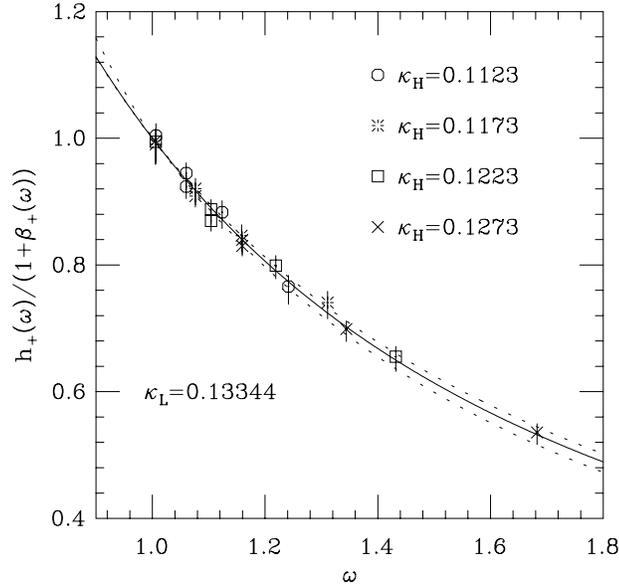


Figure 11: The radiatively corrected form factor $h_+(\omega)$ in quenched QCD from UKQCD collaboration. Different symbols correspond to different heavy quark masses, while the light quark mass is kept fixed.

Lattice calculations for these form factors are restricted to large q^2 regions only, since the momenta of B and π/ρ have to be smaller than $1/a$, to avoid large discretization errors. Therefore, experimental data for the partial decay rate in the large q^2 region are necessary for such lattice calculations to be useful. Indeed, the statistics of a recent experiment [62] is precise enough to attempt an initial study of the partial decay rate and the quality of the data is expected to be further improved.

The UKQCD collaboration calculates the form factor relevant for the differential decay rate of $B \rightarrow \rho \ell \nu$ in the quenched approximation at $a \neq 0$, using the Clover quark action [63]. As shown in Fig. 12, fitting lattice data near $q^2 = q_{\max}^2$ by the form,

$$\frac{d\Gamma(B \rightarrow \rho \ell \nu)}{dq^2} \frac{10^{12}}{|V_{ub}|^2} = c^2(1 + b(q^2 - q_{\max}^2)) \times (\text{phase space factor}) \quad (22)$$

they obtain

$$c = 4.6 \pm 0.7 \text{ GeV} \quad b = (-8_{-6}^{+4}) \times 10^{-2} \text{ GeV}^{-2}. \quad (23)$$

As the precision of the differential decay rate will be further improved in future experiments, this information will be useful in the extraction of $|V_{ub}|$.

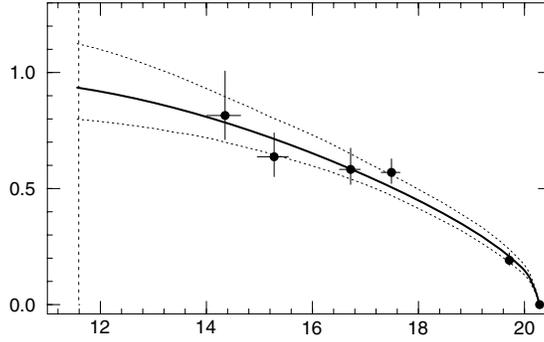


Figure 12: The form factor $B \rightarrow \rho \ell \nu$ decay as a function of q^2 in quenched QCD from the UKQCD collaboration. The solid line represents the fit given in the text, and vertical dotted line shows q_{\max}^2 for $B \rightarrow D^*$ decay.

There are two form factors for $B \rightarrow \pi \ell \nu$ decay, defined by

$$\langle \pi(k) | \bar{u} \gamma_\mu b | B(p) \rangle = f^+(q^2) \left[(p+k)_\mu - \frac{m_B^2 - m_\pi^2}{q^2} q_\mu \right] + f^0(q^2) \frac{m_B^2 - m_\pi^2}{q^2} q_\mu. \quad (24)$$

Only $f^+(q^2)$ contributes to the differential decay rate, according to

$$\frac{d\Gamma}{dq^2}(B \rightarrow \pi \ell \nu) \propto |V_{ub}|^2 |f^+(q^2)|^2. \quad (25)$$

The contribution of $f^0(q^2)$ is negligible in the decay rate, because $q_\mu L_\mu \propto m_l$, where L_μ is the leptonic weak current and m_l is the lepton mass. However, $f^0(q^2)$ may be used to check lattice calculations by the relation that $f^0(q_{\max}^2) = f_B/f_\pi$, implied by the soft pion theorem in the $m_\pi \rightarrow 0$ limit.

It has been found that the calculation of these form factors is rather difficult. The chiral extrapolation of $f^0(q^2)$ linear in the light quark mass does not appear to satisfy the relation of the soft pion theorem [5, 64]. Although it is claimed that the inclusion of a square root term in the chiral extrapolation solves the problem [65], at this moment we have to conclude that lattice results are premature for a detailed comparison with experimental data, and that more work will be needed on this form factor.

5 Impact on determination of the CKM matrix

Let us collect the lattice results and discuss their impact on the CKM matrix determination. We employ the four standard constraint relations given below to

Experimental inputs		Theoretical inputs		
quantities	value	quantities	Buras99	Lattice
$ V_{cb} $	0.040(2)	\hat{B}_K	0.80(15)	0.87(6)
$ V_{ub} $	$3.56(56) \cdot 10^{-3}$	f_B		210(30) MeV
$ \varepsilon $	$2.280(13) \cdot 10^{-3}$	\hat{B}_{B_d}		1.28(0.08)
ΔM_d	$0.471(16) \text{ ps}^{-1}$	$\sqrt{\hat{B}_B} f_B$	200(40) MeV	240(23) MeV
ΔM_s	$> 12.4 \text{ ps}^{-1}$	ξ	1.14(8)	1.17(6)
m_t	165(5) GeV			

Table 5: Input parameters.

determine the unitarity triangle (see [26] for detailed notations).

$$\sqrt{\bar{\rho}^2 + \bar{\eta}^2} = \left(1 - \frac{\lambda^2}{2}\right) \frac{1}{\lambda} \left| \frac{V_{ub}}{V_{cb}} \right|, \quad (26)$$

$$\frac{|\varepsilon|}{\lambda^{10} C_\varepsilon} = \bar{\eta} [(1 - \bar{\rho}) A^2 \eta_2 S_0(x_t) + P_0(\varepsilon)] A^2 \hat{B}_K, \quad (27)$$

$$\sqrt{(1 - \bar{\rho})^2 + \eta^2} = \frac{1}{\lambda} \left| \frac{V_{td}}{V_{cb}} \right|,$$

$$\text{with } \Delta M_d = \frac{G_F^2}{6\pi^2} \eta_B \cdot m_B \cdot \sqrt{\hat{B}_{B_d}} f_{B_d} m_W^2 S_0(x_t) |V_{td}|^2, \quad (28)$$

$$\sqrt{(1 - \bar{\rho})^2 + \eta^2} \geq \xi \sqrt{\frac{10.2 / ps}{(\Delta M_s)_{min}}}, \quad (29)$$

where $\bar{\rho} = \rho(1 - \lambda^2/2)$ and $\bar{\eta} = \eta(1 - \lambda^2/2)$. Experimental inputs, which come from the $b \rightarrow u$ transition for Eq. (26), $K_0 - \bar{K}_0$ (ε) for Eq. (27), and from $B_0 - \bar{B}_0$ mixing for Eqs. (28, 29), are summarized in Table 5. As theoretical inputs, we consider two choices, also specified in Table 5: one choice based on the lattice results summarized in this review, and the other choice from [26]. We shall refer to the latter as a standard one.

In Fig. 13 the plot on the left shows the output in the $\bar{\rho} - \bar{\eta}$ plane from the standard input. The shaded region is allowed within 1σ of each constraint. We take a flat distribution of each error in this analysis. In the plot on the right, the output from the lattice input is represented by thin curves (denoted as ‘‘present’’). Due to the larger value of $\hat{B}_B^{1/2} f_B$ than in the standard input, more positive values of $\bar{\rho}$ are favored. Thick curves (denoted as ‘‘ideal’’) are the constraints from lattice inputs without errors. In both plots, the constraint from ΔM_s is represented by dashed curves.

From these figures we conclude the following. For a better determination of the CKM matrix, lattice calculations have to reduce their errors in $\hat{B}_B^{1/2} f_B$ and

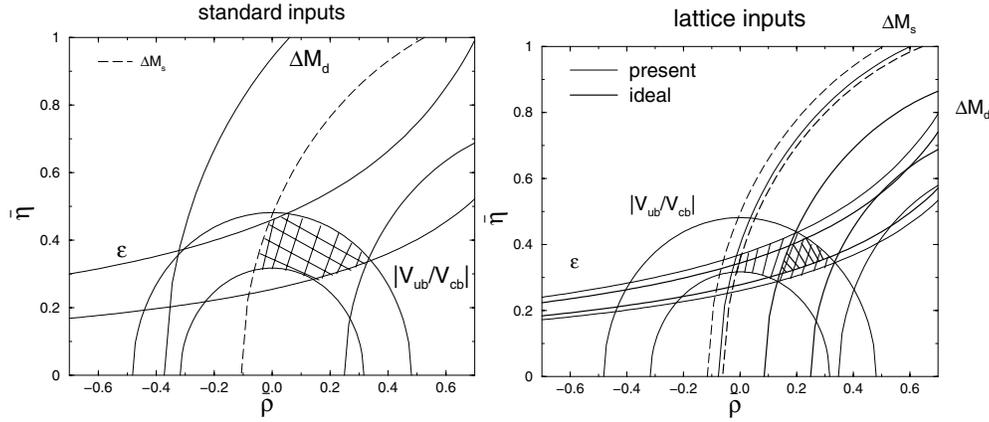


Figure 13: Constraints for CKM matrix in $\bar{\rho} - \bar{\eta}$ plane from the standard input (left) and the lattice input (right)

precisely determine the central value of both B_K and $\hat{B}_B^{1/2} f_B$ with less systematic uncertainty (in the continuum limit of full QCD). On the other hand, experiments need to reduce their errors in $|V_{cb}|$ and $|V_{ub}|$, where lattice calculations can also contribute, and provide a measurement of ΔM_s instead of the lower bound available at present.

6 Summary

In this review, we have presented the status of lattice QCD calculations of the quantities relevant for further understanding of the Standard Model and its limitations. The main results may be summarized as follows.

1. In the continuum limit of full QCD with $N_f = 2$, lattice QCD predicts that $m_s^{\overline{MS}}(m_c) = 100(15)$ MeV, which is smaller than expected. This small value of the strange quark mass tends to increase the estimate for ε'/ε in the standard analysis.
2. The continuum limit of quenched calculations for the $K_0 - \bar{K}_0$ mixing parameter gives $\hat{B}_K = 0.87(6)$. Clearly a full QCD estimate is called for.
3. Calculations in full QCD with $N_f = 2$ for the B meson decay constant estimate $f_B = 210(30)$ MeV, which is found to be larger than previous quenched results.

4. The $B_0 - \bar{B}_0$ mixing parameter $B_B(m_b) = 0.80(5)$ in the static limit of the quenched approximation. The next step is to estimate the heavy meson mass dependence.
5. Lattice methods for form factors of $D \rightarrow K^{(*)}$, π , ρ decays are now well established. There is a promising method for the calculation of $\mathcal{F}_{B \rightarrow D^{(*)}}(1)$ to extract $|V_{cb}|$. More work is still needed for $B \rightarrow \pi$ (ρ) decays.

As a conclusion, we stress that systematic investigations for all these quantities in the continuum limit of full QCD will be the next target.

I thank Drs. P. Collins, S. Hashimoto, H. Shanahan, and A. Ukawa for informative communications and useful discussions.

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Discussion

Masanori Yamauchi (KEK): Do the errors quoted represent one standard deviation, just like experimental statistical errors?

Aoki: Yes, the quoted numbers generally represent one standard deviation of statistical errors, while some of them also include systematic errors.

Jeff Richman (UCSB): I would like to comment that there is no fundamental obstacle to measuring the rate for $B \rightarrow \rho \ell \nu$ at high q^2 . In fact, the experimental acceptance is good in this region, and CLEO has now obtained a crude measurement of $d\Gamma/dq^2$ for $B \rightarrow \rho \ell \nu$, which is limited only by statistics. If we meet again in three years, perhaps the lattice will have a solid prediction for the rate at high q^2 and experimenters will have a solid measurement.

Aoki: I agree with you. I think that lattice theorists and experimentalists in this field should communicate with each other.

George Hou (National Taiwan University): The strange quark mass m_s not only enters ε'/ε , but it also enters rare B decays. In the CLEO fit discussed in Poling's talk that obtained a somewhat different value for γ (or ϕ_2) than usual, one also obtains a m_s value consistent with the lower value that you have reported.

Aoki: That is an interesting information. I would like to see the detailed analysis in the future.

Flavio Constantini (Pisa University): It is true that the decrease of the strange q mass from 130 MeV to 100 improves the agreement between the lattice result and the experimental value of ε'/ε . However, to agree with the present experimental values you need to go as low as 70 MeV. Moreover the dependence on the strange quark mass acts on the numerator of the expression and not only on the denominator.

Aoki: Since there exist ambiguities also in other parameters such as B_6 , B_8 and $\Lambda_{\overline{MS}}$, we do not need the strange quark mass as low as 70 MeV. The explicit

strange quark mass dependence in the numerator of the expression is included in my estimate. As mentioned in the talk, however, the implicit strange quark mass dependence in B_6 and B_8 is not considered here.

Matthias Neubert (SLAC): In fact the quantity ε'/ε does not depend at all on the strange quark mass. It enters only by convention when you define B_6 . It seems to me you should not use a lattice estimate of m_s unless at the same time you use a lattice estimate of B_6 .

Aoki: Rigorously speaking, I agree with you. According to this criteria, however, I have to say that there is no reliable theoretical estimate for ε'/ε . Now it seems to me that only lattice calculations can give a reliable theoretical estimate for ε'/ε , and, as I mentioned in my talk, serious efforts on such calculations have just started.

B. F. L. Ward (University of Tennessee): Could you comment on the prospects for calculating the rate of $B \rightarrow \pi\pi$ on the lattice?

Aoki: There are two main difficulties in calculating the rate of $B \rightarrow \pi\pi$ on the lattice. One is the problem of dealing with decays into 2 particles such as $K \rightarrow \pi\pi$ decays, the other is the difficulty of putting a heavy quark on the lattice. Therefore, no lattice calculation has been made for the rate of $B \rightarrow \pi\pi$ so far.

Helen Quinn (SLAC): When you put your errors on lattice quantities for which you have only a quenched calculation I believe that these errors include statistics and errors due to extrapolation but no estimate of the errors due to quenching. Is this correct, and if so how can one best estimate the additional uncertainties due to the quenched approximation?

Aoki: Yes. In general the errors due to quenching are not included, and it is difficult to estimate them unless the corresponding full QCD calculation is really performed. One might naively think that the quenching errors are 10–20%. In some calculations, the quenching errors are partly estimated by the variation of the result under the change of experimental input parameters. For example, the difference of the strange quark mass between K -input and Φ -input, which is about 20–25%, may be interpreted as the quenching errors. However this kind of the estimation is far from being quantitative.