REGULARIZATION OF GAUGE FIELD THEORIES*

William A. Bardeen†
Institute for Theoretical Physics
Stanford University
Stanford, California

Yang-Mills gauge field theories form the basis of the new class of renormalizable theories of weak and electromagnetic interactions.[1] The renormalizability of these theories[2] depends crucially upon the high degree of symmetry implied by the exact local gauge invariance of the initial lagrangian. Unfortunately, the divergences encountered in the Feynman amplitudes of perturbation theory may prevent the implementation of these symmetries in higher order. Indeed, the well-known Adler, Bell, and Jackiw[3] Ward-identity anomalies of the free spinor loop provide an example where the full gauge symmetry of the initial lagrangian cannot be maintained.

The known spinor-loop anomalies already place dynamical restrictions on "realistic" models of weak interactions. We must now ask whether it is possible to avoid new anomalies either in higher order or in the Yang-Mills structure which would destroy the renormalizability of these theories. Using the work of the past year, we will show that this question can be given a positive answer. We conclude that if the theory is chosen as to avoid the anomalies of the free spinor loop then no further anomalies are present in the theory.

Our procedure will be to devise a regularization scheme for the Feynman amplitudes which preserves the local gauge symmetry at every stage. Ignoring for a moment the problem of including spinor particles, we examine two methods proposed for regularizing meson amplitudes.

An approach discussed by B. W. Lee and J. Zinn-Justin[4] involves the addition of higher dimension, but gauge-invariant, terms in the lagrangian to both the kinetic energy and interaction terms. An analysis of the Feynman amplitudes shows that all of the more complicated amplitudes are regularized by this mechanism and the gauge invariance of the unregularized amplitudes can be directly established.

A simpler, more intuitive approach has recently been proposed by 't Hooft and Veltman[5] and others.[6] They show that Feynman amplitudes can be defined in noninteger or even complex dimensions of space-time. To any finite order in perturbation theory, the amplitudes are meromorphic functions of the dimension variable with a simple pole structure on the positive real axis. For scalar particles, this regularization corresponds to a modification of the phase-space integrals. For the gauge theories of mesons considered here, the gauge symmetry implies algebraic relations between different amplitudes. As these algebraic manipulations are not sensitive to the dimension of space-time, the gauge relations are easily established for amplitudes regularized by this method. A simple algorithm for constructing amplitudes in n dimensions will be discussed at the end of this talk.

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†Alfred P. Sloan Foundation Fellow

We now wish to discuss the problem of regularizing theories involving spinor particles and particularly the anomaly problem. We remark that the n-dimensional technique can be simply extended to theories such as electrodynamics as discussed by 't Hooft and Veltman.[5] However, weak-interaction theories usually involve the axial vector current, and no successful extension of γ_5, or more precisely the pseudotensor, to complex dimensions is possible in general.

We may avoid this problem by observing that the regularization of only the meson part of any graph is almost sufficient to regularize the whole graph. Consider the graph in Fig. 1 where an arbitrary meson blob M connects the spinor line AB with spinor loops C and D. If we have sufficiently regularized the meson integrations, we observe that there are no further divergences associated with the spinor line. Also since the spinor loops are connected by meson lines, there are no

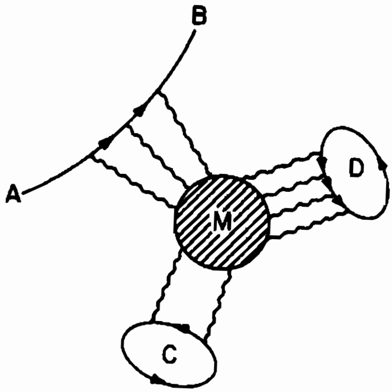

Fig. 1. A general Feynman graph with spinor line AB, spinor loops C and D, and a meson blob M.

overlapping divergences. Hence, we may study independently the divergences of each spinor-loop subgraph. There is only a polynominal ambiguity in each spinor-loop subgraph, and it is simple in principle to give a prescription for renormalizing these subgraphs. There are no further divergences.

We turn to the gauge relations between graphs such as in Fig. 1. As described previously there are at least two methods of giving a gauge-invariant regularization of the meson part of the graph. As we have not modified the structure of the spinor line, gauge relations involving manipulations along the spinor line are preserved as the regularized meson integrations are not divergent. Gauge relations also involve manipulation of the spinor loops. If the spinor loops can be given a gauge-invariant definition, then the whole graph can be given a definition consistent with the gauge relations. Hence only the lowest-order spinor-loop anomalies need be cancelled to obtain a gauge-invariant regularization procedure for the whole graph.

We can give a simple rule for regularization within the n-dimensional procedure. The calculation is to be done in n-dimensions, but the γ-matrices must be kept four dimensional (they have only the first four components of an n-vector). Hence, even if a spinor propagator carries an n-dimensional momentum, it only depends upon the first four components of that n-vector. The spinor-loop integration is therefore reduced to a four-dimensional integration. The small spinor loops must be computed consistent with all relevant gauge relations (the cancellation of the free spinor-loop anomalies is the only dynamical constraint on the theory). The gauge relations never involve momenta coming out of spinor line or loop and hence the fact that γ-matrices are four dimensional does not destroy any of the gauge relations.

We now briefly remark upon the actual construction of n-dimensional amplitudes. It is obvious that the operation $\int d^n p$ has no immediate meaning if n = -100 + 37i. 't Hooft and Veltman give a definition by integrating explicitly the first k integer components of momentum and doing the remaining n-k components via a volume integral which is defined through integrations by parts. An equivalent method has been suggested by Lautrup[7] which exploits the use of differential regularizer acting on the integrand. Hence, the n-dimensional integral may be defined in terms of a normal four-dimensional integral. The n-dimensional amplitude for a single-loop graph is defined by

$$\begin{aligned} A(n, P_i) &= \int d^n \ell \;\; I^n(\ell^2, \ell \cdot P_i) \\ &\equiv \int d^4 \ell \;\; R_n \, I^n(\ell^2, \ell \cdot P_i) \,, \end{aligned} \tag{1}$$

where $R_n = (\frac{1}{\pi}\, \partial_{\ell^2})^{2-\frac{n}{2}}$.

The integrand, I^n, is to be computed as if you were in n dimensions; in practice, it means only that $g^{\mu\nu} g_{\mu\nu} = n$ instead of 4 in doing the algebra.

This procedure can be generalized to the many-loop case. We have

$$\begin{aligned} A(n, P_j) &= \int d^n \ell_1 \,\text{--}\, \int d^n \ell_r \;\; I^n(\ell_i \cdot \ell_j, \ell_i \cdot P_j) \\ &\equiv \int d^4 \ell_1 \,\text{--}\, \int d^4 \ell_r \;\; R_n \, I^n(\ell_i \cdot \ell_j, \ell_i \cdot P_j) \,, \end{aligned} \tag{2}$$

where $R_n = (\det A)^{2-\frac{n}{2}}$

and $A = \{a_{ij}\}$, $a_{ij} = \frac{1}{2\pi}\, \partial_{\ell_i \cdot \ell_j}$, where $i \neq j$

$\quad = \frac{1}{\pi}\, \partial_{\ell_i^2}$ where $i = j$.

Again, the integrand is computed with $g^{\mu\nu} g_{\mu\nu} = n$. It should be noted that the differential operator acts only on the meson part of the graph. With these definitions it is easy to show that all n-dimensional rules for shifting momenta, symmetric integration, and relabelling loop momenta are allowed.

For simple graphs this procedure is similar to that employed by Speer.[8] The important difference, however, for Yang-Mills theories is that the n-dimensional technique regularizes the whole meson blob at once.

We have established that a gauge-invariant regularization of Yang-Mills theories can be made. The only dynamical constraint on the theory comes from cancellation of the anomalies of the free spinor loops. The n-dimensional regularization scheme is intuitively pretty and simple to use in practical calculations.[9]

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References

[1] S. Weinberg, Phys. Rev. Letters 19, 1264 (1967); A. Salam, Elementary Particle Theory, N. Svartholm, Editor (Almqvist and Wiksell, Stockholm, 1968).

[2] G. 't Hooft, Nucl. Phys. B33, 173 (1971); G. 't Hooft and M. Veltman, Combinatorics of Gauge Fields, Utrecht preprint, July (1972); B. W. Lee, Phys. Rev. D5, 823 (1972); B. W. Lee and J. Zinn-Justin, Phys. Rev. D5, 3121, 3137, 3155 (1972).

[3] J. S. Bell and R. Jackiw, Nuovo Cimento 51, 47 (1969); S. L. Adler, Phys. Rev. 177, 2426 (1969).

[4] B. W. Lee and J. Zinn-Justin, Phys. Rev. D5, 3121 (1972).

[5] G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972).

[6] C. G. Bollini, J. J. Giambiagi, and A. Gonzales Dominguez, Nuovo Cimento 31, 550 (1964); G. M. Cicuta, Analytic Continuation in the Dimension of Space-Time, submitted to this conference, #376; G. M. Cicuta, Analytic Renormalization via Continuous Space Dimension, submitted to this conference, #526.

[7] B. Lautrup, private communication.

[8] E. R. Speer, Generalized Feynman Amplitudes (Princeton University Press, Princeton, 1969).

[9] W. A. Bardeen, R. Gastmans, and B. Lautrup, Static Quantities in Weinberg's Model of Weak and Electromagnetic Interactions, Nucl. Phys., to be published.