FIELD THEORY IN LESS THAN FOUR DIMENSIONS; THE RENORMALIZATION GROUP*

Kenneth G. Wilson
Laboratory of Nuclear Studies
Cornell University
Ithaca, New York

Two topics are summarized here. The first is field theory in space-time dimension d between 2 and 4. The second is a reformulation of the renormalization group. Until now the principal model field theory exhibiting anomalous dimensions[1] has been the Thirring model[2] in one-space, one-time dimension. This is a very special model and is insufficient to give a general understanding of the phenomenon.

In the past year a vast number of models exhibiting anomalous dimensions have become soluble. The study of these models has just begun;[3,4] when it is complete one should have a much better understanding of the physics of anomalous dimensions.

The new models are field theories defined for space-time dimensions d in the range $2 < d < 4$. They unfortunately all reduce to trivial free field theories for $d = 4$. The models include the standard theories which are renormalizable in four dimensions (ϕ^4 theory, etc.) but with an unconventional renormalization procedure to prevent these theories from being super-renormalizable in less than 4 dimensions. For example, for $\lambda_0\phi^4$ theory one requires λ_0 to behave as Λ^{4-d} as the cutoff $\Lambda \to \infty$.

Figure 1 is a map showing the various types of model field theories studied to date. Mack's work on the ϕ^3 theory in $6 + \epsilon$ dimensions[4] is related to the theories discussed here.

Two new expansion techniques have been developed to solve these theories.[3] Both techniques are stolen from previous work on statistical mechanics. The simplest expansion is the $1/N$ expansion: N is the number of internal components of a scalar field ϕ or a spinor field ψ. The interactions $\lambda_0(\phi^2)^2$ and $G_0(\bar{\psi}\psi)^2$ (with $\phi^2 = \sum_{i=1}^{N} \phi_i^2$, etc.) have been studied for large N. This is possible for the whole range $2 < d < 4$. The idea that the $N \to \infty$ limit is soluble is due to Stanley.[5] The essential idea is that the easily calculable bubble graphs are the dominant graphs for large N (Fig. 2). The other technique, discovered by Wilson and Fisher[6,7] is an expansion in powers of $\epsilon = 4 - d$. This can be done for any N. Studies in statistical mechanics[6,7] show that this expansion gives good results for $d = 3$. The continuation to nonintegral dimension d has been most extensively discussed by 't Hooft and Veltman.[8]

The most interesting two results obtained to date from these models are the following.[3] Anomalous dimensions have been calculated in powers of ϵ (or $1/N$) for a number of operators. In particular the anomalous dimensions of the n^{th} rank tensor operators governing deep inelastic scattering in the $(\phi^2)^2$ theory have been computed to order ϵ^2. The result for all N is that the anomaly (departure from canonical dimension) is positive, different for different N and smaller than $\epsilon^2/96$. For $N \to \infty$ the anomaly goes to zero. This is a remarkably small anomaly since the only reasonable values of ϵ are 1 or 2. The second result is that for large N the Fermi interaction $G_0(\bar{\psi}\psi)^2$ is equivalent

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For a general discussion of domains see Ref. 10, Lecture XII; the domains for this example (for small u_0, v_0, and ϵ) were derived in Ref. 6.

Reference 10 provides an extensive introduction to the ϵ expansion as well as the renormalization group.

References

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to the Yukawa interaction $g_0\phi(\bar{\psi}\psi)$ where ϕ is a single scalar field, provided the coupling constants are renormalized so that $G_0\Lambda^{2-\epsilon}$ and $g_0\Lambda^{-\epsilon}$ are held fixed as the cutoff $\Lambda \to \infty$. This is a similar result to Bjorken's discussion of quantum electrodynamics arising from a vector Fermi interaction.[9]

A deeper development of the last few years is a complete reformulation[10] of the renormalization group ideas of Gell-Mann and Low.[11] The reformulation borrows heavily from previous ideas of Kadanoff[12] and others in statistical mechanics. It is a complex of ideas designed to meet head-on the problems which in the past have seemed hopeless both in field theory and statistical mechanics. It goes well beyond the (present) Glimm-Jaffe program[13] in that it deals with theories requiring coupling-constant renormalization. The modern renormalization group is already important in classical statistical mechanics;[14] considerable work remains before one will know whether it can solve the presently insoluble problems of strong interactions.

Two results will be quoted from the renormalization group work. The first result is an indication that the $\lambda_0\phi^4$ theory (with $N = 1$) is trivial after renormalization for any value of λ_0 in the range $0 \le \lambda_0 \le \infty$ (infinity is included).[15] In other words, the only hope to obtain a nonzero renormalized coupling constant λ_R is to allow negative or complex λ_0, in either case leading to unpleasant consequences. (This is similar to what happens in the Lee model.[16]) This result is obtained by two completely unrelated methods. One method is to calculate a form of the Gell-Mann-Low eigenvalue function $\psi(\lambda_0)$ (see Adler's talk) using the Schiff expansion.[17] In Schiff's approach one uses a lattice as a cutoff and expands in the part of the kinetic term $(\nabla\phi)^2$ which couples different lattice sites. Using modern statistical mechanical techniques[18] this expansion was calculated to 9 nontrivial orders. The other method is a direct approximate solution of the ϕ^4 theory using an approximate formulation of the renormalization group.[19] Either method by itself is unreliable, but both methods agree that λ_R is zero in the limit of infinite cutoff for all λ_0 in the range $0 \le \lambda_0 \le \infty$.

The second result is the idea of "domains". The idea is this. Consider the interaction

$$L_I = u_0\Lambda^\epsilon \left[\phi_1^{\,4} + \phi_2^{\,4}\right] + v_0\Lambda^\epsilon\phi_1^{\,2}\phi_2^{\,2}.$$

Here ϵ is $4 - d$; ϕ_1 and ϕ_2 are two scalar fields, and u_0 and v_0 are coupling constants to be held fixed as $\Lambda \to \infty$. Then there are domains in u_0, v_0 space corresponding to a unique renormalized theory. These domains are shown in Fig. 3. Any values (u_0, v_0) in the region A, for example, give the same renormalized theory (apart from mass terms) with the same anomalous dimensions in the limit $\Lambda \to \infty$. The line B is a separate domain. Any point on the line B corresponds to another renormalized theory with another unique set of anomalous dimensions. The point D $(u_0 = v_0 = 0)$ defines a free-field theory with canonical dimensions. Associated with each domain there are a definite number of free parameters (masses or coupling constants) which affect low-energy behavior only (corresponding to the "generalized mass terms" of Ref. 1). In the example there are the standard mass terms (involving $\phi_1^{\,2}$ and $\phi_2^{\,2}$); in addition there are free coupling constants whenever any point in a domain can be approached from outside. The theory associated with the domain A has no free-coupling constants; the domains B and C each give one free-coupling constant, while the domain D gives two free-coupling constants (the domain D corresponds to the standard super-renormalizable theory with $\phi_1^{\,4} + \phi_2^{\,4}$ and $\phi_1^{\,2}\phi_2^{\,2}$ as interactions).

Fig. 2. Dominant bubble graphs for the four-point function in the $N \to \infty$ limit.

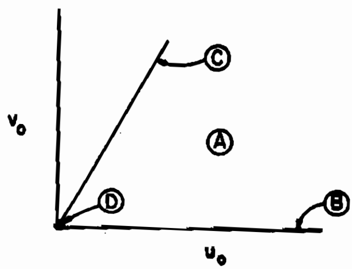

Fig. 3. Domains within which the renormalized theory is independent of the unrenormalized constants u_0 and v_0. Domains B and C are lines while D is a single point (the origin).

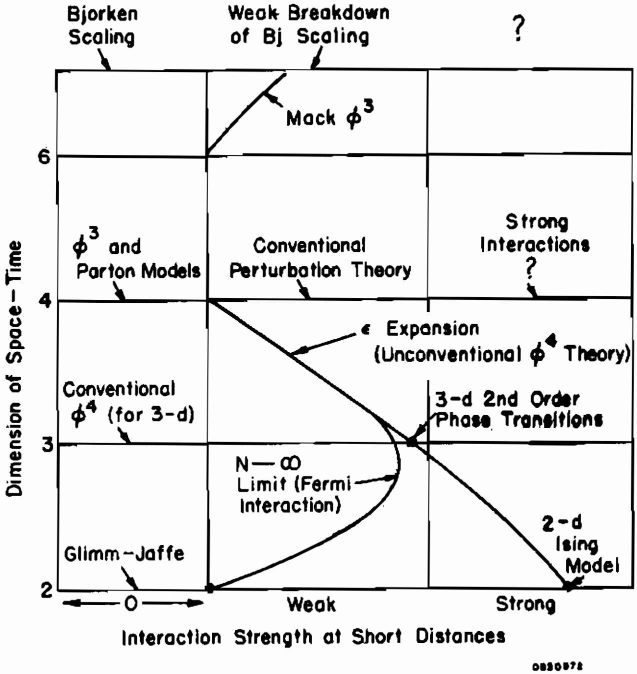

Fig. 1. Map showing various models of short distance behavior (the Thirring model, not shown, covers the entire d = 2 line). The equivalence of field theory to models of 2nd order phase transitions is reviewed in Ref. 10.