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INELASTIC SCATTERING, THE VIRTUAL RADIATOR AND ENERGY LOSS BY RELATIVISTIC MUONS

Michael J. Tannenbaum
Harvard University

ABSTRACT

Simple formulas for direct pair production are derived from the general equation for inelastic lepton scattering. Applications to energy loss by high-energy muons are discussed.

By now everyone is familiar with the general formula for inelastic scattering by muons or electrons.[1]

$$\frac{\partial^2 \sigma}{\partial q^2 \partial \nu} = \frac{4\pi\alpha^2}{q^4} W_2(\nu, q^2) \left[1 - \frac{\nu}{E} + \frac{q^2}{4E^2}\left(\frac{2W_1}{W_2} - 1\right)\right]. \tag{1}$$

E is the energy of the incident muon of mass m which scatters through angle θ, losing energy ν to a virtual photon of invariant mass $-q^2$. The final energy of the muon is $E-\nu$ and the four momentum of the photon is

$$q^2 = 4E(E-\nu)\sin^2\frac{\theta}{2} + \frac{m^2\nu^2}{E(E-\nu)}. \tag{2}$$

The structure functions $W_2(\nu, q^2)$ and $W_1(\nu, q^2)$ can be related to the cross section for virtual photon interactions with the target nucleus.[2]

$$\begin{aligned} W_2 &\equiv \frac{1}{4\pi^2\alpha}\frac{q^2}{\sqrt{\nu^2+q^2}}\left[\sigma_T(\nu, q^2) + \sigma_L(\nu, q^2)\right], \\ W_1 &\equiv \frac{1}{4\pi^2\alpha}\frac{q^2}{\sqrt{\nu^2+q^2}}\left(1 - \frac{2m^2}{q^2}\right)\frac{q^2+\nu^2}{q^2}\sigma_T(\nu, q^2), \end{aligned} \tag{3}$$

where σ_T and σ_L are the cross sections for transversely and longitudinally polarized virtual photons of energy ν and four momentum q^2. Therefore, these formulas all taken together relate any electroproduction process to its corresponding photoproduction process. In the limit $q^2 \to 0$ this relation is exact.

$$\lim_{q^2 \to 0} \sigma_L(\nu, q^2) = 0,$$

$$\lim_{q^2 \to 0} \sigma_T(\nu, q^2) = \sigma_\gamma(\nu),$$

where $\sigma_\gamma(\nu)$ is the photoproduction cross section for real photons of energy ν.

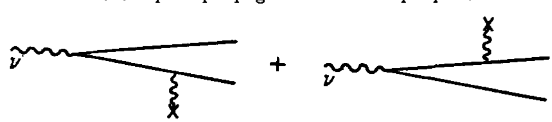
It is interesting to apply this formalism to the processes of direct pair production, $\mu + Z \to \mu + Z + e^+ + e^-$ or $\mu + Z \to \mu + Z + \mu^+ + \mu^-$.

These processes can be calculated exactly, but the calculations are extremely tedious.[3] In addition, because we know the correct answer we can check any assumption we make in trying to relate $\sigma_T(\nu, q^2)$ to $\sigma_\gamma(\nu)$ for $q^2 \neq 0$.

The cross section for producing a pair of particles of mass m_2 by an incident real photon of energy ν on a nucleus of charge Z is well known to all[4] and can be put into a simple form so that it applies to both muon and electron pairs.

$$\sigma_\gamma(\nu) = \left(\frac{Z}{6}\right)^2 1.52\ \mu b \frac{m_\mu^2}{m_2^2} \ln \frac{\nu}{6.7 m_2}. \tag{4}$$

m_μ is the muon mass and m_2 is the mass of the particles produced so that the $1/m^2$ dependence of the cross section is explicitly exhibited. If you ask why the cross section depends on m^{-2} instead of some other parameter, it turns out to be related to the minimum value of the lepton propagator which is proportional to m^2. For pair



production by virtual photons, the minimum value of the lepton propagator goes like $q^2 + m^2$. Thus it seems to be a reasonable assumption to take the virtual photon pair production cross section as

$$\sigma_T(\nu, q^2) = \left(\frac{Z}{6}\right)^2 1.52\ \mu b \frac{m_\mu^2}{q^2 + m_2^2} \ln \frac{\nu}{6.7 m_2}.$$

Note that with this assumption the ratio

$$\sigma_T(\nu, q^2)/\sigma_\gamma(\nu) = \frac{1}{1 + q^2/m_2^2}.$$

This acts like a slowly varying form factor with cutoff parameter $\Lambda^2 = m_2^2$ which will cut off integrals over q^2 when q^2 gets too high, i.e., $q^2 >> \Lambda^2$.

Substituting all this into our original formula and assuming $\sigma_L << \sigma_T$, we find the differential cross section for direct production of a pair of mass m_2 by an incident particle of mass m_μ

$$\frac{d^2\sigma}{\partial q^2 \partial\nu} = \frac{\alpha}{\pi}\frac{1}{q^2}\frac{1}{\sqrt{q^2+\nu^2}}\left(\frac{Z}{6}\right)^2 1.52\,\mu b\,\frac{m_\mu^2}{m_2^2+q^2}\left[\left(1-\frac{\nu}{E}\right)\left(1-\frac{q^2_{min}}{q^2}\right)+\frac{\nu^2}{2E^2}+\frac{q^2}{4E^2}\right]\ln\frac{\nu}{6.7m_2}. \qquad (5)$$

If we assume that $\nu^2 >> q^2$ and we integrate over q^2, we obtain the interesting formula

$$\frac{d\sigma}{d\nu} = \frac{\delta(\nu)}{\nu}\,\sigma_\gamma(\nu), \qquad (6)$$

where

$$\delta(\nu) = \frac{\alpha}{\pi}\left\{\left[1-\frac{\nu}{E}+\frac{\nu^2}{E^2}\left(\frac{1}{2}+\frac{m_1^2}{m_2^2}\right)\right]\ln\left[1+\frac{m_2^2}{m_1^2}\frac{E}{\nu}\left(\frac{E}{\nu}-1\right)\right]-\left(1\ \frac{\nu}{E}\right)\right\}.$$

$\delta(\nu)$ is the famous virtual radiator which traces its ancestry back to Weizsacker and Williams[2,5] and is interpreted as the total probability of virtual photon emission by the incident lepton.

One could now integrate $d\sigma/d\nu$ to find the total trident cross section or $\nu\, d\sigma/d\nu$ to find the energy loss due to tridents, but since we want simple formulas, we first rewrite the differential cross section in a simpler form assuming $\nu^2 >> q^2$

$$\frac{\partial^2\sigma}{\partial q^2\partial\nu} = \frac{\alpha}{\pi}\frac{1}{q^2}\frac{1}{\nu}\left(1-\frac{\nu}{E}+\frac{\nu^2}{2E^2}\right)\frac{\sigma_\gamma(\nu)}{1+q^2/\Lambda^2}. \qquad (7)$$

For direct pair production $\Lambda^2 = m_2^2$ but in fact it turns out that this formula is rather general and even work for nuclear interactions[6] if you take $\Lambda^2 = 1/3\ \mathrm{GeV}^2$.

Now, the integral over q^2 is especially simple and we find

$$\delta(\nu) = \frac{\alpha}{\pi}\left(1-\frac{\nu}{E}+\frac{\nu^2}{2E^2}\right)\ln\left[1+\frac{\Lambda^2}{m_1^2}\frac{E}{\nu}\left(\frac{E}{\nu}-1\right)\right]. \qquad (8)$$

For all cases in tridents <u>except</u> $\mu + Z \to \mu + Z + e + e$, the logarithm term is slowly varying so we can replace its energy dependence by a constant value which we guess

$$\delta(\nu) \simeq \frac{\alpha}{\pi}\left(1-\frac{\nu}{E}+\frac{\nu^2}{2E^2}\right)\ln\left(1+2\frac{m_2^2}{m_1^2}\right).$$

We then can easily integrate over ν to find the total trident cross section

$$\sigma(E) = \frac{\alpha}{2\pi} \ln \frac{E}{6.7m_2}\left(\ln \frac{E}{6.7m_2} - \frac{3}{2}\right) \ln\left(1 + 2\frac{m_2^2}{m_1^2}\right)\left(\frac{Z}{6}\right)^2 1.52\ \mu b\, \frac{m_\mu^2}{m_2^2}. \tag{9}$$

This cross section is plotted in Fig. 1 along with an older formula due to Bhabha[7] and Block[8]

$$\sigma_{Bhabha}(E) = \frac{\alpha}{2\pi}\left(\frac{Z}{6}\right)^2 1.012\ \mu b\, \frac{m_\mu^2}{m_2^2}\left(\ln^3 \frac{E}{m_2} - 6.29 \ln^2 \frac{E}{m_2} + 14.99 \ln \frac{E}{m_2} - 15.42\right).$$

The Bhabha formula seems high by a factor of 4 to 7. Also shown is an exact calculation of muon tridents in carbon.[9] This agrees quite well with our formula from 5 to 500 GeV, but our formula is about 30% below the exact value throughout the entire interval.

The case of direct electron pair production by muons is slightly tricky to evaluate, but it is most relevant for NAL since it contributes to the energy loss of high-energy muons. For this case we use μ as the muon mass and m as the electron mass. The virtual radiator then becomes

$$\delta(\nu) = \frac{\alpha}{\pi}\left(1 - \frac{\nu}{E} + \frac{\nu^2}{2E^2}\right) \ln\left[1 + \frac{m^2}{\mu^2}\frac{E}{\nu}\left(\frac{E}{\nu} - 1\right)\right] \tag{10}$$

because m^2/μ^2 is so small, the logarithm term is not slowly varying but rather is equal to

$$\left[1 + \frac{m^2}{\mu^2}\frac{E}{\nu}\left(\frac{E}{\nu} - 1\right)\right] \simeq \frac{m^2}{\mu^2}\frac{E}{\nu}\left(\frac{E}{\nu} - 1\right),$$

as long as

$$\frac{\nu}{E} \geq \frac{m}{\mu} \simeq \frac{1}{200}.$$

Hence, we break up our integration into two regions

$$1 \geq \frac{\nu}{E} \geq \frac{m}{\nu},$$

in which we assume that the above approximation is good, and

$$\frac{Em}{\mu} > \nu > 2m$$

in which the logarithm is again slowly varying. It turns out that the latter region can be ignored.

The virtual radiator then becomes

$$\delta(\nu) = \frac{\alpha}{\pi}\frac{m^2}{\mu^2}\left(\frac{E^2}{\nu^2} - \frac{2E}{\nu} + \frac{3}{2} - \frac{1}{2}\frac{\nu}{E}\right). \tag{11}$$

We only need to carry the leading term, and if we calculate the cross section, we find

$$\frac{d\sigma}{d\nu} = \frac{\alpha}{\pi}\ 1.52\ \mu b \left(\frac{Z}{6}\right)^2 \frac{E^2}{\nu^3} \ln \frac{\nu}{6.7m}. \tag{12}$$

Thus, while the formula for direct pair production by muons is quite simple, it is also quite divergent. We are saved because the logarithm in the virtual radiator cuts off the lower bound of ν at $\nu/E = m/\mu$. The total pair production cross section can be found by integration and turns out to be

$$\sigma_{pair}(E) = \frac{\alpha}{2\pi}\left(\frac{Z}{6}\right)^2 1.52\ \mu b\ \frac{\mu^2}{m^2}\ \ln \frac{E}{4.1\mu}. \tag{13}$$

Two things are worth noting about this formula. First of all, electron pair production by muons is comparable to electron pair production by electrons, both of these processes going like $1/m^2$. The reason is that both cross sections are cut off by $\Lambda^2 = m^2$, but the muons have high enough energy so that they can still radiate a substantial number of photons with $q^2 << \Lambda^2$. A second thing to note is that the total pair production cross section by muons is substantially larger than their bremsstrahlung cross section.

$$\sigma_{brem}(E) = 1.95\ \mu b \left(\frac{Z}{6}\right)^2 \ln \frac{1.43E}{\mu}, \tag{14}$$

$$\frac{\sigma_{pair}}{\sigma_{brem}} \approx \frac{7}{18}\frac{\alpha}{\pi}\frac{\mu^2}{m^2} \approx 40. \tag{15}$$

However, in spite of the fact that the pair production cross section is larger than the bremsstrahlung cross section, the average energy loss per collision by pair production is <u>less</u> than the energy loss by bremsstrahlung.

Using Eq. (12) we can find

$$<\nu> \equiv \int_{\nu_{min}}^{E} \nu \frac{d\sigma}{d\nu} d\nu,$$

for pair production which is the average energy loss per nucleus. $<\nu>$ is related to dE/dx by the number of nucleii per g/cm ^{2},

$$-\frac{dE}{dx} = \frac{N_0}{A} <\nu> .$$

For pair production by muons, we find

$$<\nu>_{pair} = \frac{\alpha}{\pi}\left(\frac{Z}{6}\right)^2 1.52\ \mu b\ \frac{E\mu}{m}\ \ln \frac{E}{2.5\mu}. \tag{16}$$

The bremsstrahlung energy loss is given by the well known formula

$$\langle\nu\rangle_{\text{brem}} = 1.95\ \mu b\left(\frac{Z}{6}\right)^2 E\left(\ln\frac{12E}{5\mu Z^{1/3}} - \frac{1}{3}\right), \tag{17}$$

so that ignoring the slight difference in logarithms we get the neat relation

$$\frac{\langle\nu\rangle_{\text{pair}}}{\langle\nu\rangle_{\text{brem}}} \simeq \frac{7}{9}\frac{\alpha}{\pi}\frac{\mu}{m} = 0.37. \tag{18}$$

Our result for the energy loss in direct pair production seems to differ significantly from that calculated by Mando and Ronchi[10] who obtained

$$\langle\nu\rangle_{\text{pair}} = \frac{\alpha}{\pi}\left(\frac{Z}{6}\right)^2 9.42\ \mu b\ \frac{E\mu}{m}\ \ln\frac{E}{16.2\mu}.$$

As I see it, there are three ways to resolve this discrepancy. The first is to use the virtual radiator without expanding the logarithm [i.e., use Eq.(10) instead of (11)] to find the differential cross section valid for all values of ν

$$\nu\frac{d\sigma}{d\nu} = \frac{\alpha}{\pi}\left(1 - \frac{\nu}{E} + \frac{\nu^2}{2E^2}\right)\ln\left[1 + \frac{m^2}{\mu^2}\frac{E}{\nu}\left(\frac{E}{\nu} - 1\right)\right]\left(\frac{Z}{6}\right)^2 1.52\ \mu b\ \frac{\mu^2}{m^2}\ \ln\frac{\nu}{6.7m}, \tag{19}$$

and then integrate this exactly. This method, however, is still sensitive to our original assumption. The second method is to integrate the exact cross section, and the third way is to perform a good experimental measurement. It is hoped that the first two methods can be tried shortly. The third method will have to wait until a 500-BeV proton hits a beam dump but will hopefully be performed shortly thereafter.

The results presented in this note show in a concise form that the energy loss and straggling by direct pair production is fundamentally different than that due to bremsstrahlung because the differential energy loss probability is peaked towards small energy loss in pair production.[11]

$$\nu\frac{d\sigma_{\text{pair}}}{d\nu} \sim \frac{1}{\nu^2}.$$

While it is flat for bremsstrahlung,

$$\nu\frac{d\sigma_{\text{brem}}}{d\nu} \sim \text{constant}.$$

It is hoped that the simple formulas presented here will somehow be of help in understanding the problem of energy loss and straggling of high-energy muons via direct pair production.

REFERENCES

[1] S. D. Drell and J. D. Walecka, Ann. Phys. (N.Y.), 28, 18 (1964).

[2] R. H. Dalitz and D. R. Yennie, Phys. Rev. 105, 1598 (1957); L. Hand and R. Wilson, Stanford Linear Accelerator Center Report #25, TID-4500 (1963); M. Gourdin, Nuovo Cimento 37, 208 (1965).

[3] See M. J. Tannenbaum, Phys. Rev. 167, 1308 (1968) for the detailed references.

[4] If you are not one of the multitude, see R. M. Sternheimer in *Methods of Experimental Physics*, Vol. 5, Yuan and Wu (Academic Press, New York, 1961).

[5] C. F. Von Weizsacker, Z. Physik 88, 612 (1934); F. J. Williams, Proc. Roy. Soc. (London) A139, 163 (1933).

[6] See Appendix C. in National Accelerator Laboratory Proposal 29, 1970.

[7] H. J. Bhabha, Proc. Roy. Soc. (London) A152, 559 (1935).

[8] M. M. Block, D. T. King, and W. W. Wada, Phys. Rev. 96, 1627 (1954).

[9] M. J. Tannenbaum, unpublished.

[10] M. Mando and L. Ronchi, Nuovo Cimento 9, 105 (1952); J. E. Cousins and N. F. Nash, Advances in Physics 11, 349 (1962).

[11] Note that we can use $\nu^2\, d\sigma/d\nu$ to find

$$\langle \overline{\nu^2} \rangle \equiv \frac{\langle \nu^2 \rangle_{pair}}{\sigma_{pair}} = 2\, E^2 \frac{m^2}{\mu^2} \ln \frac{\mu}{M},$$

$$\langle \bar{\nu} \rangle \equiv \frac{\langle \nu \rangle_{pair}}{\sigma_{pair}} = 2\, E \frac{m}{\mu},$$

and we can define a parameter

$$S^2 \equiv \langle \overline{\nu^2} \rangle - (\langle \bar{\nu} \rangle)^2,$$

which must be related to the straggling, and then we find

$$\frac{S^2}{E^2} = 2 \frac{m^2}{\mu^2} \left(\ln \frac{\mu}{m} - 2\right)$$

or

$$\frac{S}{E} = 1.25 \times 10^{-2}.$$

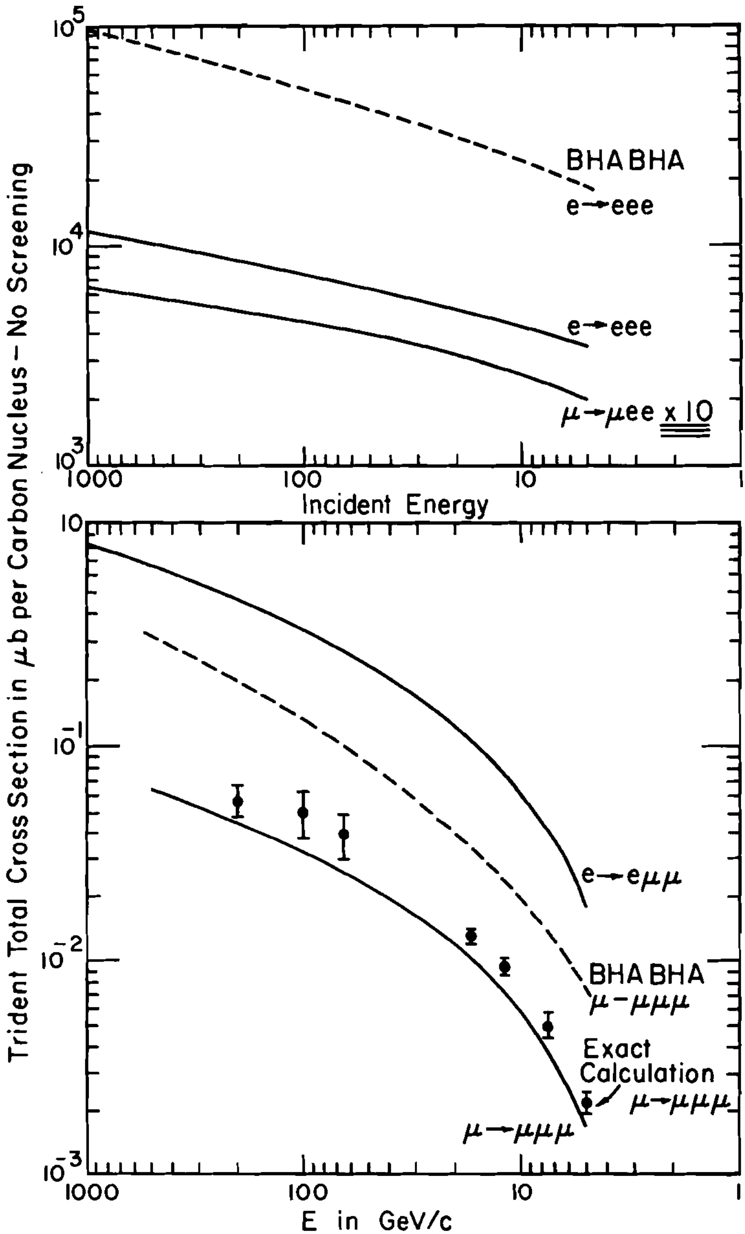


Fig. 1. Comparison of formulas given in the text for direct pair production (solid line) to the exact calculation (data points) and the BHABHA formula (broken line).

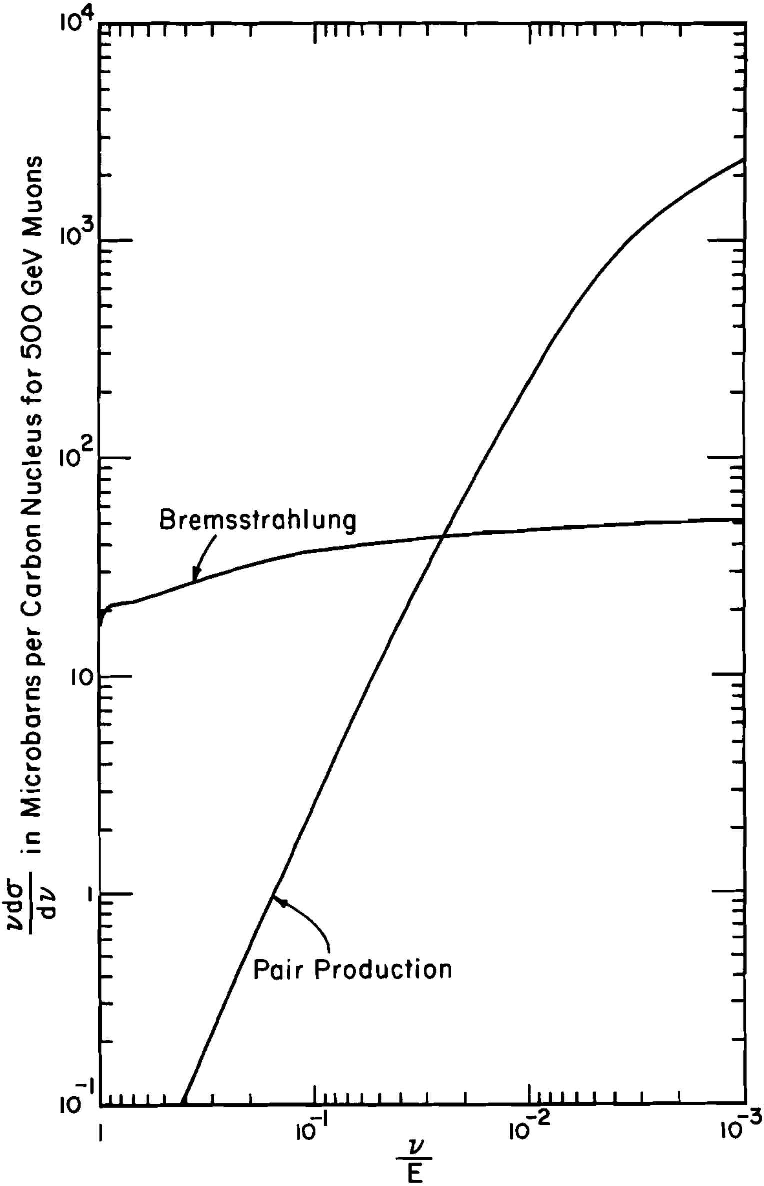


Fig. 2. Distribution of energy loss via bremsstrahlung and direct pair production for 500-GeV muons on carbon.