Radiative correction from the Wigner's little group perspective

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Abstract

Mainly motivated from the radiative energy loss calculation in quantum chromodynamics, we present a new approach to calculate the soft factor associated to a photon and gluon radiation. We show that the action of the little group on the single particle state provides the necessary condition to calculate the soft factor without Feynman diagrams. Further, this approach will lead a better understanding of the analytic structure of any radiation to any given process.

1 Introduction

In order to match the high precision physics at the Large Hadron Collider (LHC), it is required for the theoretical predictions to include high order corrections. In the anomalous magnetic moment of the electron, the quantum electrodynamics (QED) calculation matches the experimental precision with an order of 10^{12} by including multiple radiative correction; see [1]. Including such higher orders of correction is not just about precision, radiative correction may also unfold hidden phenomena. In the electroweak sector where initial state radiation leads to a reduced effective center of mass energy for hard scattering processes, which changes the shape of the measured Z resonance in $e^+e^- \to Z \to 2f$; see [2]. In heavy ion physics, a precise understanding of jet energy loss is needed to characterize several properties of the quark gluon plasma. Such an understanding requires a theoretical computation of multiple gluon emission, a mechanism that dominate jet suppression in for light flavored jets.

From classical electrodynamics it has been understood that an accelerated charge radiates photon. Similarly color charged particles radiate gluon from the sudden change of momentum caused by some hard scattering between the jet and the constituents of the bulk. Computing the effect of such radiation is very complex using the standard diagrammatic method of Feynman. However with the modern onshell method, the complexity in the multiple radiation effect is reduced to a simple factorization of the radiation effect to the born amplitude [6, 3]. This factorization is not a surprise since the scattering with the medium and soft radiation act respectively in two different scales, short and large distance scales. This is known as the factorization scales.

In the on-shell approach, the main idea is to generate higher-point amplitude from the lower-points using on-shell recursions. Among the different on-shell recursions, the most famous is the BCFW recursion relation of Britto, Cachazo, Feng and Witten [4]. Constructed from a complex deformations of the external momenta, the use of the BCFW needs a better understanding of three-point amplitudes that play the role of seeds in the recursion. And the success of the on-shell recursions relay on the fact that the seeds can be fully fixed by the symmetry of the theory, mainly by the little group action of the Lorentz group that leaves a momentum invariant [5].

In the present work, we review the computation of the soft factor from radiative correction by imposing scale factorization between the hard collision and soft radiation. Then we use the little group action to fixed the kinematics of the soft contribution part of the factorized amplitude. The advantage of this method is that we do not need the analytic expression of the hard scattering in order to fully fixed the kinematics of the soft factor of the radiation.

It is important to inform the reader that we will only consider massless theory through out of this work.

2 Wigner's little group W

2.1 Description

In quantum field theory, the Wigner's little groups are defined to be the subgroups of the Lorentz group whose transformations leave the momentum of a given particle invariant [5]. Denoted by W, the little groups are known as the symmetry groups for internal space-time structure of relativistic particles.
Let W_μ^ν be the little group associated to the momentum p_μ such that

$$W_\nu^\mu p_\mu = p_\nu. \tag{1}$$

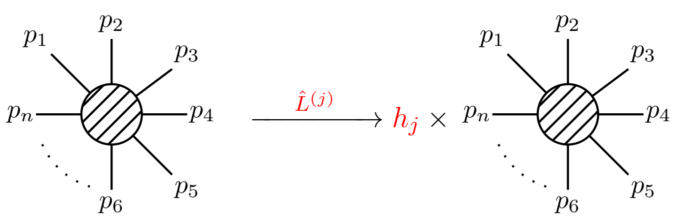
The action of the little groups on a massless particle are represented by the action of unitary groups $\hat{U}(W,\theta)$ on the associated single particle state $|p,h\rangle$. From such action, it as been shown in [6, 7] that the single particle state will just pic a phase

$$\hat{U}(W,\theta)\,|p,h\rangle = e^{i\theta h}\,|p,h\rangle\,, \tag{2}$$

where θ is a parameter associated to the little group, p the momentum of the massless particle that is invariant to the action of W as in (1), and h be its helicity. The relation (2) implies that the little group action of a massless particle state is isomorphic to a $U(1)$ Lie group generated by the helicity operator $\hat{L}$:

$$\hat{U}(W,\theta) \simeq 1 + \theta\hat{L} \quad \text{with} \quad \hat{L}\,|p,h\rangle = h\,|p,h\rangle\,. \tag{3}$$

Therefore on a generic amplitudes, the asymptotic behavior of the external particle leads to the following relation



, (4)

where $\hat{L}^{(j)}$ represents the helicity operator associated to the j-th particle with helicity h_j.

2.2 Spinor representation of W

The spinor representation is one of the key ingredients of on-shell method [8, 9]. In this representation, a momentum p_μ will be represented by a two by two matrix given by

$$P_{a\dot{a}} = \sigma^\mu_{a\dot{a}} p_\mu, \tag{5}$$

while the corresponding invariant is equal to the determinant of the 2×2 matrix $P_{a\dot{a}}$. For massless particles this determinant is zero and then the momentum can be parametrized as follow

$$P_{a\dot{a}} = \lambda_a \bar{\lambda}_{\dot{a}}, \tag{6}$$

where λ and $\bar{\lambda}$ are called the spinor helicity variables and transform, respectively, as a left and a right handed spinors. From these new variables we can define two different Lorentz invariant products that are both antisymmetrics:

$$\langle 12 \rangle = \epsilon_{ab} \lambda_1^a \lambda_2^b \qquad \text{and} \qquad [12] = \epsilon_{\dot{a}\dot{b}} \bar{\lambda}_1^{\dot{a}} \bar{\lambda}_2^{\dot{b}}, \tag{7}$$

which can be related to the four vector scalar product from $2p_1.p_2 = \langle 12 \rangle [12]$. In this representation, the $U(1)$ little group, that leaves p_μ invariant, becomes a simple scaling of the spinor variables,

$$W : \begin{cases} \lambda \longrightarrow t\, \lambda \\ \bar{\lambda} \longrightarrow t^{-1} \bar{\lambda}, \end{cases} \tag{8}$$

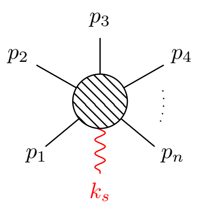
generated by the helicity operator

$$\hat{L} = -\frac{1}{2} \left(\lambda_a \frac{\partial}{\partial \lambda_a} - \bar{\lambda}_{\dot{a}} \frac{\partial}{\partial \bar{\lambda}_{\dot{a}}} \right). \tag{9}$$

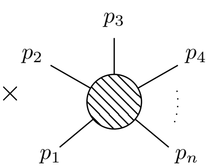
The single particle states and the amplitudes will be parametrized by the spinor variables associated to the individual particles and them helicities.

3 Helicity constraints

In this section, we will focus on the internal space-time structure of the scale factorization via the little groups action (2). Our first assumption is that the hard collision, occurs at small distance scale, does not interfere with the soft radiation at large distance scale. Such scale separation is represented by the following factorization of any given amplitude



$$\simeq \quad H\left(k_s; \{p_i\}\right) \times$$

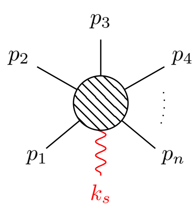


$$, \tag{10}$$

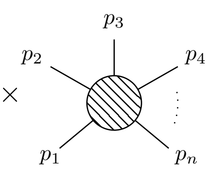
where p_i's are the momenta of the hard collision and k_s the momentum of the soft radiation in red. The function $H(k_s; \{p_i\})$ is the soft-factor that contain all the information on the soft radiation "s" and depends on the momentum k_s and some set of the hard momentum p_i.

In order for the factorization to be physical, the both sides of the relation (10) must have the same helicity, that is to say the relation has to hold under the helicity operator

$$\hat{L}^{(I)} \left[\right.$$



$$\left. \right] \simeq \hat{L}^{(I)} \left[H\left(k_s; \{p_i\}\right) \times \right.$$



$$\left. \right], \tag{11}$$

where $\hat{L}^{(I)}$ is the helicity operator associated to the I-th particule, $I = \{p_i\} \cup k_s$.

The action of any helicity operator $\hat{L}^{(I)}$ on amplitudes is given in (4), and in order to (11) to hold, the following constraint has to be satisfied by the soft-factor H,

$$\hat{L}^{(I)} H(k_s; \{p_i\}) = \delta_{I,s}\, h_s H(k_s; \{p_i\}). \tag{12}$$

Here $\delta_{I,s}$ is the Kronecker symbol; and h_s is the helicity of the soft particle. Since the helicity operator $\hat{L}$ is a scaling generator in the spinor representation (9), therefor one can say that the soft-factor only scale under a soft scaling

$$\begin{cases} \hat{L}^{(s)} H(k_s; \{p_i\}) = h_s H(k_s; \{p_i\}), \\ \hat{L}^{(i)} H(k_s; \{p_i\}) = 0. \end{cases} \tag{13}$$

The simplicity of such constraints relay in the spinor representation where it becomes a system of linear equations as in [10].

4 Radiative calculation

In this section we are going to solve the helicity constraint (13). To simplify the problem, the hard collisions between the parton and the medium will be considered as a $2 \to 2$ process. It is also well known that parton medium interactions are dominated by a t-channel exchange for high-p_T jet as shown in Fig.1.

(a) t-channel exchange QED (b) t-channel exchange QCD

Figure 1: Scattering between the hard jet in red and the constituent of the medium.

The t-channel interactions are characterized by $(p_3 - p_4)^2 = q^2$, which is equal to zero for the case where q_μ be on-shell, and such case is characterized by the proportionality of the following spinor variables

$$\lambda_a^{(3)} \sim \lambda_a^{(4)} \sim \lambda_a^{(q)} \quad \text{or} \quad \bar{\lambda}_{\dot{a}}^{(3)} \sim \bar{\lambda}_{\dot{a}}^{(4)} \sim \bar{\lambda}_{\dot{a}}^{(q)}, \tag{14}$$

where $\lambda^{(i)}$ and $\bar{\lambda}_{\dot{a}}^{(i)}$ are the spinor variables associated to the particle with momentum p_i, and "$\sim$" be the symbol of proportionality. Inspired by the MHV amplitudes where it depends only on λ's (independent of $\bar{\lambda}$'s), we choose the case where the soft-factor $H(k_s; \{p_i\})$ will depends only on λ's for $h_s = -1$.

4.1 Single emission

For the single emission, the solution of the helicity constraints (13) is given by

$$H(s) = A \frac{\langle 14 \rangle}{\langle 1s \rangle \langle s4 \rangle} + B \frac{\langle 42 \rangle}{\langle 4s \rangle \langle s2 \rangle}, \tag{15}$$

where the angle braket's are the spinor scalar products defined in (7). The label "1" and "2" are the kinematics of the parton jet, "3" and "4" the kinematics characteristic of the medium, and "s" the kinematic of the radiated gluon. A and B are the factors that contain the information on the non-kinematic degrees of freedom, such as coupling, color charge, ...

- In the case of photon radiation: we have $A = B = Qe$, electrical charge of the parton, and after simplification we obtain

$$H_\gamma(s) = e\frac{\langle 12\rangle}{\langle 1s\rangle\,\langle s2\rangle}. \tag{16}$$

- In the case of gluon radiation : $A = gT_a\hat{C}$ and $B = g\hat{C}T_a$, where g is the coupling of QCD, T_a the color charge associated to the emitted gluon, and $\hat{C}$ the color charge associated to the hard amplitude in Fig.1b. The soft-factor is given by

$$H_g(s) = gT_a\hat{C}\frac{\langle 14\rangle}{\langle 1s\rangle\,\langle s4\rangle} + g\hat{C}T_a\frac{\langle 42\rangle}{\langle 4s\rangle\,\langle s2\rangle} \tag{17}$$

4.2 Double emission

For the double emission, it is useful to decompose the double soft-factor into an independent emission piece and a correlated one. Such decomposition can be written as

$$H(s_1, s_2) = H(s_1)H(s_2) + \frac{\xi}{\langle s_1 s_2\rangle}R(s_1, s_2), \tag{18}$$

In this expression ξ is a correlation parameter that is only non-zero for non-Abelian theory since in the Abelian case the multiple radiaiton are independent [9]. $R(s_1, s_2)$ are some function that scale only on p_{s_1} and p_{s_2} and from (13), one can derive that

$$\hat{L}^{(s_i)}R(s_1, s_2) = \frac{1}{2}R(s_1, s_2) \qquad \text{and} \qquad \hat{L}^{(i)}R(s_1, s_2) = 0. \tag{19}$$

In a collinear limit where the two radiation are emitted at a very small angle from the high energetic parton, we can find

$$\frac{R(s_1, s_2)}{\langle s_1 s_2\rangle} \xrightarrow{p_{s_1}\|p_{s_2}} H(p_s)\text{Split}(z, p_{s_1}, p_{s_2}) \tag{20}$$

where $H(p_s)$ is the soft-factor of a single particle with momentum $p_s = p_{s_1} + p_{s_2}$, and the function $\text{Split}(z, p_{s_1}, p_{s_2})$ is the splitting amplitude that gives the probability for a soft radiation with momentum p_s to be composed by two collinear emission, see [11], with a fraction z, i.e

$$p_{s_1} = zp_s \quad \text{and} \quad p_{s_2} = (1 - z)p_s. \tag{21}$$

5 Conclusion

To conclude let us remind ourselves that the main goal of this work is to compute the soft-factor by assuming scale factorization. To achieve that goal we introduced the Wigner's little groups and them action on the particle state, massless particle. Then we derived the helicity constraints equations to the soft-factor by applying the little group action to the scale factorization equation. And finally we solved these kinematic constraint for the case of single and double emission.

The derivation of the helicity constraints in (13) allow us to compute the soft-factor associated to radiation induced from hard collision. Here the constraints form a system of linear equation with which we can fix the kinematics of the soft-factor. As shown in (15), the different constant in the general expression of single emission can be fixed by the different degrees of liberty of the theory. The solution for the single photon and gluon emissions respectively given by (16) and (17) agree with the known solutions [3, 9, 11]. For the double emission the introduction of the introduction of the correlation parameter ξ in our ansatz allows us to differentiate the Abelian and the non-Abelian part for the solution. by plugin the ansatz back in the helicity constraints gave to the unknown function $R(s_1, s_2)$ some constraint lead to the resolution of the double emission soft-factor.

These result showed us that our method of resolution is very efficient to compute radiation soft-factors and avoid the complication caused by the large number of diagrams in the Feynman approach.

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