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THE RAOELINIAN OPERATOR

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Dedication: This work is dedicated to my beloved wife and our 6 children,to my 50 years as professor in the University of Antanarivo,to 48 years as full member of Malagasy Academy , to the 42 years of setting up of the LPNPA from scratch,to the 26 years of founding of INSTN-Madagascar,to the 5th anniversary of the official setting up of CORANANO from scratch,and last but not least to the great number of students of mine.

Abstract: The raoelinian operator for causal and anticausal functions is introduced to unify the derivative and integral operators for any order (real,complex number).The method is based on properties of Euler's gamma and beta functions.The properties of raoelinian operator are given too (linearity,semi-group property,principle of correspondence,obtention of the integral and derivative operators from the raoelinian operator) . Liouville fractional integral,Riemann fractional integral,Caputo fractional derivative,Liouville-Caputo fractional derivative are particular cases of the raoelinian operator. Remarkable relations verified by the raoelinian operators of $\pm e, \pm\pi$ orders are derived.

1-Introduction

The underlying idea comes from the expressions of $n-$derivatives D^n and $n-$order integral J^n of trigonometric and exponential function for n positive integer.

$$\begin{cases} D^1\,(sin)(x) = sin(x+\frac{\pi}{2}) \\ D^1\,(cos)(x) = cos(x+\frac{\pi}{2}) \end{cases} \qquad D^n\,(sin)(x) = sin(x+n\frac{\pi}{2}) \quad D^n\,(cos)(x) = cos(x+n\frac{\pi}{2})$$

$$\begin{cases} J^1\,(sin)(x) = sin(x-\frac{\pi}{2}) \\ J^1\,(cos)(x) = cos(x-\frac{\pi}{2}) \end{cases} \qquad J^n\,(sin)(x) = sin(x-n\frac{\pi}{2}) \quad J^n\,(cos)(x) = cos(x-n\frac{\pi}{2})$$

We define **one operator $\mathcal{R}^s$ for any real number s** such

$$\mathcal{R}^s\,(sin)(x) = \begin{cases} D^s\,(sin)(x) & \text{for any positive real number s} \\ J^{-s}\,(sin)(x) & \text{for any negative real number s} \end{cases}$$

$$\begin{cases} D^1\,(exp)(kx) = k\,(exp)(kx) \\ J^1\,(exp)(kx) = k^{-1}\,(exp)(kx) \end{cases} \qquad D^n\,(exp)(kx) = k^n\,(exp)(kx) \quad J^n\,(exp)(kx) = k^{-n}\,(exp)(kx)$$

Let us define the **one operator R^s for any real number s** such

$$\mathcal{R}^s\,(exp)(kx) = k^s\,(exp)(kx) = \begin{cases} D^s\,(exp)(kx) & \text{for any positive real number s} \\ J^{-s}\,(exp)(kx) & \text{for any negative real number s} \end{cases}$$

2-Method

Our method is the following:

1. Define the integral operator J^1 and the derivative operator D^1 of the first order $(s = 1)$over the definition set E ,
2. Define the integral operator J^s and the derivative operator D^s for any $s \in \mathbb{N}$ (positive integer number),
3. Extend to $s \in \mathbb{Z}$ (positive and negative integer number),
4. Define $\mathcal{R}^s$,
5. Extend to any $s \in \mathbb{R}$ (real number),
6. Extend to any $s \in \mathbb{C}$ (complex number),
7. Look for in which case we have

   α) The principle of correspondence

$$\lim_{s \to n} J^s(f)(x)(a) = \int_a^x \int_a^{t_1} \int_a^{t_2} \dots \int_a^{t_M} f(t_M) dt_M dt_{M-1} \dots dt_1 = J^n(f)(x)(a) \quad for\ s \in \mathbb{N}$$

$$\lim_{s \to n} D^s(f)(x)(a) = \frac{d^n}{dx^n}(f)(x)(a) \quad for\ s \in \mathbb{N}$$

$\frac{d^n}{dx^n}$ and J^n are respectively the ordinary derivative and lntegral operators of n order

β) linear property of $\mathcal{R}^s$

$$\mathcal{R}^s(\lambda f + \mu g) = \lambda \mathcal{R}^s(f) + \mu \mathcal{R}^s(g)$$

for any λ and μ (real or complex number) and any f and g belonging to the definition set E.

3-Definition of the raoelinian operator $\mathcal{R}^s$

Theorem: Let E be the set of infinite integrable and derivable functions defined on the interval $I = [a, +\infty[, a \in \mathbb{R}$ such $f(x) = 0$ for $x \leq a$ (causal function.). The raoelinian operator $\mathcal{R}^s$ is defined by the relation

$$\mathcal{R}^s(f)(x)(a) = \begin{cases} \dfrac{1}{\Gamma(s)} \displaystyle\int_a^x (x-y)^{s-1} f(y)dy = \dfrac{x^s}{\Gamma(s)} \displaystyle\int_{\frac{a}{x}}^1 (1-u)^{s-1} f(ux)du & for\ Re(s) \geq 0 \\ \dfrac{1}{\Gamma(k+s)} \displaystyle\int_a^x (x-y)^{k+s-1} D^k f(y)dy = \dfrac{x^{s+k}}{\Gamma(k+s)} \displaystyle\int_{\frac{a}{x}}^1 (1-u)^{k+s-1} D^k f(ux)du & for\ Re(s) \leq 0 \end{cases}$$

where Γ is the extension of Euler gamma function. k is a positive integer verifying $k + s \notin \mathbb{Z}_-$ (in fact the Γ function is not defined for nonpositive integers). D^k is the ordinary derivative operator of order k. The definition $\mathcal{R}^s$ is independent of on the choice of k.

The raoelinian operator $\mathcal{R}^s$ gives at the same time the integrals and derivatives operators at any order s. If $s \in \mathbb{N} = \mathbb{Z}_+$, $\mathcal{R}^s$ is the ordinary integral of order s. If $s \in \mathbb{Z}^-$ (negative integer), $\mathcal{R}^s$ is the ordinary derivation of order $|s|$. If $s \in \mathbb{R}_+$ (positive real) or $Re(s) > 0 (s \in \mathbb{C})$, the operator $\mathcal{R}^s$ is the fractional integral operator J^s of f at any order s. If $s \in \mathbb{R}_-$ (negative real) or $Re(s) < 0$ $(s \in \mathbb{C})$, the operator $\mathcal{R}^s$ is the fractional derivative operator D^s of f at order $|s|$ or $|Re(s)|$.

Proof: The proof is given in our paper [7] for real orders, in our paper [6] for complex order s, in our paper [8] for real and complex order for causal functions.
The case of anticausal function ($f = 0$ for $x \geq a$) is studied in our paper [9]. The antiintegral and antiderivative are introduced.

4-Properties of the raoelinian operator $\mathcal{R}^s$.

4.1 Linear property

$$\mathcal{R}^s(\lambda f + \mu g) = \lambda \mathcal{R}^s(f) + \mu \mathcal{R}^s(g) \quad \forall \lambda \in \mathbb{R} \; or \; \mathbb{C}; \;\; \forall \mu \in \mathbb{R} \; or \; \mathbb{C}; \; \forall f \in E; \forall g \in E$$

Proof: The proof is very easy.

4.2 Semi group proprety of $\mathcal{R}^s$

$$\mathcal{R}^{s_1}\mathcal{R}^{s_2} = \mathcal{R}^{s_1+s_2} = \mathcal{R}^{s_2}\mathcal{R}^{s_1} \quad \forall \, s_1 \in \mathbb{R} \; or \; \mathbb{C}, \forall \, s_2 \in \mathbb{R} \; or \; \mathbb{C}$$

Proof: The proof is given in Appendix I. It is assumed that the derivative D^1 (respectively the integral operator J^1) is the inverse of the integral operator J^1 (respectively D^1).
The study of the inverse of an operator is given in Appendix II.

4.3 Principle of correspondence.

It is easily shown the following properties

$$\lim_{s \to n} \mathcal{R}^s \, (f)(x)(a) = \mathcal{R}^n(f)(x)(a) = J^n(f)(x)(a) \quad \forall \, n \in \mathbb{N}, \forall \, f \in E, \forall a \in \mathcal{R}$$

$$\lim_{s \to -m} \mathcal{R}^s \, (f)(x)(a) = \mathcal{R}^{-m}(f)(x)(a) = D^m(f)(x)(a) \quad \forall \, m \in \mathbb{N}, \forall \, f \in E, \forall a \in \mathcal{R}$$

where J^n is the ordinary integral operator of order n and D^m is the ordinary derivative operator $\frac{d^m}{dx^m}$ of order m. Then,

$$\lim_{s \to n} \mathcal{R}^s = J^n \qquad \forall \, n \in \mathbb{N} \qquad\qquad \lim_{s \to -m} \mathcal{R}^s = D^m \qquad \forall \, m \in \mathbb{N}$$

4.4 Obtention of the integral operator J^n and the derivative operator D^m from the raoelinian operator $\mathcal{R}^s$.

4.4.1 Obtention of the integral operator J^n from the raoelinian operator $\mathcal{R}^s$

It may be easily shown

$$\mathcal{R}^s(f)(x)(a) = J^s(f)(x)(a) \Leftrightarrow \mathcal{R}^s = J^s \quad \forall\, s \in \mathbb{R}_+ \; or \; Re(s) > 0 \; if \; s \in \mathbb{C}, \forall\, a \in \mathbb{R}$$

4.4.2 Obtention of the derivative operator D^n from the raoelinian operator $\mathcal{R}^s$

a) The left- hand side derivative operator D_L^m

$$D_L^m = \mathcal{R}^{-k}\mathcal{R}^{k-m} = \mathcal{D}^k\mathcal{R}^{k-m} \quad \forall\, k \in \mathbb{N}, \forall\, m \in \mathbb{R}_+ \; or \; Re(m) > 0 \; if \; m \in \mathbb{C}$$

no summation on $k,$ is independent on the choice of k.

Proof: It may be obtained easily by application of the semi-group property. The direct calculation is not difficult.

$$D_L^m(f)(x)(a) = D^k \, \frac{1}{\Gamma(k-m)} \int_a^x (x-y)^{k-m-1} \, f(y)dy$$

By successive application of the operator D^1 on the integral $(1 \leq k \leq E[m]$ with $E[m]$ the integer part of m, it is easily found that $D_L^m(f)(x)(a)$ is in fact independent on k.

Taking $k = 1$, we obtain

$$D_L^m(f)(x)(a) = D^1 \, \frac{1}{\Gamma(k-m)} \int_a^x (x-y)^{k-m-1} \, f(y)dy$$

If $f(y) = C$, where C is a constant

$$D_L^m(C)(a) = C\frac{(x-a)^m}{\Gamma(1-m)} \neq 0$$

is not null if C is not a null constant.

The Riemann m −fractionnal derivative is defined by [4]

$$D_{Ri}^m(f)(x) = \frac{d}{dx}\frac{1}{\Gamma(k-m)}\int_0^x (x-y)^{k-m-1} f(y)dy$$

which is exactly our $D_L^m(f)(x)(a)$ for $a = 0$.

b) The right-hand side derivative operator D_R^m

$$\mathcal{R}^{k-m}\mathcal{R}^{-k} = \mathcal{R}^{k-m}\mathcal{R}^{k} = D_R^m$$

$\forall\, k \in \mathbb{N}, \forall\, m \in \mathbb{R}_+ \; or\; Re(m) > 0 \; if\; m \in \mathbb{C},\; 1 \leq k \leq E[m]$ (with $E[m]$ the integer part of m.). no summation on k, is independent of the choice of k.

Proof:

$$D_R^m(f)(x)(a) = \mathcal{R}^{k-m}(D^k f)(x)(a) = \frac{1}{\Gamma(k-m)}\int_a^x (x-y)^{k-m-1}\frac{d^k}{dy^k} f(y)dy$$

By successive integration by part of the function f and taking account the limit values, we obtain that the second member is independent on the choice of k. Taking $k = 1$, we have

$$D_R^m(f)(x)(a) = \frac{1}{\Gamma(1-m)} \int_a^x (x-y)^{-m} \frac{d^1}{dy^1} f(y)dy$$

If $f(y) = C$ where C is a constant

$$D_R^m(C)(a) = 0 \quad even\, if\ C \neq 0$$

D_R^m is the good choice rather D_L^m if we require the principle of correspondence (the derivative of a of a constant is null)

Let us introduce the raoelinian operator $\mathcal{R}^{-m}$ for $m \in \mathbb{R}_+ (\ or\ Re(m) > 0\ if\ m \in \mathbb{C}\ )$ and show that

$$D_R^m = \mathcal{R}^{-m}$$

$$\mathcal{R}^s(f)(x)(a) = \frac{1}{\Gamma(s)} \int_a^x (x-y)^{\,s-1} f(y)dy$$

Let us integrate by part the function under the integral sign and take account of the limit values:

$$\mathcal{R}^s(f)(x)(a) = \frac{1}{\Gamma(s+1)} \int_a^x (x-y)^s \frac{d}{dy} f(y)dy$$

$$\mathcal{R}^{-m}(f)(x)(a) = \frac{1}{\Gamma(1-m)} \int_a^x (x-y)^{-m} \frac{d}{dy} f(y)dy = D_R^m(f)(x)(a)$$

$$\mathcal{R}^{-m} = D_R^m \qquad\qquad m \in \mathbb{R}_+ (\ or\ Re(m) > 0\ if\ m \in \mathbb{C}$$

The Liouville-Caputo fractional derivative definition is [4]

$$ {}_{LC}D_{+}^{m}(f)(x) = \frac{1}{\Gamma(1-m)} \int_{-\infty}^{x} (x-y)^{-m} \frac{d}{dy} f(y) dy $$

which is exactly $D_R^m(f)(x)(-\infty) = \mathcal{R}^{-m}(f)(x)(-\infty)$ with the particular value $a = -\infty$.

The Caputo fractional derivative is defined by [4]

$$ {}_{C}D_{+}^{m}(f)(x) = \frac{1}{\Gamma(1-m)} \int_{0}^{x} (x-y)^{-m} \frac{d}{dy} f(y) dy $$

which is exactly $D_R^m(f)(x)(0)$ with the value $a = 0$

Remarks

a) If the operatorD^1 (respectively J^1) is the inverse of the operator J^1 (respectively D^1), then we have the semi-group property for the raoelinian operator $\mathcal{R}^s$ and we have **one** derivative operator $D_L^m = D_R^m$
If it is not the case, the semi-group property does not stand for the raoelinian operator $\mathcal{R}^s$ but we have the semi-group property for J^s and D^s separately.

b) The definition of the derivative operator D^s obtained from the definition of the integral operator J^s

$$D^s(f)(x)(a) = D^k J^{k-s}(f)(x)(a)$$

introduces the choice on the positive number k $(k - s > 0)$. We have shown in fact that the final result is independent on k .
We have the relation

$$J^1(f)(x)(a) = \int_a^x f(t)dt = F(x) - F(a)$$

F being a primitive of f.

$$D^1 J^1(f)(x)(a) = F'(x) = f(x) \quad \text{or } D^1 J^1 = 1_E$$

$$J^1 D^1(f)(x)(a) = \int_a^x f'(t)\, dt = f(x) - f(a) \neq f(x)\ if\ f(a) \neq 0 \quad J^1 D^1 \neq 1_E$$

$$J^1 D^1 = D^1 J^1 = 1_E \text{ if and only if } f(a) = 0$$

If $J^1 D^1 \neq D^1 J^1$ then, we have the problem of the choice (left- hand side or right- hand side) for the derivative. We give the study about the inverse of an operator in Appendix II.

5-Applications of our results.

Let us take $f(x) = x^p$ for any positive real p

$$\mathcal{R}^s(x^p) = \frac{\Gamma(p+1)}{\Gamma(p+s+1)} x^{p+s} = \frac{p!}{(p+s)!} x^{p+s}$$

With the extension of the eulerian gamma function $\Gamma(k)$ for any positive real s , we have the following results

$$\mathcal{R}^\pi(x^e) = J^\pi(x^e) = \frac{e!}{(e+\pi)!} x^{\pi+e} \qquad \mathcal{R}^e(x^\pi) = J^e(x^\pi) = \frac{\pi!}{(e+\pi)!} x^{\pi+e}$$

The ratio

$$\frac{\mathcal{R}^\pi(x^e)}{\mathcal{R}^e(x^\pi)} = \frac{e!}{\pi!} = 0{,}592761\ 747048\ 502\ 880\ 285354\ 552\ 437\ 32\ \ldots$$

is independent of x.

$$\mathcal{R}^{-\pi}(x^e) = D_R^\pi(x^e) = \frac{e!}{(e-\pi)!} x^{e-\pi} \qquad \mathcal{R}^{-e}(x^\pi) = D_R^e(x^\pi) = \frac{\pi!}{(\pi-e)!} x^{\pi-e}$$

The product

$$\mathcal{R}^{-\pi}(x^e)\mathcal{R}^{-e}(x^\pi) = \frac{e!}{(e-\pi)!}\frac{\pi!}{(\pi-e)!} = 22.364\ 994\ 517\ 058\ 857\ 454\ 906\ 921\ 720\ 114\ \ldots$$

is independent of x. Direct calculations of these results have been given in our paper [7].

Appendix I: The proof of the semi-group property of the raoelinian operator $\mathcal{R}^s$

We have to demonstrate the relation

$$\mathcal{R}^{s_1}\mathcal{R}^{s_2} = \mathcal{R}^{s_1+s_2} = \mathcal{R}^{s_2}\mathcal{R}^{s_1}$$

It is assumed that the operator J^1 (respect. D^1) is the inverse of the operator D^1 (respect. J^1)

$$\mathcal{R}^{s_1}\mathcal{R}^{s_2}(f)(x)(a) = \frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_a^x dy(x-y)^{s_1-1}\int_a^x (y-z)^{s_2-1} f(z)dz \quad \forall f \in E, \forall s_1, s_2 \in \mathbb{R}\ or\ \mathbb{C}, \forall a \in \mathbb{R}$$

Applying Dirichlet's formula [11]

$$\int_a^x dy(x-y)^{s_1-1}\int_a^x (y-z)^{s_2-1} g(y,z)dz = \int_a^z dz\int_z^x dy\,(x-y)^{s_1-1}(y-z)^{s_2-1}$$

with $g(y,z) = f(z)$, we have

$$\mathcal{R}^{s_1}\mathcal{R}^{s_2}(f)(x)(a) = \frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_a^x dzf(z)\int_x^z dy(x-y)^{s_1-1}\,(y-z)^{s_2-1}$$

We introduce the variable $u = \frac{y-z}{x-z}$ $\quad y = z + u(x-z) \quad dy = (x-z)du$

$$\mathcal{R}^{s_1}\mathcal{R}^{s_2}(f)(x)(a) = \frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_a^x dzf(z)\,(x-z)^{s_1+s_2-1}\int_0^1 du(1-u)^{s_1}\,u^{s_2-1} = \frac{1}{\Gamma(s_1+s_2)}\int_a^x dzf(z)(x-z)^{s_1+s_2-1}$$

$$\Longrightarrow \mathcal{R}^{s_1}\mathcal{R}^{s_2}(f)(x)(a) = \frac{1}{\Gamma(s_1+s_2)}\int_a^x dzf(z)(x-z)^{s_1+s_2-1} = \mathcal{R}^{s_1+s_2}(f)(x)(a)$$

$$\Rightarrow \mathcal{R}^{s_1}\mathcal{R}^{s_2} = \mathcal{R}^{s_1+s_2}$$

The expression is symmetric in $s_1$and s_2

$$\mathcal{R}^{s_1}\mathcal{R}^{s_2} = \mathcal{R}^{s_1+s_2} = \mathcal{R}^{s_2}\mathcal{R}^{s_1}$$

Appendix II : Inverse of an operator [12]

Theorem: Any operator A has **at least** a right- hand side inverse C.

Proof: Let y be an element of the value domain $Val(A)$ of an operator A. By definition of $Val(A)$, there is at least an element x_0 belonging to the definition domain $Def(A)$ of the operator A such as

$$A(x_0) = y$$

For any element $y \in Val(A)$, we may choose an x. Let us designate it by x_0 and define the operate C such

$$C(y) = x_0$$

Then $A(C)(y) = A(x_0) = y$ for any $y \in Val(A)$

$$AC = 1_{Val(A)}$$

in which $1_{Val(A)}$ is the identity operator on $Val(A)$. **It depends on a choice**.

Example: The inverse function in the classical meaning is in fact a right hand-side inverse. For instance, Arcsin is the right-hand side inverse function of the sinus function

$$\sin(Arcsin) = 1_{Val(\sin)} = 1_{[-1,1]}$$

$\sin(Arcsin)\,(a) = sin\beta = a$in which $\beta = Arcsin(a)$ is the principal determination belonging to the angular interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We have $Arcsin(\alpha) = x_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$Arcsin(sin)(x) = Arcsin(sinx)$$

$$Arcsin(\alpha) = x_0$$

$$Arcsin(sinx) = 1_{Def\ sin}$$

$$Val\big(Arcsin(sin)\big) = [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$\big(Arcsin(sin)\big) = Def(sin) = \mathbb{R}$$

The inverse function implies a choice for the determination for a function having multideterminations (function like $\sqrt{\ }$, Arcsin, Arctg,etc,…). It is worth pointing out that Arcsin is an operator but arcsin is not an operator because there are many (infinite) values of arcsin(x) for one given x.

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