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Fermion fields and fermion states in a phase space representation of quantum theory

Ravo Tokiniaina Ranaivoson[1], Raoelina Andriambololona[2], Hanitriarivo Rakotoson[3], Roland Raboanary[4]

tokhiniaina@gmail.com[1], *tokiniainaravor13@gmail.com*[1], *raoelinasp@yahoo.fr*[2], *raoelina.andriambololona@gmail.com*[2], *jacquelineraoelina@hotmail.com*[2], *infotsara@gmail.com*[3], *r_raboanary@yahoo.fr*[4]

Information Technology and Theoretical Physics Department[1,2,3]
Institut National des Sciences et Techniques Nucléaires (INSTN- Madagascar)
BP 3901 Antananarivo101, Madagascar, *instn@moov.mg*

Mention Physique et Applications, Faculté des Sciences – University of Antananarivo[2,4]

Abstract: This work is about the study on an approach to establish a description of fermion fields and fermion states in a framework of a phase space representation of quantum theory. This approach is based on the results obtained after a series of previous works concerning the phase space representation. Some significant parts of these previous results has been reviewed and exploited and then extended. Expression of fermion fields operators fitting with the phase space representation has been established, with the corresponding field equation, from the current formulation of quantum electrodynamics. Then discussion about the description of the fermion fields and fermion states in phase space representation has been deduced from the results of previous works concerning the relation between the phase space representation, linear canonical transformation and properties of elementary fermions.

Keywords: *Phase space representation, fermion fields, fermion states, field equation, particle properties*

1-Introduction

In our paper [1], an approach has been introduced to deal with the problem of establishing a phase space representation of quantum theory which takes into account the Heisenberg uncertainty relation. The formulation corresponding to this approach was enriched and developed through our works [2-8]. This work aims at extending this formulation in order to obtain the description of fermions fields and fermions states in the framework of the phase space representation.

From then, the basic formulation of the phase space representation is reviewed. This leads to the development of an approach which permits to deduce the expression of the fermions field and its corresponding equation, compatible with the phase space representation, from the current formulation of quantum electrodynamics. It is followed by the review of the results obtained in [8] concerning the relation between the phase space representation, linear canonical transformations and properties of the elementary fermions of the Standard Model. Thus, discussion concerning the establishment of a description of fermions fields and fermions states within the phase space representation is deduced.

2-Basic formulation of the phase space representation

From the current formulation of quantum mechanics, it is well known that the main representations which are used to represent the quantum state of a particle are the coordinate and momentum representations. For the coordinate representation, the basis of the states space, used to expand a state $|\psi\rangle$, is composed by the eigenstates $|x\rangle$ of the coordinate operator $\boldsymbol{x}$

$$\begin{cases} |\psi\rangle = \int |x\rangle\,\langle x|\psi\rangle dx = \int \psi(x)|x\rangle\,dx \\ \boldsymbol{x}|x\rangle = x|x\rangle \end{cases} \tag{1.1}$$

For the momentum representation, the basis elements are the eigenstates $|p\rangle$ of the momentum operator $\boldsymbol{p}$

$$\begin{cases} |\psi\rangle = \int |p\rangle\,\langle p|\psi\rangle dx = \int \tilde{\psi}(p)|p\rangle\,dp \\ \boldsymbol{p}|p\rangle = p|p\rangle \end{cases} \tag{1.2}$$

The functions $\psi(x)$ and $\tilde{\psi}(p)$ are the wavefunctions respectively in the coordinate representation and in the momentum representation. They are linked by a Fourier transformation

$$\tilde{\psi}(p) = \langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int \langle p|x\rangle\langle x|\psi\rangle\,dx = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-i\frac{px}{\hbar}}\psi(x)\,dp \tag{1.3}$$

The phase space representation, introduced in [1] may be seen as a coordinate-momentum joint representation which takes into account the Heisenberg uncertainty principle. It permits to have simultaneously a look at the values of the coordinate and momentum of a particle without violating the uncertainty principle. The expression of a quantum state $|\psi\rangle$ in this representation is

$$|\psi\rangle = \iint \Psi(n, X, P, 𝒷)|n, X, P, 𝒷\rangle \frac{dXdP}{2\pi\hbar} = \sum_{\boldsymbol{n}} \Psi(n, X, P, 𝒷)|n, X, P, 𝒷\rangle \qquad (1.4)$$

in which the elements $|n, X, P, 𝒷\rangle$ of the basis which defines the representation are the phase space states [1]. We have the relations

$$\begin{cases} \langle n, X, P, 𝒷|\boldsymbol{x}|n, X, P, 𝒷\rangle = X \\ \langle n, X, P, 𝒷|\boldsymbol{p}|n, X, P, 𝒷\rangle = P \\ \left\langle n, X, P, 𝒷\middle|(\boldsymbol{x} - X)^{\boldsymbol{2}}\middle|n, X, P, 𝒷\right\rangle = (2n+1)(a)^2 = (2n+1)\mathcal{A} \\ \left\langle n, X, P, 𝒷\middle|(\boldsymbol{p} - P)^{\boldsymbol{2}}\middle|n, X, P, 𝒷\right\rangle = (2n+1)(𝒷)^2 = (2n+1)\mathcal{B} \end{cases} \qquad (1.5)$$

According to these relations, $X, P, (2n+1)\mathcal{A}, (2n+1)\mathcal{B}$ are respectively the means values and statistical dispersions (variances) of the coordinate and momentum in the state $|n, X, P, 𝒷\rangle$. n being a positive integer. The standard deviation (uncertainty) $a = \sqrt{\mathcal{A}}$ and $𝒷 = \sqrt{\mathcal{B}}$ relative to the state $|0, X, P, 𝒷\rangle$ corresponds to a saturation of the Heisenberg uncertainty relation that is

$$\begin{cases} a𝒷 = \frac{\hbar}{2} \\ \mathcal{A}\mathcal{B} = (\frac{\hbar}{2})^2 \end{cases} \qquad (1.6)$$

The states $|n, X, P, 𝒷\rangle$ are the eigenstates of the momentum dispersion operator $\beth^+$ [1, 3, 4]

$$\beth^+ = \frac{1}{2}[(\boldsymbol{p} - P)^{\boldsymbol{2}} + 4(\frac{\mathcal{B}}{\hbar})^2(\boldsymbol{x} - X)^{\boldsymbol{2}}] \Rightarrow \beth^+|n, X, P, 𝒷\rangle = (2n+1)\mathcal{B}|n, X, P, 𝒷\rangle$$

The functions $\Psi(n, X, P, 𝒷)$ in the relations (1.4) are the phase space wavefunctions [1, 6]. They are related to the wavefunction $\psi(x)$ in coordinate representation by the relation

$$\Psi(n, X, P, 𝒷) = \int \langle n, X, P, 𝒷|x\rangle\langle x|\psi\rangle\, dx = \int \frac{H_n\left(\frac{x - X}{\sqrt{2}a}\right)}{\sqrt{2^n n!\sqrt{2\pi}a}} e^{-\left(\frac{x-X}{2a}\right)^2 - i\frac{Px}{\hbar}} \psi(x)dx \qquad (1.7)$$

in which H_n is a Hermit polynomial of degree n.

We may remark that the wavefunction in coordinate representation corresponding to a state $|n, X, P, 𝒷\rangle$ which permits to define the phase space representation is

$$\langle x|n, X, P, 𝒷\rangle = (\langle n, X, P, 𝒷|x\rangle)^* = \frac{H_n\left(\frac{x - X}{\sqrt{2}a}\right)}{\sqrt{2^n n!\sqrt{2\pi}a}} e^{-\left(\frac{x-X}{2a}\right)^2 + i\frac{Px}{\hbar}} \qquad (1.8)$$

A multidimensional generalization of the phase space representation may be performed [4]. The relations who generalize (1.5), (1.6) and (1.8) and define the basis $\{\,|\{n_\mu\}, \{X_\mu\}, \{P_\mu\}, \{𝒷_\mu^\nu\}\rangle\}$ of the space states defining this generalization are

$$\begin{cases} \left\langle \{n_\mu\}, \{X_\mu\}, \{P_\mu\}, \{𝒷_\mu^\nu\}\middle|\boldsymbol{x}_\rho\middle|\{n_\mu\}, \{X_\mu\}, \{P_\mu\}, \{𝒷_\mu^\nu\}\right\rangle = X_\rho \\ \left\langle \{n_\mu\}, \{X_\mu\}, \{P_\mu\}, \{𝒷_\mu^\nu\}\middle|\boldsymbol{p}_\rho\middle|\{n_\mu\}, \{X_\mu\}, \{P_\mu\}, \{𝒷_\mu^\nu\}\right\rangle = P_\rho \\ \left\langle \{n_\mu = 0\}, \{X_\mu\}, \{P_\mu\}, \{𝒷_\mu^\nu\}\middle|(\boldsymbol{p}_\rho - P_\rho)(\boldsymbol{p}_\lambda - P_\lambda)\middle|\{n_\mu = 0\}, \{X_\mu\}, \{P_\mu\}, \{𝒷_\mu^\nu\}\right\rangle = \mathcal{B}_{\rho\lambda} \\ \left\langle \{n_\mu = 0\}, \{X_\mu\}, \{P_\mu\}, \{𝒷_\mu^\nu\}\middle|(\boldsymbol{x}_\rho - X_\rho)(\boldsymbol{x}_\lambda - X_\lambda)\middle|\{n_\mu = 0\}, \{X_\mu\}, \{P_\mu\}, \{𝒷_\mu^\nu\}\right\rangle = \mathcal{A}_{\rho\lambda} \end{cases}$$

$$\mathcal{A}^{\mu\rho}\mathcal{B}_{\rho\nu} = \frac{1}{4}\delta^{\mu}_{\nu}$$

$$\langle\{x_\mu\}|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}^{\nu}_{\mu}\}\rangle =$$

$$\frac{1}{[2\pi det(\mathcal{B}_{\mu\nu})]^{1/4}}(\prod_{\mu=0}^{N-1}\frac{H_{n_\mu}[\sqrt{2}\mathscr{b}^{\mu}_{\nu}(x^\nu - X^\nu)]}{\sqrt{2^{n_\mu}n_\mu!}})\,e^{-\mathcal{B}_{\mu\nu}(x^\mu - X^\mu)(x^\nu - X^\nu) - iP_\mu x^\mu}$$

2- Fermions fields in phase space representation

In the framework of the current formulation of quantum electrodynamics, the well-known equation for free fermions field $\boldsymbol{\psi}$ is the Dirac equation

$$(i\gamma^\mu\frac{\partial}{\partial x^\mu} - m)\boldsymbol{\psi} = 0 \qquad (2.1)$$

and the corresponding expression of the field operator $\boldsymbol{\psi}$ which satisfies this equation is

$$\vec{\boldsymbol{\psi}}(x) = \sum_{\sigma=1}^{2}\int\frac{d^3\vec{p}}{(2\pi)^3E(\vec{p})}[\mathbb{a}_\sigma(\vec{p})u_\sigma(\vec{p})e^{-ip^\mu x_\mu} + \mathbb{b}^\dagger_\sigma(\vec{p})v_\sigma(\vec{p})e^{ip^\mu x_\mu}] \qquad (2.2)$$

in which $u_\sigma(\vec{p})$ and $v_\sigma(\vec{p})$ are Dirac spinors which satisfies the relations

$$\begin{cases}(\gamma^\mu p_\mu - m)u_\sigma(\vec{p}) = 0\\(\gamma^\mu p_\mu + m)v_\sigma(\vec{p}) = 0\end{cases} \qquad (2.3)$$

and $\mathbb{a}_\sigma(\vec{p}), \mathbb{b}^\dagger_\sigma(\vec{p})$ are respectively the particle annihilation and antiparticle creation operators. They satisfies the well-known canonical anticommutation relations

$$\begin{cases}\{\mathbb{a}_\sigma(\vec{p}), \mathbb{a}^\dagger_{\sigma'}(\vec{p}')\} = (2\pi)^3\delta_{\sigma\sigma'}\delta(\vec{p} - \vec{p}')\\\{\mathbb{b}_\sigma(\vec{p}), \mathbb{b}^\dagger_{\sigma'}(\vec{p}')\} = (2\pi)^3\delta_{\sigma\sigma'}\delta(\vec{p} - \vec{p}')\end{cases} \qquad (2.4)$$

the expression (2.2) can be seen as a momentum representation of the field operator. A possible expression for a phase space representation is

$$\boldsymbol{\psi}(x) = \sum_{\sigma=1}^{2}\int\frac{d^3\vec{P}d^3\vec{X}}{(2\pi)^3E}[\mathbb{a}_\sigma u_\sigma e^{-iP_\mu x^\mu} + \mathbb{b}^\dagger_\sigma v_\sigma e^{iP_\mu x^\mu}]e^{-\mathcal{B}_{\mu\nu}(x^\mu - X^\mu)(x^\nu - X^\nu)} \qquad (2.5)$$

in which P_μ and X^μ are the mean values of the momentum and coordinate of a particle and the $\mathcal{B}_{\mu\nu}$ are the components of the momentum dispersion-codispersion (i.e statistical variance-covariance) tensor. E is a normalization. The expression (2.5) stands if the field equation is

$$[i(\gamma^\mu\frac{\partial}{\partial x^\mu} + 2\mathcal{B}_{\mu\nu}x^\nu) - m]\boldsymbol{\psi} = 0 \qquad (2.6)$$

and if we have the relations

$$\begin{cases}(\gamma^\mu Z_\mu - m)u_\sigma = 0\\ (\gamma^\mu Z_\mu^* + m)v_\sigma = 0\end{cases} \quad (2.7)$$

in which Z_μ is the complex variable

$$Z_\mu = P_\mu + 2i\mathcal{B}_{\mu\nu}X^\nu \quad (2.8)$$

Remark: The Dirac equation (2.1) is obtained from (2.6) in the limits $\mathcal{B}_{\mu\nu} \to 0$

3- Linear canonical transformations and properties of elementary fermions of the Standard Model

The phase space representation provides a framework for the study of linear canonical transformation (LCT) [2, 4-5, 7-8]. We have shown in [8] that the study on the spinorial representation of this transformations permits to deduce some of the properties of the elementary fermions of the Standard Model of Particle Physics.

A linear canonical transformation (LCT) can be defined as linear transformation mixing the momentum and coordinate operators and leaving invariant the canonical commutation relations of coordinate and momentum operators. With the reduced operators $\boldsymbol{\wp}_\mu$ and $\boldsymbol{\varkappa}_\mu$ corresponding to a pseudo-Euclidian space with signature(N_+, N_-), the definition of an LCT may be written as [4, 8]

$$\begin{cases}\boldsymbol{\wp}'_\mu = \Pi_\mu^\nu \boldsymbol{\wp}_\nu + \Theta_\mu^\nu \boldsymbol{\varkappa}_\nu\\ \boldsymbol{\varkappa}'_\mu = \Xi_\mu^\nu \boldsymbol{\wp}_\nu + \Lambda_\mu^\nu \boldsymbol{\varkappa}_\nu\\ [\boldsymbol{\wp}'_\mu, \boldsymbol{\varkappa}'_\nu]_- = [\boldsymbol{\wp}_\mu, \boldsymbol{\varkappa}_\nu]_- = i\eta_{\mu\nu}\end{cases} \Leftrightarrow \begin{pmatrix}\Pi & \Xi\\ \Theta & \Lambda\end{pmatrix}^t \begin{pmatrix}0 & \eta\\ -\eta & 0\end{pmatrix}\begin{pmatrix}\Pi & \Xi\\ \Theta & \Lambda\end{pmatrix} = \begin{pmatrix}0 & \eta\\ -\eta & 0\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}\Pi & \Xi\\ \Theta & \Lambda\end{pmatrix} \in Sp(2N_+, 2N_-) \quad (3.1)$$

These relations mean that an LCT corresponds to an element of the pseudosymplectic group $Sp(2N_+, 2N_-)$. A geometric parameterization is [7-8]

$$\begin{pmatrix}\Pi & \Xi\\ \Theta & \Lambda\end{pmatrix} = e^{\begin{pmatrix}\lambda+\mu & \varphi+\theta\\ \varphi-\theta & \lambda-\mu\end{pmatrix}} \Leftrightarrow \begin{cases}\theta^T = \eta\theta\eta\\ \varphi^T = \eta\varphi\eta\\ \mu^T = \eta\mu\eta\\ \lambda^T = -\eta\lambda\eta\end{cases} \Leftrightarrow \begin{pmatrix}\lambda+\mu & \varphi+\theta\\ \varphi-\theta & \lambda-\mu\end{pmatrix} \in Lie\,[Sp(2N_+, 2N_-)]$$

The LCTs corresponding to the case $\mu = \varphi = 0$ may be seen as a generalization of pseudo-orthogonal transformations (like Lorentz transforms) and Fractional Fourier transformations (rotations in coordinate –momentum plane) [7-8]. We have a spinorial representation according to the following relations [8]

$$\mathcal{S} = \varrho(e^{\begin{pmatrix}\lambda & -\theta\\ \theta & \lambda\end{pmatrix}}) \Leftrightarrow \begin{cases}(\boldsymbol{\wp}' \quad \boldsymbol{\varkappa}') = (\boldsymbol{\wp} \quad \boldsymbol{\varkappa})e^{\begin{pmatrix}\lambda & -\theta\\ \theta & \lambda\end{pmatrix}}\\ \alpha^\mu \boldsymbol{\wp}'_\mu + \beta^\mu \boldsymbol{\wp}'_\mu = \mathcal{S}(\alpha^\mu \boldsymbol{\wp}_\mu + \beta^\mu \boldsymbol{\varkappa}_\mu)\mathcal{S}^{-1}\end{cases} \quad (3.2)$$

$$\mathcal{S} = e^{[\frac{1}{4}(\eta_{\mu\rho}\lambda_\nu^\rho + \eta_{\nu\rho}\lambda_\mu^\rho)(\alpha^\mu\alpha^\nu + \beta^\mu\beta^\nu) + \frac{1}{2}\eta_{\mu\rho}\theta_\nu^\rho\alpha^\mu\beta^\nu]} \quad (3.3)$$

The operator $\mathcal{S}$ defining this spinorial representation is an element of the spin group $Spin(2N_+, 2N_-)$. And the operators α^μ and β^μ are the generators of the Clifford algebra $\mathfrak{C}(2N_+, 2N_-)$ i.e. they verify the following anticommutation relations

$$\begin{cases} \alpha^\mu\alpha^\nu + \alpha^\nu\alpha^\mu = 2\eta^{\mu\nu} \\ \beta^\mu\beta^\nu + \beta^\nu\beta^\mu = 2\eta^{\mu\nu} \\ \alpha^\mu\beta^\nu + \beta^\nu\alpha^\mu = 0 \end{cases} \tag{3.4}$$

For the case of $(N_+, N_-) = (1,4)$ (pentadimensional theory), we may choose the following matrices representation [8]

$$\begin{cases} \alpha^0 = \sigma^1 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\ \alpha^1 = i\sigma^3 \otimes \sigma^1 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\ \alpha^2 = i\sigma^3 \otimes \sigma^3 \otimes \sigma^1 \otimes \sigma^0 \otimes \sigma^0 \\ \alpha^3 = i\sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^1 \otimes \sigma^0 \\ \alpha^4 = i\sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^1 \\ \beta^0 = \sigma^2 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\ \beta^1 = -i\sigma^3 \otimes \sigma^2 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\ \beta^2 = -i\sigma^3 \otimes \sigma^3 \otimes \sigma^2 \otimes \sigma^0 \otimes \sigma^0 \\ \beta^3 = -i\sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^2 \otimes \sigma^0 \\ \beta^4 = -i\sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^2 \end{cases} \tag{3.5}$$

Then, in defining the operators

$$\begin{cases} \mathcal{Y}^0 = \frac{i}{4}[\alpha^0, \beta^0] = \frac{1}{2}i\alpha^0\beta^0 = -\frac{1}{2}\sigma^3 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\ \mathcal{Y}^1 = \frac{i}{6}[\alpha^1, \beta^1] = \frac{1}{3}i\alpha^1\beta^1 = -\frac{1}{3}\sigma^0 \otimes \sigma^3 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\ \mathcal{Y}^2 = \frac{i}{6}[\alpha^2, \beta^2] = \frac{1}{3}i\alpha^2\beta^2 = -\frac{1}{3}\sigma^0 \otimes \sigma^0 \otimes \sigma^3 \otimes \sigma^0 \otimes \sigma^0 \\ \mathcal{Y}^3 = \frac{i}{6}[\alpha^3, \beta^3] = \frac{1}{3}i\alpha^3\beta^3 = -\frac{1}{3}\sigma^0 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^3 \otimes \sigma^0 \\ \mathcal{Y}^4 = \frac{i}{6}[\alpha^4, \beta^4] = \frac{1}{2}i\alpha^4\beta^4 = -\frac{1}{2}\sigma^0 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^3 \end{cases} \tag{3.6}$$

and

$$\begin{cases} I_3 = \frac{1}{2}\mathcal{Y}^0 - \frac{1}{2}\mathcal{Y}^4 \\ Y_W = \mathcal{Y}^0 + \mathcal{Y}^1 + \mathcal{Y}^2 + \mathcal{Y}^3 + \mathcal{Y}^4 \\ Q = \mathcal{Y}^0 + \frac{1}{2}\mathcal{Y}^1 + \frac{1}{2}\mathcal{Y}^2 + \frac{1}{2}\mathcal{Y}^3 \\ \mathcal{C} = \mathcal{Y}^1 + \mathcal{Y}^2 + \mathcal{Y}^3 \end{cases} \tag{3.7}$$

it can be seen (c.f table 1 below) that the eigenvalues of the operators I_3, Y_W and Q correspond respectively to the weak isospin, weak hypercharge and electric charge of a single fermion generation of the Standard Model.

N°	$\mathcal{Y}^0$	$\mathcal{Y}^1$	$\mathcal{Y}^2$	$\mathcal{Y}^3$	$\mathcal{Y}^4$	I_3	Y_W	Q	$\mathcal{C}$	Particle
1	$-1/2$	$-1/3$	$-1/3$	$-1/3$	$-1/2$	0	-2	-1	-1	e_R
2	$-1/2$	$-1/3$	$-1/3$	$-1/3$	$1/2$	$-1/2$	-1	-1	-1	e_L
3	$-1/2$	$-1/3$	$-1/3$	$1/3$	$-1/2$	0	$-4/3$	$-2/3$	$-1/3$	$\bar{u}_R^{blue}$
4	$-1/2$	$-1/3$	$-1/3$	$1/3$	$1/2$	$-1/2$	$-1/3$	$-2/3$	$-1/3$	$\bar{u}_L^{blue}$
5	$-1/2$	$-1/3$	$1/3$	$-1/3$	$-1/2$	0	$-4/3$	$-2/3$	$-1/3$	$\bar{u}_R^{green}$
6	$-1/2$	$-1/3$	$1/3$	$-1/3$	$1/2$	$-1/2$	$-1/3$	$-2/3$	$-1/3$	$\bar{u}_L^{green}$
7	$-1/2$	$-1/3$	$1/3$	$1/3$	$-1/2$	0	$-2/3$	$-1/3$	$1/3$	d_R^{red}
8	$-1/2$	$-1/3$	$1/3$	$1/3$	$1/2$	$-1/2$	$1/3$	$-1/3$	$1/3$	d_L^{red}
9	$-1/2$	$1/3$	$-1/3$	$-1/3$	$-1/2$	0	$-4/3$	$-2/3$	$-1/3$	$\bar{u}_R^{red}$
10	$-1/2$	$1/3$	$-1/3$	$-1/3$	$1/2$	$-1/2$	$-1/3$	$-2/3$	$-1/3$	$\bar{u}_L^{red}$
11	$-1/2$	$1/3$	$-1/3$	$1/3$	$-1/2$	0	$-2/3$	$-1/3$	$1/3$	d_R^{green}
12	$-1/2$	$1/3$	$-1/3$	$1/3$	$1/2$	$-1/2$	$1/3$	$-1/3$	$1/3$	d_L^{green}
13	$-1/2$	$1/3$	$1/3$	$-1/3$	$-1/2$	0	$-2/3$	$-1/3$	$1/3$	d_R^{blue}
14	$-1/2$	$1/3$	$1/3$	$-1/3$	$1/2$	$-1/2$	$1/3$	$-1/3$	$1/3$	d_L^{blue}
15	$-1/2$	$1/3$	$1/3$	$1/3$	$-1/2$	0	0	0	1	$\bar{\nu}_R$
16	$-1/2$	$1/3$	$1/3$	$1/3$	$1/2$	$-1/2$	1	0	1	$\bar{\nu}_L$
17	$1/2$	$-1/3$	$-1/3$	$-1/3$	$-1/2$	$1/2$	-1	0	-1	ν_L
18	$1/2$	$-1/3$	$-1/3$	$-1/3$	$1/2$	0	0	0	-1	ν_R
19	$1/2$	$-1/3$	$-1/3$	$1/3$	$-1/2$	$1/2$	$-1/3$	$1/3$	$-1/3$	$\bar{d}_L^{blue}$
20	$1/2$	$-1/3$	$-1/3$	$1/3$	$1/2$	0	$2/3$	$1/3$	$-1/3$	$\bar{d}_R^{blue}$
21	$1/2$	$-1/3$	$1/3$	$-1/3$	$-1/2$	$1/2$	$-1/3$	$1/3$	$-1/3$	$\bar{d}_L^{gree}$
22	$1/2$	$-1/3$	$1/3$	$-1/3$	$1/2$	0	$2/3$	$1/3$	$-1/3$	$\bar{d}_R^{green}$
23	$1/2$	$-1/3$	$1/3$	$1/3$	$-1/2$	$1/2$	$1/3$	$2/3$	$1/3$	u_L^{red}
24	$1/2$	$-1/3$	$1/3$	$1/3$	$1/2$	0	$4/3$	$2/3$	$1/3$	u_R^{red}
25	$1/2$	$1/3$	$-1/3$	$-1/3$	$-1/2$	$1/2$	$1/3$	$1/3$	$-1/3$	$\bar{d}_L^{red}$
26	$1/2$	$1/3$	$-1/3$	$-1/3$	$1/2$	0	$2/3$	$1/3$	$-1/3$	$\bar{d}_R^{red}$
27	$1/2$	$1/3$	$-1/3$	$1/3$	$-1/2$	$1/2$	$1/3$	$2/3$	$1/3$	u_L^{green}
28	$1/2$	$1/3$	$-1/3$	$1/3$	$1/2$	0	$4/3$	$2/3$	$1/3$	u_R^{green}
29	$1/2$	$1/3$	$1/3$	$-1/3$	$-1/2$	$1/2$	$1/3$	$2/3$	$1/3$	u_L^{blue}
30	$1/2$	$1/3$	$1/3$	$-1/3$	$1/2$	0	$4/3$	$2/3$	$1/3$	u_R^{blue}
31	$1/2$	$1/3$	$1/3$	$1/3$	$-1/2$	$1/2$	1	1	1	$\bar{e}_L$
32	$1/2$	$1/3$	$1/3$	$1/3$	$1/2$	0	2	1	1	$\bar{e}_R$

$$\begin{cases} I_3 = \frac{1}{2}\mathcal{Y}^0 - \frac{1}{2}\mathcal{Y}^4 \\ Y_W = \mathcal{Y}^0 + \mathcal{Y}^1 + \mathcal{Y}^2 + \mathcal{Y}^3 + \mathcal{Y}^4 \\ Q = \mathcal{Y}^0 + \frac{1}{2}\mathcal{Y}^1 + \frac{1}{2}\mathcal{Y}^2 + \frac{1}{2}\mathcal{Y}^3 \\ \mathcal{C} = \mathcal{Y}^1 + \mathcal{Y}^2 + \mathcal{Y}^3 \end{cases} \qquad Q = I_3 + \frac{1}{2}Y_W = \mathcal{Y}^0 + \frac{1}{2}\mathcal{C}$$

The operator $\mathcal{C}$ has also some interesting properties related to the nature of the particles. The eigenvalues of $\mathcal{C}$ are integer numbers for leptons and fractional numbers for quarks.
As it may be expected, the relationship between the electric charge Q, the weak isospin I_3 ,the weak hypercharge Y_W is an analog of the Gellmann-Nishijima. The existence of right handed (sterile) neutrino is obtained.

The previous results show that all of the particles belonging to a single generation of the Standard Model can be described with a single general spinor field $\overrightarrow{\boldsymbol{\psi}}$

$$\overrightarrow{\boldsymbol{\psi}} = \boldsymbol{\psi}^a \vec{\zeta}_a = \overrightarrow{\boldsymbol{\psi}}_{\boldsymbol{e}} + \overrightarrow{\boldsymbol{\psi}}_{\boldsymbol{u}} + \overrightarrow{\boldsymbol{\psi}}_{\boldsymbol{\nu}} + +\overrightarrow{\boldsymbol{\psi}}_{\boldsymbol{d}} = (\boldsymbol{\Pi}^e + \boldsymbol{\Pi}^u + \boldsymbol{\Pi}^\nu + \boldsymbol{\Pi}^d)\overrightarrow{\boldsymbol{\psi}} \tag{3.8}$$

the index a runs from 1 to 32.The elements $\vec{\zeta}_a$ of the considered basis are eigenspinors of the operators $\mathcal{Y}^0, \mathcal{Y}^1, \mathcal{Y}^2, \mathcal{Y}^3, \mathcal{Y}^4, I_3, Y_W, Q$ and $\mathcal{C}$. $\boldsymbol{\Pi}^e, \boldsymbol{\Pi}^u, \boldsymbol{\Pi}^\nu, \boldsymbol{\Pi}^d$are respectively the operators corresponding to the projections in the spinor subspaces of electron type field, quark up type field, neutrino type field and quark down type field. The explicit expressions of these projections operators are given in [8].

4-General fermions fields and fermions states in phase space representation

The results obtained in the section 2 and 3 suggest that we may consider a phase space representation of the general spinor field $\overrightarrow{\boldsymbol{\psi}}$ in the form

$$\overrightarrow{\boldsymbol{\psi}}(x) = \sum_s \int \frac{d^3\vec{P} d^3\vec{X}}{(2\pi)^3} [\mathbb{a}(\vec{P},\vec{X},\mathcal{B}_{\mu\nu},s)\vec{u}(\vec{P},\vec{X},\mathcal{B}_{\mu\nu},s)e^{-iP^\mu x_\mu} + \mathbb{b}^\dagger(\vec{P},\vec{X},\mathcal{B}_{\mu\nu},s)\vec{v}(\vec{P},\vec{X},\mathcal{B}_{\mu\nu},s)e^{iP^\mu x_\mu}]e^{-\mathcal{B}_{\mu\nu}(x^\mu - X^\mu)(x^\nu - X^\nu)} \tag{4.1}$$

The index s refers to the set of quantum number describing the particle nature and internal state (isospin, electric charge, colors,...). According to this decomposition a fermion particle state in the phase space representation is of the form $|\vec{P},\vec{X},\mathcal{B}_{\mu\nu},s\rangle$. $\vec{P}$ and $\vec{X}$ being the means of its momentum and coordinate vectors and $\mathcal{B}_{\mu\nu}$ the momentum dipersion-codispersion (statistical variance –covariance) tensor. It corresponds to $\mathcal{B}_{\mu\nu}$ a coordinate dispersion-codispersion tensor $\mathcal{A}_{\mu\nu}$ so that we have a saturation of the Heisenberg principle

$$\mathcal{A}^{\mu\rho}\mathcal{B}_{\rho\nu} = \frac{1}{4}\delta^\mu_\nu(\hbar)^2 \tag{4.2}$$

When a particle propagates, we may associate to it a “mean trajectory” with equation $\vec{P} = \vec{P}(X^0)$,$\vec{X} = \vec{X}(X^0)$. This mean trajectory may be related to the classical concept of trajectory in the classical limit.

5-Conclusion

The phase space representation provides the possibility to consider at the same time the momentum and coordinate of a particle, taking into account the Heisenberg uncertainty principle. It provides an interesting framework for the study of fermions and their properties. It permits in particular to describe fundamental properties, like isospin, electric charge and color, as a natural consequence of the spinorial representation of linear canonical transformations like spin is described as a natural consequence of the spinorial representation of Lorentz transformations. These results suggest the possibility of extending our approach to establish an unified theory of interactions. The phase space representation may also be used in the study of the relation between classical and quantum theories.

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