Fermions states and phase space representation of quantum theory

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1- Phase space representation

We consider the example of one dimension quantum mechanics. For a quantum state $|\psi\rangle$ of a particle we can have the coordinate representation, the momentum representation and the phase space representation.

- **Coordinate representation**

$$|\psi\rangle = \int |x\rangle \langle x|\psi\rangle dx = \int \psi(x)|x\rangle\, dx$$

- **Momentum representation**

$$|\psi\rangle = \int |p\rangle \langle p|\psi\rangle dx = \int \tilde{\psi}(p)|p\rangle\, dp$$

The wavefunctions $\psi(x)$ and $\tilde{\psi}(p)$ are linked by a Fourier Transforms

$$\tilde{\psi}(p) = \langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int \langle p|x\rangle\langle x|\psi\rangle\, dx = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-i\frac{px}{\hbar}}\psi(x)\, dx$$

- **Phase space representation (coordinate and momentum conjoint representation)**

The phase space representation is a coordinate-momentum conjoint representation taking into account the Heisenberg uncertainty relation (and saturate it). In this representation:

$$\begin{cases} |\psi\rangle = \sum_{\boldsymbol{n}} \Psi(n, X, P, \mathscr{b})\, |n, X, P, \mathscr{b}\rangle = \iint \Psi(n, X, P, \mathscr{b})\, |n, X, P, \mathscr{b}\rangle \frac{dXdP}{2\pi\hbar} \\ \Psi(n, X, P, \mathscr{b}) = \int \langle n, X, P, \mathscr{b}|x\rangle\langle x|\psi\rangle\, dx = \int \frac{H_n(\frac{x-X}{\sqrt{2}a})}{\sqrt{2^n n! \sqrt{2\pi}a}}\psi(x) e^{-\left(\frac{x-X}{2a}\right)^2 - i\frac{Px}{\hbar}} dx \end{cases} \qquad a\mathscr{b} = \frac{\hbar}{2}$$

$$\begin{cases} \langle n,X,P,\mathscr{b}|\boldsymbol{x}|n,X,P,\mathscr{b}\rangle = X \\ \langle n,X,P,\mathscr{b}|\boldsymbol{p}|n,X,P,\mathscr{b}\rangle = P \\ \langle n,X,P,\mathscr{b}\left|(\boldsymbol{x}-X)^{\boldsymbol{2}}\right|n,X,P,\mathscr{b}\rangle = (2n+1)(\mathscr{a})^2 = (2n+1)\mathcal{A} \\ \langle n,X,P,\mathscr{b}\left|(\boldsymbol{p}-P)^{\boldsymbol{2}}\right|n,X,P,\mathscr{b}\rangle = (2n+1)(\mathscr{b})^2 = (2n+1)\mathcal{B} \end{cases} \qquad \mathscr{a}\mathscr{b} = \frac{\hbar}{2} \Leftrightarrow \mathcal{A}\mathcal{B} = (\frac{\hbar}{2})^2$$

$X, P, (2n+1)\mathcal{A}, (2n+1)\mathcal{B}$ are respectively the means values and statistical dispersions (variances) of the coordinate and momentum operators $\boldsymbol{x}$ and $\boldsymbol{p}$ in the state $|n,X,P,\mathscr{b}\rangle$. The standard deviation (uncertainty) $\mathscr{a} = \sqrt{\mathcal{A}}$ and $\mathscr{b} = \sqrt{\mathcal{B}}$ relative to the state $|0,X,P,\mathscr{b}\rangle$ corresponds to a minimalization in the Heisenberg uncertainty relation. The states $|n,X,P,\mathscr{b}\rangle$ are the eigenstates of the momentum dispersion operator $\beth^{+}$

$$\beth^{+} = \frac{1}{2}[(\boldsymbol{p}-P)^{\boldsymbol{2}} + 4(\frac{\mathcal{B}}{\hbar})^2(\boldsymbol{x}-X)^{\boldsymbol{2}}] \Rightarrow \beth^{+}|n,X,P,\mathscr{b}\rangle = (2n+1)\mathcal{B}|n,X,P,\mathscr{b}\rangle$$

A N −dimensional generalization can be performed. The basis $\{\,\left|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\right\rangle\}$ of the space states defining this generalization can be defined by the relation

$$\left\langle\{x_\mu\}\middle|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\right\rangle = \frac{1}{[2\pi det(\mathcal{B}_{\mu\nu})]^{1/4}}(\prod_{\mu=0}^{N-1}\frac{H_{n_\mu}[\sqrt{2}\mathscr{b}_\nu^\mu(x^\nu - X^\nu)]}{\sqrt{2^{n_\mu}n_\mu!}})\, e^{-\mathcal{B}_{\mu\nu}(x^\mu-X^\mu)(x^\nu-X^\nu)-i\frac{P_\mu x^\mu}{\hbar}}$$

$$\begin{cases} \left\langle\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\middle|\boldsymbol{x}_\rho\middle|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\right\rangle = X_\rho \\ \left\langle\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\middle|\boldsymbol{p}_\rho\middle|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\right\rangle = P_\rho \\ \left\langle\{n_\mu=0\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\middle|(\boldsymbol{p}_\rho-P_\rho)(\boldsymbol{p}_\lambda-P_\lambda)\middle|\{n_\mu=0\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\right\rangle = \mathcal{B}_{\rho\lambda} \\ \left\langle\{n_\mu=0\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\middle|(\boldsymbol{x}_\rho-X_\rho)(\boldsymbol{x}_\lambda-X_\lambda)\middle|\{n_\mu=0\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\right\rangle = \mathcal{A}_{\rho\lambda} \end{cases}$$

$$\mathcal{A}^{\mu\rho}\mathcal{B}_{\rho\nu} = \frac{1}{4}\delta_\nu^\mu(\hbar)^2 \quad \mathscr{a}_\rho^\mu\mathscr{b}_\nu^\rho = \frac{1}{2}\delta_\nu^\mu\hbar$$

2- Field equation and fermions states in phase space representation

In framework of quantum field theory, the well-known equation for free fermionic field is the Dirac equation

$$(i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}-m)\overrightarrow{\boldsymbol{\psi}}=0$$

The expression of the field $\overrightarrow{\boldsymbol{\psi}}$ is

$$\overrightarrow{\boldsymbol{\psi}}(x)=\sum_{\sigma=1}^{2}\int\frac{d^3\vec{p}}{(2\pi)^3E(\vec{p})}[\mathbb{a}_{\sigma}(\vec{p})\vec{u}_{\sigma}(\vec{p})e^{-ip^{\mu}x_{\mu}}+\mathbb{b}_{\sigma}^{\dagger}(\vec{p})\vec{v}_{\sigma}(\vec{p})e^{ip^{\mu}x_{\mu}}]\quad\begin{cases}\{\mathbb{a}_{\sigma}(\vec{p}),\mathbb{a}_{\sigma'}^{\dagger}(\vec{p}')\}=(2\pi)^3\delta_{\sigma\sigma'}\delta(\vec{p}-\vec{p}')\\\{\mathbb{a}_{\sigma}(\vec{p}),\mathbb{b}_{\sigma'}^{\dagger}(\vec{p}')\}=(2\pi)^3\delta_{\sigma\sigma'}\delta(\vec{p}-\vec{p}')\end{cases}$$

$$\begin{cases}(\gamma^{\mu}p_{\mu}-m)\vec{u}_{\sigma}(\vec{p})=0\\(\gamma^{\mu}p_{\mu}+m)\vec{v}_{\sigma}(\vec{p})=0\end{cases}$$

This field equation can be seen as a momentum representation decomposition of the field operator. A possible expression for a phase space representation is (the $\mathcal{B}_{\mu\nu}$ being the momentum dispersion, i.e statistical variance-covariance, tensor)

$$\overrightarrow{\boldsymbol{\psi}}(x)=\sum_{\sigma=1}^{2}\int\frac{d^3\vec{P}d^3\vec{X}}{(2\pi)^3}[\mathbb{a}_{\sigma}(\vec{P},\vec{X},\mathcal{B}_{\mu\nu})u_{\sigma}(\vec{P},\vec{X},\mathcal{B}_{\mu\nu})e^{-iP^{\mu}x_{\mu}}+\mathbb{b}_{\sigma}^{\dagger}(\vec{P},\vec{X},\mathcal{B}_{\mu\nu})v_{\sigma}(\vec{P},\vec{X},\mathcal{B}_{\mu\nu})e^{iP^{\mu}x_{\mu}}]e^{-\mathcal{B}_{\mu\nu}(x^{\mu}-X^{\mu})(x^{\nu}-X^{\nu})}$$

this expression stands if the field equation is (the Dirac equation is recovered in the limit $\mathcal{B}_{\mu\nu}\to 0$)

$$[i(\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}+2\mathcal{B}_{\mu\nu}x^{\nu})-m]\overrightarrow{\boldsymbol{\psi}}=0$$

and we have

$$Z_{\mu}=P_{\mu}+2i\mathcal{B}_{\mu\nu}X^{\nu}\Rightarrow\begin{cases}(\gamma^{\mu}Z_{\mu}-m)\vec{u}_{\sigma}(\vec{P},\vec{X},\mathcal{B}_{\mu\nu})=0\\(\gamma^{\mu}Z_{\mu}^{*}+m)\vec{v}_{\sigma}(\vec{P},\vec{X},\mathcal{B}_{\mu\nu})=0\end{cases}\quad\Rightarrow\quad(\gamma^{\mu}Z_{\mu})(\gamma^{\nu}Z_{\nu}^{*})=m^2\Leftrightarrow\begin{cases}\eta^{\mu\nu}(P_{\mu}P_{\nu}+4\mathcal{B}_{\mu\rho}\mathcal{B}_{\nu\lambda}X^{\rho}X^{\lambda})=m^2\\(P_{\mu}\mathcal{B}_{\nu\rho}X^{\rho}-P_{\nu}\mathcal{B}_{\mu\rho}X^{\rho})=0\end{cases}$$

3- Linear canonical transformations and properties of elementary fermions of the Standard Model

The phase space representation provide a framework for the study of linear canonical transformation (LCT). And we have shown that the study on the spinorial representation of this transformations permits to deduce some of the properties of the elementary fermions of the Standard Model of Particle Physics.

A linear canonical transformation can be defined as linear transformations mixing the momentum and coordinate operators and leaving invariant the canonical commutation relations. With the reduced centered operators,

$$\begin{cases} \boldsymbol{\wp}_{\boldsymbol{\mu}} = \sqrt{2}a_\mu^\nu(\boldsymbol{p}_{\boldsymbol{\nu}} - P_\nu) \\ \boldsymbol{\varkappa}_{\boldsymbol{\mu}} = \sqrt{2}b_\mu^\nu(\boldsymbol{x}_{\boldsymbol{\nu}} - X_\nu) \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{p}_{\boldsymbol{\mu}} = \sqrt{2}b_\mu^\nu \boldsymbol{\wp}_{\boldsymbol{\nu}} + P_\mu \\ \boldsymbol{x}_{\boldsymbol{\mu}} = \sqrt{2}a_\mu^\nu \boldsymbol{\varkappa}_{\boldsymbol{\nu}} + X_\mu \end{cases} \quad a_\mu^\rho b_\rho^\nu = \frac{1}{2}\delta_\mu^\nu \Rightarrow [\boldsymbol{\wp}_\mu, \boldsymbol{\varkappa}_{\boldsymbol{\nu}}]_- = [\boldsymbol{p}_\mu, \boldsymbol{x}_{\boldsymbol{\nu}}]_- = i\eta_{\mu\nu}$$

corresponding to a pseudo-Euclidian space with signature (N_+, N_-), the definition is

$$\begin{cases} \boldsymbol{\wp}'_\mu = \Pi_\mu^\nu \boldsymbol{\wp}_{\boldsymbol{\nu}} + \Theta_\mu^\nu \boldsymbol{\varkappa}_{\boldsymbol{\nu}} \\ \boldsymbol{\varkappa}'_\mu = \Xi_\mu^\nu \boldsymbol{\wp}_{\boldsymbol{\nu}} + \Lambda_\mu^\nu \boldsymbol{\varkappa}_{\boldsymbol{\nu}} \\ \left[\boldsymbol{\wp}'_\mu, \boldsymbol{\varkappa}'_\nu\right]_- = \left[\boldsymbol{\wp}_\mu, \boldsymbol{\varkappa}_\nu\right]_- = i\eta_{\mu\nu} \end{cases} \Leftrightarrow \begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix}^t \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} \begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} \in Sp(2N_+, 2N_-)$$

An LCT corresponds to an element of the pseudosymplectic group $Sp(2N_+, 2N_-)$. A geometric parameterization is

$$\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} = e^{\begin{pmatrix} \lambda+\mu & \varphi+\theta \\ \varphi-\theta & \lambda-\mu \end{pmatrix}} \Leftrightarrow \begin{cases} \theta^T = \eta\theta\eta \\ \varphi^T = \eta\varphi\eta \\ \mu^T = \eta\mu\eta \\ \lambda^T = -\eta\lambda\eta \end{cases} \Leftrightarrow \begin{pmatrix} \lambda+\mu & \varphi+\theta \\ \varphi-\theta & \lambda-\mu \end{pmatrix} \in \mathfrak{sp}(2N_+, 2N_-) = Lie\ [Sp(2N_+, 2N_-)]$$

The LCTs corresponding to the case $\mu = \varphi = 0$ may be seen as generalization of pseudo-orthogonal transformations (like Lorentz transformation) and Fractional Fourier transformations (rotations in coordinate –momentum planes). It can be established that they have a spinorial representation according to the following relations

$$\mathcal{S} = \varrho\left(e^{\begin{pmatrix}\lambda & -\theta\\ \theta & \lambda\end{pmatrix}}\right) \Leftrightarrow \begin{cases} (\boldsymbol{p}' \quad \boldsymbol{x}') = (\boldsymbol{p} \quad \boldsymbol{x})e^{\begin{pmatrix}\lambda & -\theta\\ \theta & \lambda\end{pmatrix}} \\ \alpha^\mu \boldsymbol{p}'_{\boldsymbol{\mu}} + \beta^\mu \boldsymbol{p}'_{\boldsymbol{\mu}} = \mathcal{S}(\alpha^\mu \boldsymbol{p}_{\boldsymbol{\mu}} + \beta^\mu \boldsymbol{x}_{\boldsymbol{\mu}})\mathcal{S}^{-1} \end{cases}$$

$$\mathcal{S} = e^{\left[\frac{1}{4}(\eta_{\mu\rho}\lambda_\nu^\rho + \eta_{\nu\rho}\lambda_\mu^\rho)(\alpha^\mu\alpha^\nu + \beta^\mu\beta^\nu) + \frac{1}{2}\eta_{\mu\rho}\theta_\nu^\rho\alpha^\mu\beta^\nu\right]}$$

The operator $\mathcal{S}$ defining this spinorial representation is an element of the spin group $Spin(2N_+, 2N_-)$. And the operators α^μ and β^μ are the generators of the Clifford algebra $\mathfrak{C}(2N_+, 2N_-)$ i.e. they verify the following anticommutation relations

$$\begin{cases} \alpha^\mu\alpha^\nu + \alpha^\nu\alpha^\mu = 2\eta^{\mu\nu} \\ \beta^\mu\beta^\nu + \beta^\nu\beta^\mu = 2\eta^{\mu\nu} \\ \alpha^\mu\beta^\nu + \beta^\nu\alpha^\mu = 0 \end{cases}$$

For the case of $(N_+, N_-) = (1,4)$ (pentadimensional theory), we may choose the following matrices representation

$$\begin{cases} \alpha^0 = \sigma^1 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\ \alpha^1 = i\sigma^3 \otimes \sigma^1 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\ \alpha^2 = i\sigma^3 \otimes \sigma^3 \otimes \sigma^1 \otimes \sigma^0 \otimes \sigma^0 \\ \alpha^3 = i\sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^1 \otimes \sigma^0 \\ \alpha^4 = i\sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^1 \\ \beta^0 = \sigma^2 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\ \beta^1 = -i\sigma^3 \otimes \sigma^2 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\ \beta^2 = -i\sigma^3 \otimes \sigma^3 \otimes \sigma^2 \otimes \sigma^0 \otimes \sigma^0 \\ \beta^3 = -i\sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^2 \otimes \sigma^0 \\ \beta^4 = -i\sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^2 \end{cases} \quad \sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \sigma^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

If we then define the operators

$$\begin{cases} \mathcal{Y}^0 = \frac{i}{4}[\alpha^0,\beta^0] = \frac{1}{2}i\alpha^0\beta^0 = -\frac{1}{2}\sigma^3 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\ \mathcal{Y}^1 = \frac{i}{6}[\alpha^1,\beta^1] = \frac{1}{3}i\alpha^1\beta^1 = -\frac{1}{3}\sigma^0 \otimes \sigma^3 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \\ \mathcal{Y}^2 = \frac{i}{6}[\alpha^2,\beta^2] = \frac{1}{3}i\alpha^2\beta^2 = -\frac{1}{3}\sigma^0 \otimes \sigma^0 \otimes \sigma^3 \otimes \sigma^0 \otimes \sigma^0 \\ \mathcal{Y}^3 = \frac{i}{6}[\alpha^3,\beta^3] = \frac{1}{3}i\alpha^3\beta^3 = -\frac{1}{3}\sigma^0 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^3 \otimes \sigma^0 \\ \mathcal{Y}^4 = \frac{i}{6}[\alpha^4,\beta^4] = \frac{1}{2}i\alpha^4\beta^4 = -\frac{1}{2}\sigma^0 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^3 \end{cases}$$

and

$$\begin{cases} I_3 = \frac{1}{2}\mathcal{Y}^0 - \frac{1}{2}\mathcal{Y}^4 \\ Y_W = \mathcal{Y}^0 + \mathcal{Y}^1 + \mathcal{Y}^2 + \mathcal{Y}^3 + \mathcal{Y}^4 \\ Q = \mathcal{Y}^0 + \frac{1}{2}\mathcal{Y}^1 + \frac{1}{2}\mathcal{Y}^2 + \frac{1}{2}\mathcal{Y}^3 \\ \mathcal{C} = \mathcal{Y}^1 + \mathcal{Y}^2 + \mathcal{Y}^3 \end{cases} \qquad Q = I_3 + \frac{1}{2}Y_W = \mathcal{Y}^0 + \frac{1}{2}\mathcal{C}$$

It can be observed (table in the next page) that the eigenvalues of the operators I_3, Y_W and Q corresponds to the values of weak isospin, weak hypercharge and electric charge of a single fermion generation of the Standard Model. The operator $\mathcal{C}$ has also some interesting properties related to the nature of the particles.

N°	y^0	y^1	y^2	y^3	y^4	I_3	Y_W	Q	C	Particle
1	$-1/2$	$-1/3$	$-1/3$	$-1/3$	$-1/2$	0	-2	-1	-1	e_R
2	$-1/2$	$-1/3$	$-1/3$	$-1/3$	$1/2$	$-1/2$	-1	-1	-1	e_L
3	$-1/2$	$-1/3$	$-1/3$	$1/3$	$-1/2$	0	$-4/3$	$-2/3$	$-1/3$	$\bar{u}_R^{blue}$
4	$-1/2$	$-1/3$	$-1/3$	$1/3$	$1/2$	$-1/2$	$-1/3$	$-2/3$	$-1/3$	$\bar{u}_L^{blue}$
5	$-1/2$	$-1/3$	$1/3$	$-1/3$	$-1/2$	0	$-4/3$	$-2/3$	$-1/3$	$\bar{u}_R^{green}$
6	$-1/2$	$-1/3$	$1/3$	$-1/3$	$1/2$	$-1/2$	$-1/3$	$-2/3$	$-1/3$	$\bar{u}_L^{green}$
7	$-1/2$	$-1/3$	$1/3$	$1/3$	$-1/2$	0	$-2/3$	$-1/3$	$1/3$	d_R^{red}
8	$-1/2$	$-1/3$	$1/3$	$1/3$	$1/2$	$-1/2$	$1/3$	$-1/3$	$1/3$	d_L^{red}
9	$-1/2$	$1/3$	$-1/3$	$-1/3$	$-1/2$	0	$-4/3$	$-2/3$	$-1/3$	$\bar{u}_R^{red}$
10	$-1/2$	$1/3$	$-1/3$	$-1/3$	$1/2$	$-1/2$	$-1/3$	$-2/3$	$-1/3$	$\bar{u}_L^{red}$
11	$-1/2$	$1/3$	$-1/3$	$1/3$	$-1/2$	0	$-2/3$	$-1/3$	$1/3$	d_R^{green}
12	$-1/2$	$1/3$	$-1/3$	$1/3$	$1/2$	$-1/2$	$1/3$	$-1/3$	$1/3$	d_L^{green}
13	$-1/2$	$1/3$	$1/3$	$-1/3$	$-1/2$	0	$-2/3$	$-1/3$	$1/3$	d_R^{blue}
14	$-1/2$	$1/3$	$1/3$	$-1/3$	$1/2$	$-1/2$	$1/3$	$-1/3$	$1/3$	d_L^{blue}
15	$-1/2$	$1/3$	$1/3$	$1/3$	$-1/2$	0	0	0	1	$\bar{\nu}_R$
16	$-1/2$	$1/3$	$1/3$	$1/3$	$1/2$	$-1/2$	1	0	1	$\bar{\nu}_L$
17	$1/2$	$-1/3$	$-1/3$	$-1/3$	$-1/2$	$1/2$	-1	0	-1	ν_L
18	$1/2$	$-1/3$	$-1/3$	$-1/3$	$1/2$	0	0	0	-1	ν_R
19	$1/2$	$-1/3$	$-1/3$	$1/3$	$-1/2$	$1/2$	$-1/3$	$1/3$	$-1/3$	$\bar{d}_L^{blue}$
20	$1/2$	$-1/3$	$-1/3$	$1/3$	$1/2$	0	$2/3$	$1/3$	$-1/3$	$\bar{d}_R^{blue}$
21	$1/2$	$-1/3$	$1/3$	$-1/3$	$-1/2$	$1/2$	$-1/3$	$1/3$	$-1/3$	$\bar{d}_L^{green}$
22	$1/2$	$-1/3$	$1/3$	$-1/3$	$1/2$	0	$2/3$	$1/3$	$-1/3$	$\bar{d}_R^{green}$
23	$1/2$	$-1/3$	$1/3$	$1/3$	$-1/2$	$1/2$	$1/3$	$2/3$	$1/3$	u_L^{red}
24	$1/2$	$-1/3$	$1/3$	$1/3$	$1/2$	0	$4/3$	$2/3$	$1/3$	u_R^{red}
25	$1/2$	$1/3$	$-1/3$	$-1/3$	$-1/2$	$1/2$	$1/3$	$1/3$	$-1/3$	$\bar{d}_L^{red}$
26	$1/2$	$1/3$	$-1/3$	$-1/3$	$1/2$	0	$2/3$	$1/3$	$-1/3$	$\bar{d}_R^{red}$
27	$1/2$	$1/3$	$-1/3$	$1/3$	$-1/2$	$1/2$	$1/3$	$2/3$	$1/3$	u_L^{green}
28	$1/2$	$1/3$	$-1/3$	$1/3$	$1/2$	0	$4/3$	$2/3$	$1/3$	u_R^{green}
29	$1/2$	$1/3$	$1/3$	$-1/3$	$-1/2$	$1/2$	$1/3$	$2/3$	$1/3$	u_L^{blue}
30	$1/2$	$1/3$	$1/3$	$-1/3$	$1/2$	0	$4/3$	$2/3$	$1/3$	u_R^{blue}
31	$1/2$	$1/3$	$1/3$	$1/3$	$-1/2$	$1/2$	1	1	1	$\overline{e_L}$
32	$1/2$	$1/3$	$1/3$	$1/3$	$1/2$	0	2	1	1	$\overline{e_R}$

$$\begin{cases} I_3 = \frac{1}{2}y^0 - \frac{1}{2}y^4 \\ Y_W = y^0 + y^1 + y^2 + y^3 + y^4 \\ Q = y^0 + \frac{1}{2}y^1 + \frac{1}{2}y^2 + \frac{1}{2}y^3 \\ C = y^1 + y^2 + y^3 \end{cases} \qquad Q = I_3 + \frac{1}{2}Y_W = y^0 + \frac{1}{2}C$$

y^1, y^2, y^3 describes colors. The eigenvalues of C are integer for leptons and fractional for quarks. Existence of right handed (sterile) neutrinos is suggested.

The previous results show that all of these particles can be described with a unique spinor field $\overrightarrow{\boldsymbol{\psi}}$

$$\overrightarrow{\boldsymbol{\psi}} = \boldsymbol{\psi}^{\boldsymbol{a}}\vec{\zeta}_a = \overrightarrow{\boldsymbol{\psi}}_{\boldsymbol{e}} + \overrightarrow{\boldsymbol{\psi}}_{\boldsymbol{u}} + \overrightarrow{\boldsymbol{\psi}}_{\boldsymbol{\nu}} + +\overrightarrow{\boldsymbol{\psi}}_{\boldsymbol{d}} = (\boldsymbol{\Pi}^{\boldsymbol{e}} + \boldsymbol{\Pi}^{\boldsymbol{u}} + \boldsymbol{\Pi}^{\boldsymbol{\nu}} + \boldsymbol{\Pi}^{\boldsymbol{d}})\overrightarrow{\boldsymbol{\psi}}$$

the index a run from 1 to 32. The elements $\vec{\zeta}_a$ of the considered basis in the 32 dimension spinor spaces are eigenspinors of the operators $\mathcal{Y}^0, \mathcal{Y}^1, \mathcal{Y}^2, \mathcal{Y}^3, \mathcal{Y}^4, I_3, Y_W, Q$ and $\mathcal{C}$. $\boldsymbol{\Pi}^{\boldsymbol{e}}, \boldsymbol{\Pi}^{\boldsymbol{u}}, \boldsymbol{\Pi}^{\boldsymbol{\nu}}, \boldsymbol{\Pi}^{\boldsymbol{d}}$ are respectively the operators corresponding to the projections in the spinor subspaces of electron type field, quark up type field, neutrino type field and quark down type field.

- Electron-type field

$$\overrightarrow{\boldsymbol{\psi}}_{\boldsymbol{e}} = \boldsymbol{\Pi}^{\boldsymbol{e}}\overrightarrow{\boldsymbol{\psi}} = \boldsymbol{\psi}^{\boldsymbol{1}}\vec{\zeta}_1 + \boldsymbol{\psi}^{\boldsymbol{2}}\vec{\zeta}_2 + \boldsymbol{\psi}^{\boldsymbol{31}}\vec{\zeta}_{31} + \boldsymbol{\psi}^{\boldsymbol{32}}\vec{\zeta}_{32}$$

$$\boldsymbol{\Pi}^{\boldsymbol{e}} = \frac{(1-2\mathcal{Y}^0)(1-3\mathcal{Y}^1)(1-3\mathcal{Y}^2)(1-3\mathcal{Y}^3)}{16} + \frac{(1+2\mathcal{Y}^0)(1+3\mathcal{Y}^1)(1+3\mathcal{Y}^2)(1+3\mathcal{Y}^3)}{16}$$

- Up Quark-type field

$$\overrightarrow{\boldsymbol{\psi}}_{\boldsymbol{u}} = \boldsymbol{\Pi}^{\boldsymbol{u}}\overrightarrow{\boldsymbol{\psi}} = \boldsymbol{\psi}^{\boldsymbol{3}}\vec{\zeta}_3 + \boldsymbol{\psi}^{\boldsymbol{4}}\vec{\zeta}_4 + \boldsymbol{\psi}^{\boldsymbol{5}}\vec{\zeta}_5 + \boldsymbol{\psi}^{\boldsymbol{6}}\vec{\zeta}_6 + \boldsymbol{\psi}^{\boldsymbol{9}}\vec{\zeta}_9 + \boldsymbol{\psi}^{\boldsymbol{10}}\vec{\zeta}_{10} + \boldsymbol{\psi}^{\boldsymbol{23}}\vec{\zeta}_{23}, \boldsymbol{\psi}^{\boldsymbol{24}}\vec{\zeta}_{24} + \boldsymbol{\psi}^{\boldsymbol{27}}\vec{\zeta}_{27} + \boldsymbol{\psi}^{\boldsymbol{28}}\vec{\zeta}_{28} + \boldsymbol{\psi}^{\boldsymbol{29}}\vec{\zeta}_{29} + \boldsymbol{\psi}^{\boldsymbol{30}}\vec{\zeta}_{30}$$

$$\begin{aligned}\boldsymbol{\Pi}^{\boldsymbol{u}} = {} & \frac{(1-2\mathcal{Y}^0)(1-3\mathcal{Y}^1)(1-3\mathcal{Y}^2)(1+3\mathcal{Y}^3)}{16} + \frac{(1-2\mathcal{Y}^0)(1-3\mathcal{Y}^1)(1+3\mathcal{Y}^2)(1-3\mathcal{Y}^3)}{16} \\ & + \frac{(1-2\mathcal{Y}^0)(1+3\mathcal{Y}^1)(1-3\mathcal{Y}^2)(1-3\mathcal{Y}^3)}{16} + \frac{(1+2\mathcal{Y}^0)(1-3\mathcal{Y}^1)(1+3\mathcal{Y}^2)(1+3\mathcal{Y}^3)}{16} \\ & + \frac{(1+2\mathcal{Y}^0)(1+3\mathcal{Y}^1)(1-3\mathcal{Y}^2)(1+3\mathcal{Y}^3)}{16} + \frac{(1+2\mathcal{Y}^0)(1+3\mathcal{Y}^1)(1+3\mathcal{Y}^2)(1-3\mathcal{Y}^3)}{16}\end{aligned}$$

- Neutrino-type field

$$\overrightarrow{\boldsymbol{\psi}}_{\boldsymbol{\nu}} = \boldsymbol{\Pi}^{\boldsymbol{\nu}}\overrightarrow{\boldsymbol{\psi}} = \boldsymbol{\psi}^{\mathbf{15}}\vec{\zeta}_{15} + \boldsymbol{\psi}^{\mathbf{16}}\vec{\zeta}_{16} + \boldsymbol{\psi}^{\mathbf{17}}\vec{\zeta}_{17} + \boldsymbol{\psi}^{\mathbf{18}}\vec{\zeta}_{18}$$

$$\boldsymbol{\Pi}^{\boldsymbol{\nu}} = \frac{(1-2\mathcal{Y}^0)(1+3\mathcal{Y}^1)(1+3\mathcal{Y}^2)(1+3\mathcal{Y}^3)}{16} + \frac{(1+2\mathcal{Y}^0)(1-3\mathcal{Y}^1)(1-3\mathcal{Y}^2)(1-3\mathcal{Y}^3)}{16}$$

- Down quark-type field

$$\overrightarrow{\boldsymbol{\psi}}_{\boldsymbol{d}} = \boldsymbol{\Pi}^{\boldsymbol{d}}\overrightarrow{\boldsymbol{\psi}} = \boldsymbol{\psi}^{\mathbf{7}}\vec{\zeta}_{7} + \boldsymbol{\psi}^{\mathbf{8}}\vec{\zeta}_{8} + \boldsymbol{\psi}^{\mathbf{11}}\vec{\zeta}_{11} + \boldsymbol{\psi}^{\mathbf{12}}\vec{\zeta}_{12} + \boldsymbol{\psi}^{\mathbf{13}}\vec{\zeta}_{13} + \boldsymbol{\psi}^{\mathbf{14}}\vec{\zeta}_{14} + \boldsymbol{\psi}^{\mathbf{19}}\vec{\zeta}_{19}, \boldsymbol{\psi}^{\mathbf{20}}\vec{\zeta}_{20} + \boldsymbol{\psi}^{\mathbf{21}}\vec{\zeta}_{21} + \boldsymbol{\psi}^{\mathbf{22}}\vec{\zeta}_{22} + \boldsymbol{\psi}^{\mathbf{25}}\vec{\zeta}_{25} + \boldsymbol{\psi}^{\mathbf{26}}\vec{\zeta}_{26}$$

$$\begin{aligned}\boldsymbol{\Pi}^{\boldsymbol{d}} = &\frac{(1-2\mathcal{Y}^0)(1-3\mathcal{Y}^1)(1+3\mathcal{Y}^2)(1+3\mathcal{Y}^3)}{16} + \frac{(1-2\mathcal{Y}^0)(1+3\mathcal{Y}^1)(1-3\mathcal{Y}^2)(1+3\mathcal{Y}^3)}{16} \\ &+ \frac{(1-2\mathcal{Y}^0)(1+3\mathcal{Y}^1)(1+3\mathcal{Y}^2)(1-3\mathcal{Y}^3)}{16} + \frac{(1+2\mathcal{Y}^0)(1-3\mathcal{Y}^1)(1-3\mathcal{Y}^2)(1+3\mathcal{Y}^3)}{16} \\ &+ \frac{(1+2\mathcal{Y}^0)(1-3\mathcal{Y}^1)(1+3\mathcal{Y}^2)(1-3\mathcal{Y}^3)}{16} + \frac{(1+2\mathcal{Y}^0)(1+3\mathcal{Y}^1)(1-3\mathcal{Y}^2)(1-3\mathcal{Y}^3)}{16}\end{aligned}$$

The above results suggest that we may consider a phase space representation of the spinor field $\vec{\boldsymbol{\psi}}$ in the form

$$\vec{\boldsymbol{\psi}}(x) = \sum_{s} \int \frac{d^3\vec{P}d^3\vec{X}}{(2\pi)^3} [\mathbb{a}(\vec{P},\vec{X},\mathcal{B}_{\mu\nu},s)\vec{u}(\vec{P},\vec{X},\mathcal{B}_{\mu\nu},s)e^{-iP^{\mu}x_{\mu}} + \mathbb{b}^{\dagger}(\vec{P},\vec{X},\mathcal{B}_{\mu\nu},s)\vec{v}(\vec{P},\vec{X},\mathcal{B}_{\mu\nu},s)e^{iP^{\mu}x_{\mu}}]e^{-\mathcal{B}_{\mu\nu}(x^{\mu}-X^{\mu})(x^{\nu}-X^{\nu})}$$

The index s refer to the set of quantum number describing the particle nature and internal state (isospin, electric charge, colors,…). According to this decomposition a particle state in the phase space representation is of the form $|\vec{P},\vec{X},\mathcal{B}_{\mu\nu},s\rangle$. $\vec{P}$ and $\vec{X}$ being the means of its momentum and coordinate vectors and $\mathcal{B}_{\mu\nu}$ the momentum variance –covariance tensors. It corresponds to $\mathcal{B}_{\mu\nu}$ a coordinate variance-covariance tensors $\mathcal{A}_{\mu\nu}$ so that we have a saturation of the Heisenberg principle

$$\mathcal{A}^{\mu\rho}\mathcal{B}_{\rho\nu} = \frac{1}{4}\delta^{\mu}_{\nu}(\hbar)^2$$

when a particle propagate without the change of its internal state, we may associate to it a "mean trajectory" with equation $\vec{P} = \vec{P}(X^0)$,$\vec{X} = \vec{X}(X^0)$. This mean trajectory may be related to the classical concept of trajectory in the classical limit.

4-Conclusion

The phase space representation provides the possibility to look at the same time on the momentum and coordinate of a particle taking into account the Heisenberg uncertainty principle. It provides an interesting framework for the study of fermions and their properties. It permits in particular to describe fundamental properties, like isospin, electric charge and color, as a natural consequence of the spinorial representation of linear canonical transformations like spin is described as a natural consequence of the spinorial representation of Lorentz transformations. This results can help in the construction of unified theory of interactions. The phase space representation may also be utilized in the study of classical limit in quantum theory.

BIBLIOGRAPHY

1. Ravo Tokiniaina Ranaivoson: Raoelina Andriambololona, Rakotoson Hanitriarivo, Roland Raboanary: Study on a Phase Space Representation of Quantum Theory, arXiv:1304.1034 [quant-ph], International Journal of Latest Research in Science andTechnology, ISSN(Online):2278-5299, Volume 2,Issue 2 :Page No.26-35, March-April 2013
2. Raoelina Andriambololona, Ravo Tokiniaina Ranaivoson, Rakotoson Hanitriarivo, Victor Harison, Derivation of equations for scalar and fermion fields using properties of dispersion-codispersion operators, arXiv:1401.6526 [quant-ph], International Journal of Latest Research in Science and Technology ISSN(Online):2278-5299, Volume 3, Issue 1 :Page No. 48-52,January-February 2014
3. Raoelina Andriambololona, Ravo Tokiniaina Ranaivoson, Rakotoson Hanitriarivo, Wilfrid Chrysante Solofoarisina: Study on Linear Canonical Transformation in a Framework of a Phase Space Representation of Quantum Mechanics, arXiv:1503.02449 [quant-ph], International Journal of Applied Mathematics and Theoretical Physics. Vol.1, No. 1, 2015, pp. 1-8, 2015
4. Hanitriarivo Rakotoson, Raoelina Andriambololona, Ravo Tokiniaina Ranaivoson, Raboanary Roland , coordinate, momentum and dispersion operators in phase space representation,arXiv:1707.02223 [quant-ph], International Journal of Latest Research in Science and Technology, ISSN (Online):2278-5299 Volume 6, Issue 4: Page No. 8-13, July-August 2017
5. Raoelina Andriambololona, Ravo Tokiniaina Ranaivoson, Hanitriarivo Rakotoson, Hasimbola Damo Emile Randriamisy, Dispersion Operators Algebra and Linear Canonical Transformations , arXiv:1608.02268 [quant-ph], International Journal of Theoretical Physics,Volume 56, Issue 4, pp 1258–1273, Springer, April 2017.
6. Raoelina Andriambololona, Ravo Tokiniaina Ranaivoson, Hanitriarivo Rakotoson, Clifford Algebras and Spinorial Representation of Linear Canonical Transformations in Quantum Theory, arXiv:1711.04975 [quant-ph], 2017
7. Ravo Tokiniaina Ranaivoson, Raoelina Andriambololona, Hanitriarivo Rakotoson, Properties of Phase Space Wavefunctions and Eigenvalue Equation of Momentum DispersionOperator, arXiv:1711.07308 arXiv:1711.07308 [quant-ph], International Journal of Applied Mathematics and Theoretical Physics. Vol. 4, No.1, 2018, pp. 8-14,. 2018
8. Hanitriarivo Rakotoson, Raoelina Andriambololona, Ravo Tokiniaina Ranaivoson, Roland Raboanary, Linear Canonical Transformations in Relativistic Quantum Physics, arXiv:1804.10053 [quant-ph], 2018
9. Ravo Tokiniaina Ranaivoson: Raoelina Andriambololona, Rakotoson Hanitriarivo, Properties of Elementary Fermions of the Standard Model deduced from Linear Canonical Transformations ,arXiv:1806.07228 [physics.gen-ph]