Dispersion Operators Algebra and its representation

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1-Dispersion operators and their uses

1.1Phase space representation of quantum theory and dispersion operators

In one dimensional quantum mechanics, a quantum state $|\psi\rangle$ of a particle can have the coordinate representation, the momentum representation and the phase space representation.

- **Coordinate representation**

$$|\psi\rangle = \int |x\rangle\langle x|\psi\rangle dx = \int \psi(x)|x\rangle\, dx$$

- $\langle x|\psi\rangle = \psi(x)$
- The basis is $\{|x\rangle\}$ with $\boldsymbol{x}|x\rangle = x|x\rangle$

- **Momentum representation**

$$|\psi\rangle = \int |p\rangle\langle p|\psi\rangle dp = \int \tilde{\psi}(p)|p\rangle\, dp$$

- $\langle p|\psi\rangle = \tilde{\psi}(p)$
- The basis is $\{|p\rangle\}$ with $\boldsymbol{p}|p\rangle = p|p\rangle$

The wave functions are linked by a Fourier Transforms

$$\tilde{\psi}(p) = \langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int \langle p|x\rangle\langle x|\psi\rangle\, dx = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-i\frac{px}{\hbar}}\psi(x)\, dx$$

- **Phase space representation (coordinate and momentum conjoint representation)**

  The phase space representation is the result of an attempt that we have performed to establish a coordinate-momentum conjoint representation taking into account the Heisenberg uncertainty principle. In this representation:

$$|\psi\rangle = \sum_{\boldsymbol{n}} \Psi(n, X, P, \mathscr{b})|n, X, P, \mathscr{b}\rangle = \iint \Psi(n, X, P, \mathscr{b})|n, X, P, \mathscr{b}\rangle \frac{dXdP}{2\pi\hbar}$$

- $\langle n, X, P, \mathscr{b}|\psi\rangle = \Psi(n, X, P, \mathscr{b})$ is the wavefunction in the phase representation
- The element $|n, X, P, \mathscr{b}\rangle$ of the basis $\{|n, X, P, \mathscr{b}\rangle\}$ defining the phase space representation satisfies the relations

$$\langle x|n, X, P, \mathscr{b}\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int \langle p|n, X, P, \mathscr{b}\rangle\, e^{i\frac{Px}{\hbar}} dp = \frac{H_n(\frac{x-X}{\sqrt{2}a})}{\sqrt{2^n n! \sqrt{2\pi} a}} e^{-\left(\frac{x-X}{2a}\right)^2 + i\frac{Px}{\hbar}}$$

$$\langle p|n, X, P, \mathscr{b}\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-i\frac{px}{\hbar}} \langle x|n, X, P, \mathscr{b}\rangle dx = \frac{(i)^n H_n(\frac{p-P}{\sqrt{2}\mathscr{b}})}{\sqrt{2^n n! \sqrt{2\pi}\mathscr{b}}} e^{-\left(\frac{p-P}{2\mathscr{b}}\right)^2 - iX\frac{(p-P)}{\hbar}}$$

$$\begin{cases} \langle n, X, P, \mathscr{b}|\boldsymbol{x}|n, X, P, \mathscr{b}\rangle = X \\ \langle n, X, P, \mathscr{b}|\boldsymbol{p}|n, X, P, \mathscr{b}\rangle = P \\ \left\langle n, X, P, \mathscr{b}\middle|(\boldsymbol{x} - X)^{\mathbf{2}}\middle|n, X, P, \mathscr{b}\right\rangle = (2n + 1)(a)^2 = (2n + 1)\mathcal{A} \\ \left\langle n, X, P, \mathscr{b}\middle|(\boldsymbol{p} - P)^{\mathbf{2}}\middle|n, X, P, \mathscr{b}\right\rangle = (2n + 1)(\mathscr{b})^2 = (2n + 1)\mathcal{B} \end{cases} \qquad a\mathscr{b} = \frac{\hbar}{2} \Leftrightarrow \mathcal{A}\mathcal{B} = (\frac{\hbar}{2})^2$$

$X, P, (2n + 1)\mathcal{A}, (2n + 1)\mathcal{B}$ are respectively the means values and statistical dispersions (variances) of the coordinate and momentum operators $\boldsymbol{x}$ and $\boldsymbol{p}$ in the state $|n, X, P, \mathscr{b}\rangle$. The standard deviation (uncertainty) $a = \sqrt{\mathcal{A}}$ and $\mathscr{b} = \sqrt{\mathcal{B}}$ relative to the state $|0, X, P, \mathscr{b}\rangle$ corresponds to a minimalization in the Heisenberg uncertainty relation.

The states $|n, X, P, \mathscr{b}\rangle$ are the eigenstates of the **momentum dispersion operator** $\beth^{+}$

$$\beth^{+} = \frac{1}{2}[(\boldsymbol{p} - P)^{\mathbf{2}} + 4(\frac{\mathcal{B}}{\hbar})^{2}(\boldsymbol{x} - X)^{\mathbf{2}}] \Rightarrow \beth^{+}|n, X, P, \mathscr{b}\rangle = (2n+1)\mathcal{B}|n, X, P, \mathscr{b}\rangle$$

The dispersion operator $\beth^{+}$ belongs to a Lie algebra called the **dispersion operators algebra**. This algebra is generated by three hermitians operators denoted respectively $\beth^{+}, \beth^{-}$ and $\beth^{\times}$

$$\begin{cases} \beth^{+} = \frac{1}{2}[(\boldsymbol{p} - P)^{\mathbf{2}} + 4(\frac{\mathcal{B}}{\hbar})^{2}(\boldsymbol{x} - X)^{\mathbf{2}}] \\ \beth^{-} = \frac{1}{2}[(\boldsymbol{p} - P)^{\mathbf{2}} - 4(\frac{\mathcal{B}}{\hbar})^{2}(\boldsymbol{x} - X)^{\mathbf{2}}] \\ \beth^{\times} = \frac{\mathcal{B}}{\hbar}[(\boldsymbol{p} - P)(\boldsymbol{x} - X) + (\boldsymbol{x} - X)(\boldsymbol{p} - P)] \end{cases} \Rightarrow \begin{cases} [\beth^{+}, \beth^{-}] = 4i\mathcal{B}\beth^{\times} \\ [\beth^{-}, \beth^{\times}] = -4i\mathcal{B}\beth^{+} \\ [\beth^{\times}, \beth^{+}] = 4i\mathcal{B}\beth^{-} \end{cases}$$

By introducing the **reduced coordinate and momentum** operators $\not\boldsymbol{p}$ and $\not\boldsymbol{x}$

$$\begin{cases} \not\boldsymbol{p} = \sqrt{2}a(\boldsymbol{p} - P) \\ \not\boldsymbol{x} = \sqrt{2}\mathscr{b}(\boldsymbol{x} - X) \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{p} = \sqrt{2}\mathscr{b}\not\boldsymbol{p} + P \\ \boldsymbol{x} = \sqrt{2}a\not\boldsymbol{x} + X \end{cases} \Rightarrow [\not\boldsymbol{x}, \not\boldsymbol{p}] = \frac{1}{\hbar}[\boldsymbol{x}, \boldsymbol{p}] = i$$

We can define the **reduced dispersion operators** $\not\beth^{+}, \not\beth^{-}, \not\beth^{\times}$

$$\begin{cases} \beth^{+} = 4\mathcal{B}\not\beth^{+} \\ \beth^{-} = 4\mathcal{B}\not\beth^{-} \\ \beth^{\times} = 4\mathcal{B}\not\beth^{\times} \end{cases} \Leftrightarrow \begin{cases} \not\beth^{+} = \frac{1}{4}[(\not\boldsymbol{p})^{2} + (\not\boldsymbol{x})^{2}] \\ \not\beth^{-} = \frac{1}{4}[(\not\boldsymbol{p})^{2} - (\not\boldsymbol{x})^{2}] \\ \not\beth^{\times} = \frac{1}{4}[\not\boldsymbol{p}\not\boldsymbol{x} + \not\boldsymbol{x}\not\boldsymbol{p}] \end{cases} \Rightarrow \begin{cases} [\not\beth^{+}, \not\beth^{-}] = i\not\beth^{\times} \\ [\not\beth^{-}, \not\beth^{\times}] = -i\not\beth^{+} \\ [\not\beth^{\times}, \not\beth^{+}] = i\not\beth^{-} \end{cases}$$

The set $\{\not\beth^{+}, \not\beth^{-}, \not\beth^{\times}\}$ is also a basis of the dispersion operator algebra

If we consider the coordinate and momentum operators $\boldsymbol{p}_{\mu}$ and $\boldsymbol{x_{\mu}}$ defined in a N-dimensional pseudo Euclidian space with signature (N_{+}, N_{-}), $N_{+} + N_{-} = N$,which satisfies the canonical commutation relations

$$[\boldsymbol{p}_{\mu}, \boldsymbol{x_{\nu}}]_{-} = i\eta_{\mu\nu}\hbar = \begin{cases} 1 \;\; if \;\; \mu = \nu = 0, \dots, N_{+} - 1 \\ 0 \;\; if \;\; \mu \neq \nu \\ -1 \;\; if \;\; \mu = \nu = N_{+}, \dots, N \end{cases}$$

We have a **generalization of the dispersion operator algebra** generated by the operators $\not{\beth}_{\mu\nu}^{+}, \not{\beth}_{\mu\nu}^{-}$ and $\not{\beth}_{\mu\nu}^{\times}$ defined as

$$\begin{cases} \beth_{\boldsymbol{\mu\nu}}^{+} = \dfrac{1}{2}[(\boldsymbol{p_{\mu}} - P_{\mu})(\boldsymbol{p_{\nu}} - P_{\nu}) + 4\mathcal{B}_{\mu\alpha}\mathcal{B}_{\nu\beta}(\boldsymbol{x}^{\alpha} - X^{\alpha})(\boldsymbol{x}^{\beta} - X^{\beta})] \\ \beth_{\boldsymbol{\mu\nu}}^{-} = \dfrac{1}{2}[\,(\boldsymbol{p_{\mu}} - P_{\mu})(\boldsymbol{p_{\nu}} - P_{\nu}) - 4\mathcal{B}_{\mu\alpha}\mathcal{B}_{\nu\beta}(\boldsymbol{x}^{\alpha} - X^{\alpha})(\boldsymbol{x}^{\beta} - X^{\beta})] \\ \beth_{\boldsymbol{\mu\nu}}^{\times} = \mathcal{B}_{\mu\alpha}[(\boldsymbol{p_{\nu}} - P_{\nu})(\boldsymbol{x}^{\alpha} - X^{\alpha}) + (\boldsymbol{x}^{\alpha} - X^{\alpha})(\boldsymbol{p_{\nu}} - P_{\nu})] \end{cases}$$

We have also the following generalization

$$\begin{cases} \not{\boldsymbol{p}}_{\boldsymbol{\mu}} = \sqrt{2}\mathscr{a}_{\mu}^{\nu}(\boldsymbol{p_{\nu}} - P_{\nu}) \\ \not{\boldsymbol{x}}_{\boldsymbol{\mu}} = \sqrt{2}\mathscr{b}_{\mu}^{\nu}(\boldsymbol{x_{\nu}} - X_{\nu}) \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{p_{\mu}} = \sqrt{2}\mathscr{b}_{\mu}^{\nu}\not{\boldsymbol{p}}_{\boldsymbol{\nu}} + P_{\mu} \\ \boldsymbol{x_{\mu}} = \sqrt{2}\mathscr{a}_{\mu}^{\nu}\not{\boldsymbol{x}}_{\boldsymbol{\nu}} + X_{\mu} \end{cases} \Rightarrow [\not{\boldsymbol{p}}_{\mu}, \not{\boldsymbol{x}}_{\boldsymbol{\nu}}]_{-} = [\boldsymbol{p}_{\mu}, \boldsymbol{x_{\nu}}]_{-} = i\eta_{\mu\nu}$$

$$\begin{cases} \beth_{\boldsymbol{\mu\nu}}^{+} = 4\mathscr{b}_{\mu}^{\rho}\mathscr{b}_{\nu}^{\lambda}\not{\beth}_{\boldsymbol{\rho\lambda}}^{+} \\ \beth_{\boldsymbol{\mu\nu}}^{-} = 4\mathscr{b}_{\mu}^{\rho}\mathscr{b}_{\nu}^{\lambda}\not{\beth}_{\boldsymbol{\rho\lambda}}^{-} \\ \beth_{\boldsymbol{\mu\nu}}^{\times} = 4\mathscr{b}_{\mu}^{\rho}\mathscr{b}_{\nu}^{\lambda}\not{\beth}_{\boldsymbol{\rho\lambda}}^{\times} \end{cases} \Leftrightarrow \begin{cases} \not{\beth}_{\boldsymbol{\mu\nu}}^{+} = \dfrac{1}{4}(\not{\boldsymbol{p}}_{\boldsymbol{\mu}}\not{\boldsymbol{p}}_{\boldsymbol{\nu}} + \not{\boldsymbol{x}}_{\boldsymbol{\mu}}\not{\boldsymbol{x}}_{\boldsymbol{\nu}}) \\ \not{\beth}_{\boldsymbol{\mu\nu}}^{-} = \dfrac{1}{4}(\not{\boldsymbol{p}}_{\boldsymbol{\mu}}\not{\boldsymbol{p}}_{\boldsymbol{\nu}} - \not{\boldsymbol{x}}_{\boldsymbol{\mu}}\not{\boldsymbol{x}}_{\boldsymbol{\nu}}) \\ \not{\beth}_{\boldsymbol{\mu\nu}}^{\times} = \dfrac{1}{4}(\not{\boldsymbol{p}}_{\boldsymbol{\mu}}\not{\boldsymbol{x}}_{\boldsymbol{\nu}} + \not{\boldsymbol{x}}_{\boldsymbol{\nu}}\not{\boldsymbol{p}}_{\boldsymbol{\mu}}) \end{cases}$$

The dimension of the dispersion operator algebra as vectorial space is equal to $N(2N + 1)$.

1.2Linear canonical transformations and dispersion operators

Linear canonical transformation (LCT) can be defined as linear transformations mixing the momentum and coordinate operators and leaving invariant the canonical commutation relations. With the reduced operators, the definition is

$$\begin{cases} \boldsymbol{p}'_\mu = \Pi_\mu^\nu \boldsymbol{p}_\nu + \Theta_\mu^\nu \boldsymbol{\varkappa}_\nu \\ \boldsymbol{\varkappa}'_\mu = \Xi_\mu^\nu \boldsymbol{p}_\nu + \Lambda_\mu^\nu \boldsymbol{\varkappa}_\nu \\ [\boldsymbol{p}'_\mu, \boldsymbol{\varkappa}'_\nu]_- = [\boldsymbol{p}_\mu, \boldsymbol{\varkappa}_\nu]_- = i\eta_{\mu\nu} \end{cases} \Leftrightarrow \begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix}^t \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} \begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} \in Sp(2N_+, 2N_-)$$

An LCT corresponds to an element of the pseudosymplectic group $Sp(2N_+, 2N_-)$. A geometric parameterization is

$$\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} = e^{\begin{pmatrix} \lambda+\mu & \varphi+\theta \\ \varphi-\theta & \lambda-\mu \end{pmatrix}} \Leftrightarrow \begin{cases} \theta^T = \eta\theta\eta \\ \varphi^T = \eta\varphi\eta \\ \mu^T = \eta\mu\eta \\ \lambda^T = -\eta\lambda\eta \end{cases} \Leftrightarrow \begin{pmatrix} \lambda+\mu & \varphi+\theta \\ \varphi-\theta & \lambda-\mu \end{pmatrix} \in \boldsymbol{\mathfrak{sp}}(2N_+, 2N_-) = Lie\ [Sp(2N_+, 2N_-)]$$

For $\mu = \varphi = \theta = 0$ and $\lambda \neq 0$, an LCT corresponds to a Lorentz-like (pseudorthogonal) Transformation. For $\mu = \varphi = \lambda = 0$ and $\theta \neq 0$, an LCT corresponds to a fractional Fourier transformations (rotations in the coordinate-momentum planes).

There is an isomorphism between the Lie algebra $\boldsymbol{\mathfrak{sp}}(2N_+, 2N_-)$ and the **dispersion operator algebra**. This isomorphism corresponds to a unitary representation of Linear canonical transformations using dispersion operators

$$\begin{cases} \boldsymbol{p}'_\mu = \Pi_\mu^\nu \boldsymbol{p}_\nu + \Theta_\mu^\nu \boldsymbol{\varkappa}_\nu \\ \boldsymbol{\varkappa}'_\mu = \Xi_\mu^\nu \boldsymbol{p}_\nu + \Lambda_\mu^\nu \boldsymbol{\varkappa}_\nu \\ [\boldsymbol{p}'_\mu, \boldsymbol{\varkappa}'_\nu]_- = [\boldsymbol{p}_\mu, \boldsymbol{\varkappa}_\nu]_- = i\eta_{\mu\nu} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{p}'_\mu = \boldsymbol{U} \boldsymbol{p}_{\boldsymbol{\mu}} \boldsymbol{U}^\dagger \\ \boldsymbol{\varkappa}'_\mu = \boldsymbol{U} \boldsymbol{\varkappa}_{\boldsymbol{\mu}} \boldsymbol{U}^\dagger \\ [\boldsymbol{p}'_\mu, \boldsymbol{\varkappa}'_\nu]_- = [\boldsymbol{p}_\mu, \boldsymbol{\varkappa}_\nu]_- = i\eta_{\mu\nu} \end{cases} \Leftrightarrow \boldsymbol{U} = e^{-2i\eta^{\mu\rho}[\theta_\rho^\nu \beth_{\boldsymbol{\mu\nu}}^+ + \varphi_\rho^\nu \beth_{\boldsymbol{\mu\nu}}^- + (\lambda_\rho^\nu - \mu_\rho^\nu) \beth_{\boldsymbol{\mu\nu}}^\times]}$$

2-Representation of the dispersion operators algebra

The purpose is to find the representation of the dispersion operators algebra in the quantum states space. It consists to find the representations of the operators $\beth_{\mu\nu}^{+}, \beth_{\mu\nu}^{-}$ and $\beth_{\mu\nu}^{\times}$ in a basis of the quantum state space. We know three basis:

- The basis $\{ |\{x_\mu\}\rangle\}$ constituted by the eigenstates of coordinates operators (coordinate representation)
- The basis $\{ |\{p_\mu\}\rangle\}$ constituted by the eigenstates of momentum operators (momentum representation)
- The basis $\{ |\{n_\mu\}, \{X_\mu\}, \{P_\mu\}, \{b_\mu^{\nu}\}\rangle\}$ constituted by the eigenstates of the dispersion operators $\beth_{\boldsymbol{\mu\mu}}^{\times}$(phase space representation)

2.1 Coordinate representation

In the coordinate representation, we have for the representations of the coordinates ad momentum operators

$$\begin{cases} \hat{p}_\mu = i\dfrac{\partial}{\partial x^\mu} \\ \hat{x}_\mu = x_\mu \end{cases} \Rightarrow \begin{cases} \hat{\boldsymbol{p}}_{\boldsymbol{\mu}} = \sqrt{2}a_\mu^\nu(\hat{\boldsymbol{p}}_{\boldsymbol{\nu}} - P_\nu) = \sqrt{2}a_\mu^\nu(i\dfrac{\partial}{\partial x^\nu} - P_\nu) \\ \hat{\boldsymbol{x}}_{\boldsymbol{\mu}} = \sqrt{2}b_\mu^\nu(\hat{\boldsymbol{x}}_{\boldsymbol{\nu}} - X_\nu) = \sqrt{2}b_\mu^\nu(x_\nu - X_\nu) \end{cases}$$

The operators $\beth_{\mu\nu}^{+}, \beth_{\mu\nu}^{-}$ and $\beth_{\mu\nu}^{\times}$ are then represented by the linear differential operators

$$\begin{cases} \hat{\beth}_{\boldsymbol{\mu\nu}}^{+} = \dfrac{1}{4}(\hat{\boldsymbol{p}}_{\boldsymbol{\mu}}\hat{\boldsymbol{p}}_{\boldsymbol{\nu}} + \hat{\boldsymbol{x}}_{\boldsymbol{\mu}}\hat{\boldsymbol{x}}_{\boldsymbol{\nu}}) = \dfrac{1}{2}[a_\mu^\rho a_\nu^\lambda(i\dfrac{\partial}{\partial x^\rho} - P_\rho)(i\dfrac{\partial}{\partial x^\rho} - P_\lambda) + b_\mu^\rho b_\nu^\lambda(x_\rho - X_\rho)(x_\lambda - X_\lambda)] \\ \hat{\beth}_{\boldsymbol{\mu\nu}}^{-} = \dfrac{1}{4}\left(\hat{\boldsymbol{p}}_{\boldsymbol{\mu}}\hat{\boldsymbol{p}}_{\boldsymbol{\nu}} - \hat{\boldsymbol{x}}_{\boldsymbol{\mu}}\hat{\boldsymbol{x}}_{\boldsymbol{\nu}}\right) = \dfrac{1}{2}[a_\mu^\rho a_\nu^\lambda(i\dfrac{\partial}{\partial x^\rho} - P_\rho)(i\dfrac{\partial}{\partial x^\rho} - P_\lambda) - b_\mu^\rho b_\nu^\lambda(x_\rho - X_\rho)(x_\lambda - X_\lambda)] \\ \hat{\beth}_{\boldsymbol{\mu\nu}}^{\times} = \dfrac{1}{4}\left(\hat{\boldsymbol{p}}_{\boldsymbol{\mu}}\hat{\boldsymbol{x}}_{\boldsymbol{\nu}} + \hat{\boldsymbol{x}}_{\boldsymbol{\nu}}\hat{\boldsymbol{p}}_{\boldsymbol{\mu}}\right) = \dfrac{1}{2}a_\mu^\rho b_\nu^\lambda[(i\dfrac{\partial}{\partial x^\rho} - P_\rho)(x_\lambda - X_\lambda) + (x_\lambda - X_\lambda)(i\dfrac{\partial}{\partial x^\rho} - P_\rho)] \end{cases}$$

2.2 Momentum representation

In the momentum representation, we have for the representations of the coordinates ad momentum operators

$$\begin{cases} \tilde{\boldsymbol{p}}_\mu = p_\mu \\ \tilde{\boldsymbol{x}}_\mu = -i\dfrac{\partial}{\partial p^\mu} \end{cases} \Rightarrow \begin{cases} \tilde{\not{\boldsymbol{p}}}_\mu = \sqrt{2}a_\mu^\nu(\tilde{\boldsymbol{p}}_\nu - P_\nu) = \sqrt{2}a_\mu^\nu(p_\nu - P_\nu) \\ \tilde{\not{\boldsymbol{x}}}_\mu = \sqrt{2}\not{b}_\mu^\nu(-i\dfrac{\partial}{\partial p^\nu} - X_\nu) = \sqrt{2}\not{b}_\mu^\nu(-i\dfrac{\partial}{\partial p^\nu} - X_\nu) \end{cases}$$

The operators $\beth_{\mu\nu}^{+}, \beth_{\mu\nu}^{-}$ and $\beth_{\mu\nu}^{\times}$ are then represented by the linear differential operators

$$\begin{cases} \tilde{\beth}_{\mu\nu}^{+} = \dfrac{1}{4}(\tilde{\not{\boldsymbol{p}}}_\mu\tilde{\not{\boldsymbol{p}}}_\nu + \tilde{\not{\boldsymbol{x}}}_\mu\tilde{\not{\boldsymbol{x}}}_\nu) = \dfrac{1}{2}[a_\mu^\rho a_\nu^\lambda(p_\rho - P_\rho)(p_\lambda - P_\lambda) + \not{b}_\mu^\rho \not{b}_\nu^\lambda(-i\dfrac{\partial}{\partial p^\rho} - X_\rho)(-i\dfrac{\partial}{\partial p^\lambda} - X_\lambda)] \\ \tilde{\beth}_{\mu\nu}^{-}, = \dfrac{1}{4}\left(\tilde{\not{\boldsymbol{p}}}_\mu\tilde{\not{\boldsymbol{p}}}_\nu - \tilde{\not{\boldsymbol{x}}}_\mu\tilde{\not{\boldsymbol{x}}}_\nu\right) = \dfrac{1}{2}[a_\mu^\rho a_\nu^\lambda\left(p_\rho - P_\rho\right)(p_\lambda - P_\lambda) - \not{b}_\mu^\rho \not{b}_\nu^\lambda(-i\dfrac{\partial}{\partial p^\rho} - X_\rho)(-i\dfrac{\partial}{\partial p^\rho} - X_\lambda)] \\ \tilde{\beth}_{\mu\nu}^{\times}, = \dfrac{1}{4}\left(\tilde{\not{\boldsymbol{p}}}_\mu\tilde{\not{\boldsymbol{x}}}_\nu + \tilde{\not{\boldsymbol{x}}}_\nu\tilde{\not{\boldsymbol{p}}}_\nu\right) = \dfrac{1}{2}a_\mu^\rho \not{b}_\nu^\lambda[(p_\rho - P_\rho)(-i\dfrac{\partial}{\partial p^\lambda} - X_\lambda) + (-i\dfrac{\partial}{\partial p^\lambda} - X_\lambda)(p_\rho - P_\rho)] \end{cases}$$

2.3 Phase space representation

The elements of the basis $\{\left|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\not{b}_\mu^\nu\}\right\rangle\}$ in which the representation is to be performed are the eigenstates of the operators $\beth_{\boldsymbol{\rho\rho}}^{+}$. They are characterized by the relations

$$\begin{cases} \left\langle\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\not{b}_\mu^\nu\}\right|\boldsymbol{x}_\rho\left|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\not{b}_\mu^\nu\}\right\rangle = X_\rho \\ \left\langle\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\not{b}_\mu^\nu\}\right|\boldsymbol{p}_\rho\left|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\not{b}_\mu^\nu\}\right\rangle = P_\rho \\ \left\langle\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\not{b}_\mu^\nu\}\right|(\not{\boldsymbol{x}}_{\boldsymbol{\rho}})^2\left|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\not{b}_\mu^\nu\}\right\rangle = n_\rho + \dfrac{1}{2} \\ \left\langle\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\not{b}_\mu^\nu\}\right|(\not{\boldsymbol{p}}_{\boldsymbol{\rho}})^2\left|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\not{b}_\mu^\nu\}\right\rangle = n_\rho + \dfrac{1}{2} \end{cases}$$

$$\langle\{x_\mu\}|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\rangle = \frac{1}{[2\pi det(\mathcal{B}_{\mu\nu})]^{1/4}}\left(\prod_{\mu=0}^{N-1}\frac{H_{n_\mu}[\sqrt{2}\mathscr{b}_\nu^\mu(x^\nu - X^\nu)]}{\sqrt{2^{n_\mu}n_\mu!}}\right) e^{-\mathcal{B}_{\mu\nu}(x^\mu-X^\mu)(x^\nu-X^\nu)-iP_\mu x^\mu}$$

and we have the eigenvalue equation

$$\beth_{\rho\rho}^{+}|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\rangle = (2n_\rho + 1)|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\rangle$$

The purpose is to find the representation of all the generators $\beth_{\mu\nu}^{+}, \beth_{\mu\nu}^{-}$ and $\beth_{\mu\nu}^{\times}$of the dispersion operator algebra in the basis{ $|\{n_\mu\},\{X_\mu\},\{P_\mu\},\{\mathscr{b}_\mu^\nu\}\rangle$}. For sake of convenience, we split an operator $\beth_{\mu\nu}^{\times}$ in two parts: a symmetric part denoted $\beth_{\mu\nu}^{\bowtie}$ and an antisymmetric part denoted $\beth_{\mu\nu}^{\ltimes}$ i.e we consider the set

$$\begin{cases} \beth_{\mu\nu}^{+} = \frac{1}{4}(\boldsymbol{p}_{\mu}\boldsymbol{p}_{\nu} + \boldsymbol{x}_{\mu}\boldsymbol{x}_{\nu}) \\ \beth_{\mu\nu}^{-} = \frac{1}{4}(\boldsymbol{p}_{\mu}\boldsymbol{p}_{\nu} - \boldsymbol{x}_{\mu}\boldsymbol{x}_{\nu}) \\ \beth_{\mu\nu}^{\bowtie} = \frac{1}{2}(\beth_{\mu\nu}^{\times} + \beth_{\nu\mu}^{\times}) = \frac{1}{8}(\boldsymbol{p}_{\mu}\boldsymbol{x}_{\nu} + \boldsymbol{x}_{\nu}\boldsymbol{p}_{\mu} + \boldsymbol{p}_{\nu}\boldsymbol{x}_{\mu} + \boldsymbol{x}_{\mu}\boldsymbol{p}_{\nu}) \\ \beth_{\mu\nu}^{\ltimes} = \frac{1}{2}(\beth_{\mu\nu}^{\times} - \beth_{\nu\mu}^{\times}) = \frac{1}{4}(\boldsymbol{p}_{\mu}\boldsymbol{x}_{\nu} - \boldsymbol{p}_{\nu}\boldsymbol{x}_{\mu}) \end{cases}$$

We introduce the operators $\boldsymbol{z}_{\mu}$ and $\boldsymbol{z}_{\mu}^{\dagger}$ defined by the relations

$$\begin{cases} \boldsymbol{z}_\mu = \frac{1}{\sqrt{2}}(\boldsymbol{p}_\mu + i\boldsymbol{x}_\mu) \\ \boldsymbol{z}_\mu^\dagger = \frac{1}{\sqrt{2}}(\boldsymbol{p}_\mu - i\boldsymbol{x}_\mu) \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{p}_\mu = \frac{1}{\sqrt{2}}(\boldsymbol{z}_\mu^\dagger + \boldsymbol{z}_\mu) \\ \boldsymbol{x}_\mu = \frac{i}{\sqrt{2}}(\boldsymbol{z}_\mu^\dagger - \boldsymbol{z}_\mu) \end{cases}$$

We have then

$$\begin{cases} \beth^{+}_{\boldsymbol{\mu\nu}} = \dfrac{1}{4}(\boldsymbol{z}^{\dagger}_{\mu}\boldsymbol{z}_{\nu} + \boldsymbol{z}_{\mu}\boldsymbol{z}^{\dagger}_{\nu}) \\ \beth^{-}_{\boldsymbol{\mu\nu}} = \dfrac{1}{4}\left(\boldsymbol{z}^{\dagger}_{\mu}\boldsymbol{z}^{\dagger}_{\nu} + \boldsymbol{z}_{\mu}\boldsymbol{z}_{\nu}\right) \\ \beth^{\bowtie}_{\mu\nu} = \dfrac{i}{4}(\boldsymbol{z}^{\dagger}_{\mu}\boldsymbol{z}^{\dagger}_{\nu} - \boldsymbol{z}_{\mu}\boldsymbol{z}_{\nu}) \\ \beth^{\ltimes}_{\mu\nu} = -\dfrac{i}{4}(\boldsymbol{z}^{\dagger}_{\mu}\boldsymbol{z}_{\nu} - \boldsymbol{z}^{\dagger}_{\nu}\boldsymbol{z}_{\mu}) \end{cases}$$

These relations can be also written in the form

$$\begin{cases} \beth^{+}_{\boldsymbol{\mu\nu}} + i\beth^{\ltimes}_{\mu\nu} = \dfrac{1}{4}(2\boldsymbol{z}^{\dagger}_{\mu}\boldsymbol{z}_{\nu} + \eta_{\mu\nu}) \\ \beth^{+}_{\boldsymbol{\mu\nu}} - i\beth^{\ltimes}_{\mu\nu} = \dfrac{1}{4}(2\boldsymbol{z}_{\mu}\boldsymbol{z}^{\dagger}_{\nu} - \eta_{\mu\nu}) \\ \beth^{-}_{\boldsymbol{\mu\nu}} + i\beth^{\bowtie}_{\mu\nu} = \dfrac{1}{2}\boldsymbol{z}_{\mu}\boldsymbol{z}_{\nu} \\ \beth^{-}_{\boldsymbol{\mu\nu}} - i\beth^{\bowtie}_{\mu\nu} = \dfrac{1}{2}\boldsymbol{z}^{\dagger}_{\mu}\boldsymbol{z}^{\dagger}_{\nu} \end{cases}$$

We have

$$[\boldsymbol{p}_{\mu}, \boldsymbol{x}_{\nu}] = i\eta_{\mu\nu} \Rightarrow \begin{cases} \left[\beth^{+}_{\mu\mu}, \boldsymbol{z}_{\nu}\right] = -\dfrac{1}{2}\eta_{\mu\nu}\boldsymbol{z}_{\mu} \\ \left[\beth^{+}_{\mu\mu}, \boldsymbol{z}^{\dagger}_{\nu}\right] = \dfrac{1}{2}\eta_{\mu\nu}\boldsymbol{z}^{\dagger}_{\mu} \end{cases}$$

These commutation relation permits to establish the relations

$$\begin{cases} \mathbf{z}_\nu \left|\{n_\rho\},\{X_\rho\},\{P_\rho\},\{b_\rho^\lambda\}\right\rangle = \sqrt{n_\nu + \frac{1-\eta_{\nu\nu}}{2}} \left|n_\nu - \eta_{\nu\nu},\{n_\rho,\rho \neq \nu\},\{X_\rho\},\{P_\rho\},\{b_\rho^\lambda\}\right\rangle \\ \mathbf{z}_\nu^\dagger \left|\{n_\rho\},\{X_\rho\},\{P_\rho\},\{b_\rho^\lambda\}\right\rangle = \sqrt{n_\nu + \frac{1+\eta_{\nu\nu}}{2}} \left|n_\nu + \eta_{\nu\nu},\{n_\rho,\rho \neq \nu\},\{X_\rho\},\{P_\rho\},\{b_\rho^\lambda\}\right\rangle \end{cases}$$

We have then for the relations defining the representation of the dispersion operator algebra using the basis $\{\left|\{n_\rho\},\{X_\rho\},\{P_\rho\},\{b_\rho^\lambda\}\right\rangle\}$:

$$(\beth_{\boldsymbol{\mu\nu}}^{+} + i\beth_{\mu\nu}^{\ltimes})\left|\{n_\rho\},\{X_\rho\},\{P_\rho\},\{b_\rho^\lambda\}\right\rangle = \frac{1}{4}(2\mathbf{z}_\mu^\dagger \mathbf{z}_\nu + \eta_{\mu\nu})\left|\{n_\rho\},\{X_\rho\},\{P_\rho\},\{b_\rho^\lambda\}\right\rangle$$

$$= \begin{cases} \frac{1}{4}\left(2n_\mu + 1\right)\left|\{n_\rho\},\{X_\rho\},\{P_\rho\},\{b_\rho^\lambda\}\right\rangle & if\ \mu = \nu \\ \frac{1}{4}\sqrt{(2n_\nu + 1 - \eta_{\nu\nu})(2n_\mu + 1 + \eta_{\mu\mu})}\left|n_\nu - \eta_{\nu\nu}, n_\mu + \eta_{\mu\mu},\{n_\rho/\rho \neq \mu,\nu\},\{X_\rho\},\{P_\rho\},\{b_\rho^\lambda\}\right\rangle & if\ \mu \neq \nu \end{cases}$$

$$(\beth_{\boldsymbol{\mu\nu}}^{+} - i\beth_{\mu\nu}^{\ltimes})\left|\{n_\rho\},\{X_\rho\},\{P_\rho\},\{b_\rho^\lambda\}\right\rangle = \frac{1}{4}(2\mathbf{z}_\mu \mathbf{z}_\nu^\dagger - \eta_{\mu\nu})\left|\{n_\rho\},\{X_\rho\},\{P_\rho\},\{b_\rho^\lambda\}\right\rangle$$

$$= \begin{cases} \frac{1}{4}\left(2n_\mu + 1\right)\left|\{n_\rho\},\{X_\rho\},\{P_\rho\},\{b_\rho^\lambda\}\right\rangle & if\ \mu = \nu \\ \frac{1}{4}\sqrt{(2n_\nu + 1 + \eta_{\nu\nu})(2n_\mu + 1 - \eta_{\mu\mu})}\left|n_\nu + \eta_{\nu\nu}, n_\mu - \eta_{\mu\mu},\{n_\rho/\rho \neq \mu,\nu\},\{X_\rho\},\{P_\rho\},\{b_\rho^\lambda\}\right\rangle & if\ \mu \neq \nu \end{cases}$$

$$(\sqsupseteq^{-}_{\boldsymbol{\mu\nu}} + i\sqsupseteq^{\bowtie}_{\mu\nu})\left|\{n_\rho\},\{X_\rho\},\{P_\rho\},\{\mathscr{b}^{\lambda}_{\rho}\}\right\rangle = \frac{1}{2}\boldsymbol{z}_\mu \boldsymbol{z}_\nu \left|\{n_\rho\},\{X_\rho\},\{P_\rho\},\{\mathscr{b}^{\lambda}_{\rho}\}\right\rangle$$

$$= \begin{cases} \frac{1}{4}\sqrt{(2n_\nu + 1 - \eta_{\nu\nu})(2n_\nu + 1 - 3\eta_{\nu\nu})}\left|n_\nu - 2\eta_{\nu\nu},\{n_\rho/\rho \neq \nu\},\{X_\rho\},\{P_\rho\},\{\mathscr{b}^{\lambda}_{\rho}\}\right\rangle & if\ \mu = \nu \\ \frac{1}{4}\sqrt{(2n_\nu + 1 - \eta_{\nu\nu})(2n_\mu + 1 - \eta_{\mu\mu})}\left|n_\nu - \eta_{\nu\nu}, n_\mu - \eta_{\mu\mu}\{n_\rho/\rho \neq \mu,\nu\},\{X_\rho\},\{P_\rho\},\{\mathscr{b}^{\lambda}_{\rho}\}\right\rangle & if\ \mu \neq \nu \end{cases}$$

$$(\sqsupseteq^{-}_{\boldsymbol{\mu\nu}} - i\sqsupseteq^{\bowtie}_{\mu\nu})\left|\{n_\rho\},\{X_\rho\},\{P_\rho\},\{\mathscr{b}^{\lambda}_{\rho}\}\right\rangle = \frac{1}{2}\boldsymbol{z}^{\dagger}_\mu \boldsymbol{z}^{\dagger}_\nu \left|\{n_\rho\},\{X_\rho\},\{P_\rho\},\{\mathscr{b}^{\lambda}_{\rho}\}\right\rangle$$

$$= \begin{cases} \frac{1}{4}\sqrt{(2n_\nu + 1 + \eta_{\nu\nu})(2n_\nu + 1 + 3\eta_{\nu\nu})}\left|n_\nu + 2\eta_{\nu\nu},\{n_\rho/\rho \neq \nu\},\{X_\rho\},\{P_\rho\},\{\mathscr{b}^{\lambda}_{\rho}\}\right\rangle & if\ \mu = \nu \\ \frac{1}{4}\sqrt{(2n_\nu + 1 + \eta_{\nu\nu})(2n_\mu + 1 + \eta_{\mu\mu})}\left|n_\nu + \eta_{\nu\nu}, n_\mu + \eta_{\mu\mu}\{n_\rho/\rho \neq \mu,\nu\},\{X_\rho\},\{P_\rho\},\{\mathscr{b}^{\lambda}_{\rho}\}\right\rangle & if\ \mu \neq \nu \end{cases}$$

According to these relations, we have a matrix representation with matrices having infinite row and infinite column. In fact the index n_μ can take any positive integer values between 0 and $+\infty$.

We have for instance for the case of $N = 1$ with a signature (1,0) i.e $\eta_{00} = 1$

$$\begin{cases} \sqsupseteq^{+}_{\boldsymbol{00}} = \frac{1}{4}(\boldsymbol{z}^{\dagger}_0 \boldsymbol{z}_0 + \boldsymbol{z}_0 \boldsymbol{z}^{\dagger}_0) \\ \sqsupseteq^{-}_{\boldsymbol{00}} = \frac{1}{4}[(\boldsymbol{z}^{\dagger}_0)^2 + (\boldsymbol{z}_0)^2] \\ \sqsupseteq^{\bowtie}_{00} = \frac{i}{4}[(\boldsymbol{z}^{\dagger}_0)^2 - (\boldsymbol{z}_0)^2] = \sqsupseteq^{\times}_{00} \\ \sqsupseteq^{\ltimes}_{00} = 0 \end{cases} \Rightarrow \begin{cases} \sqsupseteq^{+}_{\boldsymbol{00}}|n_0, X_0, P_0, \mathscr{b}^{0}_{0}\rangle = \frac{1}{4}(2n_0 + 1)|n_0, X_0, P_0, \mathscr{b}^{0}_{0}\rangle \\ (\sqsupseteq^{-}_{\boldsymbol{00}} + i\sqsupseteq^{\times}_{\boldsymbol{00}})|n_0, X_0, P_0, \mathscr{b}^{0}_{0}\rangle = \frac{1}{2}\sqrt{n_0(n_0 - 1)}|n_0 - 2, X_0, P_0, \mathscr{b}^{0}_{0}\rangle \\ (\sqsupseteq^{-}_{\boldsymbol{00}} - i\sqsupseteq^{\times}_{\boldsymbol{00}})|n_0, X_0, P_0, \mathscr{b}^{0}_{0}\rangle = \frac{1}{2}\sqrt{(n_0 + 1)(n_0 + 2)}|n_0 + 2, X_0, P_0, \mathscr{b}^{0}_{0}\rangle \end{cases}$$

3-Conclusion

The dispersion operator algebra is closely linked to the phase space representation of quantum theory and linear canonical transformations. The states which permits to define the phase space representation are the eigenstates of some dispersion operators. There is an isomorphism between the dispersion operator algebra and the Lie algebra of the Lie group corresponding to the linear canonical transformations (LCTs). This isomorphism permits to have a unitary representation of the LCTs

The representation of the dispersion algebra in the quantum states space can be established using either the coordinate representation, the momentum representation, or the phase space representation. For the two first cases one has a representation with linear second order differential operators. For the phase space representation, we can have a matrix representation with matrices having infinite rows and columns.

The calculations and results that we have established may have interesting applications in domains related to phase space representation of quantum theory and linear canonical transformations.