The exact transition Probability for Two Flavor Neutrino Oscillations in Vacuum*

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Abstract

In this paper, transition probability for two flavor neutrino oscillations has been calculated in using the Dirac equation and compared with the exact transition probability obtained in using quantum field theory.

Keywords: Dirac equation, Negative energy, Kronecker product.

Neutrino oscillation is usually understood as the transition from a neutrino flavor state to another neutrino flavor state depending on the distance traveled [1]. Neutrino flavor states ν_e, ν_μ, ν_τ are superpositions of the neutrino mass eigenstates. Although there are three neutrino flavors two flavor neutrino oscillations exists in nature: for example $\nu_\mu \longrightarrow \nu_\tau$, in atmospheric neutrinos [2]. We suppose that a neutrino is Dirac neutrino, that is neutrino is not its own antiparticle.

Using Quantum field theory the exact transition probability $P_{QFT}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right)$ has been obtained in [3]. This transition probability recovers the Pontecorvo's one $P_{NRQM}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right)$ obtained in using nonrelativistic quantum mechanics [4]. In this paper we look for a transition probability $P_{RQM}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right)$ in using relativistic quantum mechanics, more precisely the Dirac equation. It will be a realization of the possibility of utilization of an hamiltonian from the Dirac equation [5].

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1 Dirac wave functions

The Dirac equation

$$i\hbar\gamma^{\mu}\partial_{\mu}\nu\left(t,\vec{x}\right)-mc\nu\left(t,\vec{x}\right)=0$$

where γ^{μ}'s are the gamma matrices, $\hbar$ is the Planck constant, c the speed of light, m the mass of the mass eigenstate of the spin-$\frac{1}{2}$ fermion.
The wave function $\nu=\nu\left(t,\vec{x}\right)$ solution of the Dirac equation may be written as Kronecker product or tensor product

$$\nu(t,\vec{x})=\xi\otimes se^{-\frac{i}{\hbar}(\pm Et-\vec{p}.\vec{x})} \tag{1}$$

with s is an eigeinstate of the helicity operator $\frac{\hbar}{2}\vec{\sigma}.\vec{n}=\frac{\hbar}{2}n_1\sigma_1+\frac{\hbar}{2}n_2\sigma_2+\frac{\hbar}{2}n_3\sigma_3$, with $\vec{n}=\frac{\vec{p}}{\|\vec{p}\|}=\frac{\vec{p}}{p}$, ξ is an eigeinstate of what we call the operator of sign of energy $h_D=\epsilon cp\sigma^1+mc^2\sigma^3$ [6], with ϵ is the sign of the helicity. There should be the probabilities of having positive energy and negative energy [7]. These probabilities should have impact on the transition probability.

2 Transition probability

From now on let us use the natural unit $c=1$ and $\hbar=1$. The first factors of the two mass eigenstates with positive energies are

$$\left|\xi_2(0,\vec{0})\right\rangle=\sqrt{\frac{E_2+m_2}{2E_2}}\begin{pmatrix}1\\ \frac{\epsilon_2p_2}{E_2+m_2}\end{pmatrix},\quad \left|\xi_3(0,\vec{0})\right\rangle=\sqrt{\frac{E_3+m_3}{2E_3}}\begin{pmatrix}1\\ \frac{\epsilon_3p_3}{E_3+m_3}\end{pmatrix}$$

and with negative energies are

$$\left|\bar{\xi}_2(0,\vec{0})\right\rangle=\sqrt{\frac{E_2+m_2}{2E_2}}\begin{pmatrix}-\frac{\epsilon_2p_2}{E_2+m_2}\\ 1\end{pmatrix},\quad \left|\bar{\xi}_3(0,\vec{0})\right\rangle=\sqrt{\frac{E_3+m_3}{2E_3}}\begin{pmatrix}-\frac{\epsilon_3p_3}{E_3+m_3}\\ 1\end{pmatrix}$$

at time $t=0$.

The probabilities for having the same sign of energies and different sign of energies at $t=0$ are respectively

$$P\left\{SigE_2=SigE_3,t=0\right\}=\frac{\left(E_2+m_2\right)\left(E_3+m_3\right)}{4E_2E_3}\left(1+\frac{\epsilon_2\epsilon_3p_2p_3}{\left(E_2+m_2\right)\left(E_3+m_3\right)}\right)^2$$

and

$$P\left\{SigE_2\neq SigE_3,t=0\right\}=\frac{\left(E_2+m_2\right)\left(E_3+m_3\right)}{4E_2E_3}\left(\frac{\epsilon_3p_3}{E_3+m_3}-\frac{\epsilon_2p_2}{E_2+m_2}\right)^2$$

with ϵ_2 and ϵ_3 are the helicity signs of the mass eigenstates at $(t, \vec{x}) = (0, \vec{0})$ [8].

For the second factor of the mass eigenstates, the eigenstates of helicity sign operator $\frac{\vec{\sigma}.\vec{p}}{2p} = \frac{p_1}{p}\sigma_1 + \frac{p_2}{p}\sigma_2 + \frac{p_3}{p}\sigma_3$, in spherical coordinates (ρ, θ, φ), are

$$|s_2(\vec{p}_2)\rangle = \begin{pmatrix} \cos\frac{\theta_2}{2} \\ \sin\frac{\theta_2}{2} e^{i\varphi_2} \end{pmatrix} \quad \text{and} \quad |s_3(\vec{p}_3)\rangle = \begin{pmatrix} \cos\frac{\theta_3}{2} \\ \sin\frac{\theta_3}{2} e^{i\varphi_3} \end{pmatrix}$$

for positive helicity and

$$|\bar{s}_2(\vec{p}_2)\rangle = \begin{pmatrix} -\sin\frac{\theta_2}{2} e^{-i\varphi_2} \\ \cos\frac{\theta_2}{2} \end{pmatrix} \quad \text{and} \quad |\bar{s}_3(\vec{p}_3)\rangle = \begin{pmatrix} -\sin\frac{\theta_3}{2} e^{-i\varphi_3} \\ \cos\frac{\theta_3}{2} \end{pmatrix}$$

for negative helicity.

$$P(\epsilon_2 = \epsilon_3) = \cos^2\left(\frac{\theta_2 + \theta_3}{2}\right) + \sin\theta_2 \sin\theta_3 \cos^2\left(\frac{\varphi_3 - \varphi_2}{2}\right)$$

and

$$P(\epsilon_2 \neq \epsilon_3) = \sin^2\left(\frac{\theta_2 + \theta_3}{2}\right) - \sin\theta_2 \sin\theta_3 \cos^2\left(\frac{\varphi_3 - \varphi_2}{2}\right)$$

are respectively the probabilities for having respectively same sign of helicities and different signs of helicities at time $t = 0$ for the mass eigenstates.
We suppose that $\vec{p}_2$ and $\vec{p}_3$ are in the same direction, then $P(\epsilon_2 = \epsilon_3) = 1$.

$$P\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x}) / SigE_2 = SigE_3\right) = \sin^2(2\theta_{23}) \sin^2\left(\frac{(E_3 - E_2)t + (\vec{p}_2 - \vec{p}_3).\vec{x}}{2}\right)$$

with $E_2 = \sqrt{p_2^2 + m_2^2}$ and $E_3 = \sqrt{p_3^2 + m_3^2}$.

$$P\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x}) / SigE_2 \neq SigE_3\right) = \sin^2(2\theta_{23}) \sin^2\left(\frac{(E_3 + E_2)t + (\vec{p}_2 - \vec{p}_3).\vec{x}}{2}\right)$$

Probability calculus with these probabilities yield

$$P_{RQM}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right) = \sin^2(2\theta_{23}) \frac{(E_2 + m_2)(E_3 + m_3)}{4E_2 E_3}$$
$$\left[\left(1 + \frac{p_2 p_3}{(E_2 + m_2)(E_3 + m_3)}\right)^2 \sin^2\left(\frac{(E_3 - E_2)\,t + (\vec{p}_2 - \vec{p}_3)\,\vec{x}}{2}\right)\right.$$
$$\left. + \left(\frac{p_3}{E_3 + m_3} - \frac{p_2}{E_2 + m_2}\right)^2 \sin^2\left(\frac{(E_3 + E_2)\,t + (\vec{p}_2 - \vec{p}_3)\,\vec{x}}{2}\right)\right]$$

which is equal to the exact transition probability $P_{QFT}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right)$.

Conclusion

Supposing that in the Dirac theory a spin-$\frac{1}{2}$, has the probabilities of having positive energy and negative energy and the moments $\vec{p}_2$ and $\vec{p}_3$ of the mass eigenstates are in the same direction, then

$$P_{RQM}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right) = P_{QFT}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right)$$

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