The exact transition Probability for Two Flavor Neutrino Oscillations in Vacuum

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- Neutrino flavors : ν_e, ν_μ, ν_τ :superpositions of the neutrino mass eigenstates.
- Neutrino oscillations : transition from a neutrino flavor state to another neutrino flavor state depending on the distance traveled. (B. Pontecorvo, 1957)
- Two flavor neutrino oscillations exists in nature : for example $\nu_\mu \longrightarrow \nu_\tau$, in atmospheric neutrinos (Fukuda. Y et al. (Super-Kamiokande Collaboration), Phys. Rev. Lett.,1998)

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- We suppose that a neutrino is Dirac neutrino : neutrino is not neutral particle

- Using Nonrelativistic quantum mechanics(V. Gribov and B. Pontecorvo, 1969)

$$P_{NRQM}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right) = \sin^2(2\theta_{23})\sin^2\left(\frac{(E_3 - E_2)t + (\vec{p}_3 - \vec{p}_2)\,\vec{x}}{2}\right)$$

- Using Quantum field theory(M. Blasone et al.1998)
  The exact transition probability is

$$P_{QFT}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right) = \sin^2(2\theta_{23})\frac{(E_2 + m_2)(E_3 + m_3)}{4E_2E_3}$$
$$\left[\left(1 + \frac{p_2p_3}{(E_2 + m_2)(E_3 + m_3)}\right)^2 \sin^2\left(\frac{(E_3 - E_2)\,t + (\vec{p}_3 - \vec{p}_2)\,\vec{x}}{2}\right)\right.$$
$$\left. + \left(\frac{p_3}{E_3 + m_3} - \frac{p_2}{E_2 + m_2}\right)^2 \sin^2\left(\frac{(E_3 + E_2)\,t + (\vec{p}_3 - \vec{p}_2)\,\vec{x}}{2}\right)\right]$$

- Using Relativistic quantum mechanics

$$P_{RQM}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right) = ?$$

Dirac wave function

The Dirac equation

$$i\hbar\gamma^{\mu}\partial_{\mu}\nu(t,\vec{x}) - mc\nu(t,\vec{x}) = 0$$

γ^{μ}'s are the gamma matrices ;
$\hbar$ is the Planck constant ;
c the speed of light ;
m the mass of the mass eigenstate of the spin-$\frac{1}{2}$ fermion.
The wave function $\nu = \nu(t,\vec{x})$ solution of the Dirac equation may be written as Kronecker product or tensor product

$$\nu(t,\vec{x}) = \xi \otimes s e^{-\frac{i}{\hbar}(\pm Et - \vec{p}.\vec{x})} \tag{1}$$

ξ is an eigeinstate of the hamiltonian operator $h_D = \epsilon cp\sigma^1 + mc^2\sigma^3$ (Raoelina Andriambololona et al.2011), with ϵ is the sign of the helicity.
s is an eigeinstate of the helicity operator $\frac{\hbar}{2}\vec{\sigma}.\vec{n} = \frac{\hbar}{2}n_1\sigma_1 + \frac{\hbar}{2}n_2\sigma_2 + \frac{\hbar}{2}n_3\sigma_3$, with $\vec{n} = \frac{\vec{p}}{||\vec{p}||} = \frac{\vec{p}}{p}$.

Dirac wave function

For the first factor ξ

$\xi = |\xi(E,p)\rangle = \sqrt{\frac{E+mc^2}{2E}} \begin{pmatrix} 1 \\ \frac{\epsilon c p}{E+mc^2} \end{pmatrix}$ is the eigenvector associated to the positive energy $E = +\sqrt{c^2 p^2 + m^2 c^4}$

$\xi = |\bar{\xi}(E,p)\rangle = \sqrt{\frac{E+mc^2}{2E}} \begin{pmatrix} -\frac{\epsilon c p}{E+mc^2} \\ 1 \end{pmatrix}$, is the eigenvector associated to the negative energy $-E = -\sqrt{c^2 p^2 + m^2 c^4}$

of the hamiltonian operator $h_D = \epsilon c p \sigma^1 + mc^2 \sigma^3$.

For the second factor s

In spherical coordinates (ρ, θ, φ), $\vec{n} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$,

$s = |s(\vec{p})\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix}$ is the eigenvector of $\frac{\hbar}{2}\vec{\sigma}.\vec{n}$ associated to the helicity $+\frac{1}{2}\hbar = \frac{1}{2}\epsilon\hbar$, right handed particle

$s = |\bar{s}(\vec{p})\rangle = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{pmatrix}$ is the eigenvector of $\frac{\hbar}{2}\vec{\sigma}.\vec{n}$ associated to the helicity $-\frac{1}{2}\hbar = \frac{1}{2}\epsilon\hbar$, left handed neutrino.

So, we have four Dirac particle wave functions

$$\nu_{pp}(t,\vec{x}) = |\xi(E,p)\rangle \otimes |s(\vec{p})\rangle\, e^{-\frac{i}{\hbar}(Et-\vec{p}.\vec{x})} \qquad \nu_{pn}(t,\vec{x}) = |\xi(E,p)\rangle \otimes |\bar{s}(\vec{p})\rangle\, e^{-\frac{i}{\hbar}(Et-\vec{p}}$$

$$\bar{\nu}_{np}(t,\vec{x}) = \left|\bar{\xi}(E,p)\right\rangle \otimes |s(\vec{p})\rangle\, e^{\frac{i}{\hbar}(Et+\vec{p}.\vec{x})} \qquad \bar{\nu}_{nn}(t,\vec{x}) = \left|\bar{\xi}(E,p)\right\rangle \otimes |\bar{s}(\vec{p})\rangle\, e^{\frac{i}{\hbar}(Et+\vec{p}.\vec{x}}$$

respectively for particle with positive energy/positive helicity, positive energy/negative helicity, negative energy/positive helicity and negative energy/negative helicity.

Energy sign operator and helicity operator

$h_D = \epsilon c p \sigma^1 + mc^2\sigma^3$ hamiltonian operator

let $\vec{E} = \begin{pmatrix} \epsilon c p \\ 0 \\ mc^2 \end{pmatrix}$ energy vector

$E = \left\| \vec{E} \right\| = \sqrt{m^2c^4 + c^2p^2}$ the energy

$\frac{\hbar}{2E} h_D = \frac{\hbar}{2E}\epsilon c p \sigma^1 + \frac{\hbar}{2E} mc^2\sigma^3$

spin operator in the direction of $\vec{E}$

$\frac{h_D}{E} = \frac{\epsilon c p}{E}\sigma^1 + \frac{mc^2}{E}\sigma^3$ Energy sign operator

Probability for having positive or negative energy

$\vec{\sigma}.\vec{p} = p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3$

with $\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ momentum vector

$p = \|\vec{p}\| = \sqrt{p_1^2 + p_2^2 + p_3^2}$

$\frac{\hbar}{2p}\vec{\sigma}.\vec{p} = \frac{\hbar}{2p}p_1\sigma_1 + \frac{\hbar}{2p}p_2\sigma_2 + \frac{\hbar}{2p}p_3\sigma_3$ helicity operator or

spin operator in the direction of $\vec{p}$

$\frac{\vec{\sigma}.\vec{p}}{p} = \frac{p_1}{p}\sigma_1 + \frac{p_2}{p}\sigma_2 + \frac{p_3}{p}\sigma_3$ helicity sign operator

Probability for having positive or negative helicity

These probabilities should have impact on the transition probability.

Transition probability

From now on let us use the natural unit $c = 1$ and $\hbar = 1$.

ξ eigenstate of the operator sign of energy $\frac{1}{E} h_D = \frac{\epsilon c p}{E}\sigma_1 + \frac{mc^2}{E}\sigma_3$

The two mass eigenstates with positive energies are

$$\xi_2(0,\vec{0}) = \left|\xi_2(0,\vec{0})\right\rangle = \sqrt{\frac{E_2 + m_2}{2E_2}} \begin{pmatrix} 1 \\ \frac{\epsilon_2 p_2}{E_2 + m_2} \end{pmatrix}$$

$$\xi_3(0,\vec{0}) = \left|\nu_3(0,\vec{0})\right\rangle = \sqrt{\frac{E_3 + m_3}{2E_3}} \begin{pmatrix} 1 \\ \frac{\epsilon_3 p_3}{E_3 + m_3} \end{pmatrix}$$

The two mass eigenstates with negative energies are

$$\bar{\xi}_2(0,\vec{0}) = \left|\bar{\xi}_2(0,\vec{0})\right\rangle = \sqrt{\frac{E_2 + m_2}{2E_2}} \begin{pmatrix} -\frac{\epsilon_2 p_2}{E_2 + m_2} \\ 1 \end{pmatrix}$$

$$\bar{\xi}_3(0,\vec{0}) = \left|\bar{\xi}_3(0,\vec{0})\right\rangle = \sqrt{\frac{E_3 + m_3}{2E_3}} \begin{pmatrix} -\frac{\epsilon_3 p_3}{E_3 + m_3} \\ 1 \end{pmatrix}$$

at time $t = 0$.

The probabilities for having the same sign energies and different sign of energies at $t = 0$ are respectively

$$P\left\{SigE_2 = SigE_3, t = 0\right\} = \frac{(E_2 + m_2)(E_3 + m_3)}{4E_2E_3}\left(1 + \frac{\epsilon_2\epsilon_3 p_2 p_3}{(E_2 + m_2)(E_3 + m_3)}\right)^2$$

and

$$P\left\{SigE_2 \neq SigE_3, t = 0\right\} = \frac{(E_2 + m_2)(E_3 + m_3)}{4E_2E_3}\left(\frac{\epsilon_3 p_3}{E_3 + m_3} - \frac{\epsilon_2 p_2}{E_2 + m_2}\right)^2$$

with ϵ_2 and ϵ_3 are the helicity signs of the mass eigenstates at $(t, \vec{x}) = (0, \vec{0})$.

s eigenstate of helicity sign operator $\frac{\vec{\sigma}.\vec{p}}{2p} = \frac{p_1}{p}\sigma_1 + \frac{p_2}{p}\sigma_2 + \frac{p_3}{p}\sigma_3$

In the helicity states plane,

$$|s_2(\vec{p}_2)\rangle = \begin{pmatrix} \cos\frac{\theta_2}{2} \\ \sin\frac{\theta_2}{2} e^{i\varphi_2} \end{pmatrix} \quad \text{and} \quad |s_3(\vec{p}_3)\rangle = \begin{pmatrix} \cos\frac{\theta_3}{2} \\ \sin\frac{\theta_3}{2} e^{i\varphi_3} \end{pmatrix}$$

the mass eigenstates for positive helicity and

$$|\bar{s}_2(\vec{p}_2)\rangle = \begin{pmatrix} -\sin\frac{\theta_2}{2} e^{-i\varphi_2} \\ \cos\frac{\theta_2}{2} \end{pmatrix} \quad \text{and} \quad |\bar{s}_3(\vec{p}_3)\rangle = \begin{pmatrix} -\sin\frac{\theta_3}{2} e^{-i\varphi_3} \\ \cos\frac{\theta_3}{2} \end{pmatrix}$$

the mass eigenstates for negative helicity.

$$P(\epsilon_2 = \epsilon_3) = \cos^2\left(\frac{\theta_2 + \theta_3}{2}\right) + \sin\theta_2 \sin\theta_3 \cos^2\left(\frac{\varphi_3 - \varphi_2}{2}\right)$$

and

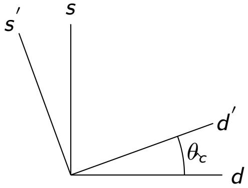
$$P(\epsilon_2 \neq \epsilon_3) = \sin^2\left(\frac{\theta_2 + \theta_3}{2}\right) - \sin\theta_2 \sin\theta_3 \cos^2\left(\frac{\varphi_3 - \varphi_2}{2}\right)$$

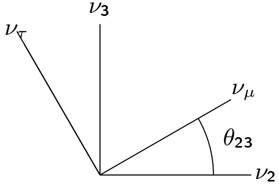
are respectively the probabilities for having respectively same sign of helicities and different signs of helicities at time $t = 0$ for the mass eigenstates.
We suppose that $\vec{p}_2 = \vec{p}_3$, thus $P(\epsilon_2 = \epsilon_3) = 1$.

Transition probability

The Dirac wave function $\nu(t,\vec{x}) = \xi \otimes s e^{-\frac{i}{\hbar}(\pm Et - \vec{p}.\vec{x})}$

In the Cabbibo plan





$$|\nu_\mu(t,\vec{x})\rangle = \cos\theta_{23}\,|\nu_2(t,\vec{x})\rangle + \sin\theta_{23}\,|\nu_3(t,\vec{x})\rangle$$
$$|\nu_\tau(,\vec{x})\rangle = -\sin\theta_{23}\,|\nu_2(t,\vec{x})\rangle + \cos\theta_{23}\,|\nu_3(t,\vec{x})\rangle$$

where $|\nu_2(t,\vec{x})\rangle = \left|\nu_2(0,\vec{0})\right\rangle e^{-i(E_2 t - \vec{p}_2.\vec{x})}$ and $|\nu_3(t,\vec{x})\rangle = \left|\nu_3(0,\vec{0})\right\rangle e^{-i(E_3 t - \vec{p}_3.\vec{x})}$.

$$P\left(\nu_\mu \longrightarrow \nu_\tau, (t,\vec{x})/SigE_2 = SigE_3\right) = \sin^2(2\theta_{23})\sin^2\left(\frac{(E_3 - E_2)t + (\vec{p}_2 - \vec{p}_3).\vec{x}}{2}\right)$$

(Pontecorvo et al.) with $E_2 = \sqrt{p_2^2 + m_2^2}$ and $E_3 = \sqrt{p_3^2 + m_3^2}$.

By using the same method

$$P\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x}) / SigE_2 \neq SigE_3\right) = \sin^2(2\theta_{23}) \sin^2\left(\frac{(E_3 + E_2)t + (\vec{p}_2 - \vec{p}_3).\vec{x}}{2}\right)$$

Probability calculus with these probabilities

$$P_{RQM}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right) = \sin^2(2\theta_{23}) \frac{(E_2 + m_2)(E_3 + m_3)}{4 E_2 E_3}$$

$$\left[\left(1 + \frac{p_2 p_3}{(E_2 + m_2)(E_3 + m_3)} \right)^2 \sin^2 \left(\frac{(E_3 - E_2)\, t + (\vec{p}_2 - \vec{p}_3)\, \vec{x}}{2} \right) \right.$$

$$\left. + \left(\frac{p_3}{E_3 + m_3} - \frac{p_2}{E_2 + m_2} \right)^2 \sin^2 \left(\frac{(E_3 + E_2)\, t + (\vec{p}_2 - \vec{p}_3)\, \vec{x}}{2} \right) \right]$$

which is equal to the exact transition probability $P_{QFT}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right)$

We have proved that for a Dirac particle there are probabilities for having positive energy and negative energy, that is, a Dirac particle is in a state, superposition of positive energy eigenstate and negative energy eigenstate. Then if at $t = 0$, $\vec{p}_2 = \vec{p}_3$

$$P_{RQM}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right) = P_{QFT}\left(\nu_\mu \longrightarrow \nu_\tau, (t, \vec{x})\right)$$

THANK YOU !