Quantum resonant systems, integrable & chaotic

arXiv: 1808.09173

Oleg Evnin

Chulalongkorn University (Bangkok)
Solvay Institutes (Brussels)

@ HEPMAD'18, Antananarivo, Madagascar
September 10, 2018

International Solvay Institutes
Brussels

VRIJE UNIVERSITEIT BRUSSEL

Resonant systems

$$\hat{H} = \frac{1}{2} \sum_{\substack{n,m,k,l=0,\\ n+m=k+l}}^{\infty} C_{nmkl} \hat{\alpha}_n^\dagger \hat{\alpha}_m^\dagger \hat{\alpha}_k \hat{\alpha}_l, \qquad [\hat{\alpha}_n, \hat{\alpha}_m^\dagger] = \delta_{nm}$$

- Identical to 2-body interaction Hamiltonian in many-body systems (condensed matter, nuclear, high-energy physics)

- But: the resonance condition n+m=k+l is imposed on the sum (this is what makes the rest of the story possible)

- Such systems emerge in weakly nonlinear analysis of systems with highly resonant frequencies of linearized perturbations (precise statement later!)

- Solvable in terms of diagonalizing finite-sized numerical matrices

- The simplest interacting quantum field theories known to man!

1d Bose-Einstein condensate

$$i\,\frac{\partial\Psi}{\partial t} = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} + x^2\right)\Psi + g|\Psi|^2\Psi$$

(Gross-Pitaevskii nonrelativistic field)

$$\Psi = \sum_{n=0}^{\infty} \alpha_n \psi_n(x) e^{-iE_n t}, \qquad E_n = n + \frac{1}{2}, \qquad \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} + x^2\right)\psi_n = E_n \psi_n$$

$$i\,\frac{d\alpha_n}{dt} = g \sum_{k,l,m=0}^{\infty} C_{nmkl}\,\bar{\alpha}_m \alpha_k \alpha_l\, e^{i(E_n+E_m-E_k-E_l)t}$$

$$C_{nmkl} = \int dx\, \psi_n \psi_m \psi_k \psi_l$$

Resonant approximation: discard all oscillatory terms!

(accurate on time scales 1/g)

Rename "slow time" gt to new time t.

Only terms with n+m=k+l are left, giving a resonant system.

A few systems of interest

Many resonant systems of our basic form (and greatly varying complexity) emerge in the physics of Bose-Einstein condensates in harmonic traps,

and in Anti-de Sitter spacetime.

Some relevant particular cases:

$C^{(Sz)}_{nmkl} = 1$ designed for turbulence studies, Lax-integrable, many exact results

$$C^{(MRS)}_{nmkl} = \frac{1}{1+(n+m+k+l)/2} \qquad C^{(LLL)}_{nmkl} = \frac{((n+m+k+l)/2)!}{2^{n+m}\sqrt{n!m!k!l!}}$$

$$C^{(CF)}_{nmkl} = \frac{1+\min(n,m,k,l)}{\sqrt{(1+n)(1+m)(1+k)(1+l)}}$$

special analytic solutions are known
full integrability unknown

Random C??? (chaotic behavior should be expected)

Block-diagonal structure

Two conserved quantities: $\hat{N} = \sum_{k=0}^{\infty} \hat{\alpha}_k^\dagger \hat{\alpha}_k, \qquad \hat{M} = \sum_{k=1}^{\infty} k\, \hat{\alpha}_k^\dagger \hat{\alpha}_k$

The Hamiltonian is block diagonal in the Fock basis $|n_0, n_1, \ldots\rangle$

Transitions only between states with equal $N = n_0 + \sum_{k=1}^{\infty} n_k \qquad M = \sum_{k=1}^{\infty} k\, n_k$

All blocks of finite sizes!

The size is given in as $p_N(M)$, the number of partitions of M into at most N parts, a standard and much-studied number-theoretical function.

One simply has to diagonalize $p_N(M)$-by-$p_N(M)$ matrices of real numbers.
(All the complexity of classical dynamics must emerge from this “trivial” structure)

Diagonalization

Two-particle case (N=2) can be solved explicitly for our sample systems, and is simple in general. “Random” resonant systems are identical to Wigner’s random matrices in this sector.

For higher N, apply numerical diagonalization after computing the matrix elements of the resonant system Hamiltonian between Fock basis states

$$\frac{1}{\sqrt{n_0!\,n_1!\cdots n_M!}}\left(\hat{\alpha}_0^\dagger\right)^{n_0}\left(\hat{\alpha}_1^\dagger\right)^{n_1}\cdots\left(\hat{\alpha}_M^\dagger\right)^{n_M}|0\rangle$$

Many patterns for the integrable/partially integrable cases:
explicit formulas for the maximal energy eigenvalue in each block,
simple-looking eigenvalue distributions at large N and M.
For the integrable case C=1, all eigenvalues are **integer** – it’s the “harmonic oscillator” of interacting quantum field theories!

Quantum chaos

Distributions of ("unfolded") level spacings of quantum systems is a good indicator of the integrable/chaotic properties of their classical limit.

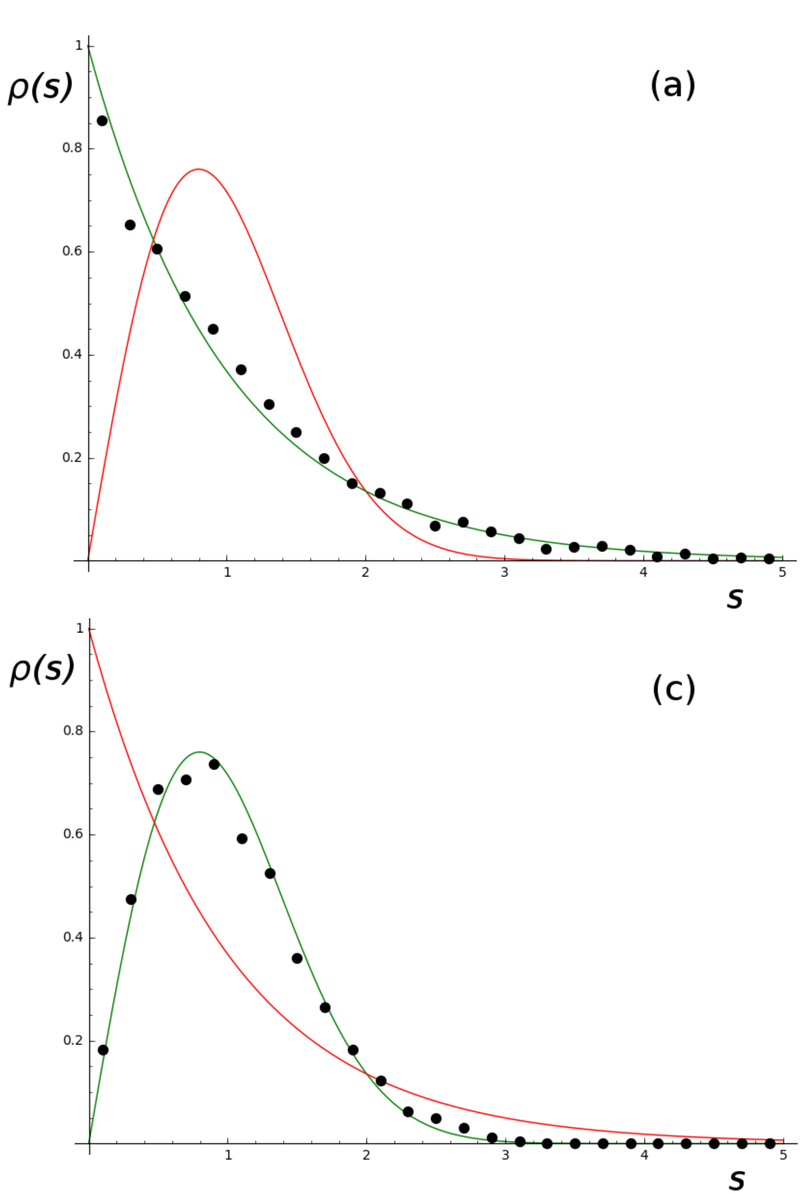
Berry-Tabor conjecture: for a generic integrable system, the distribution is the same as for points randomly thrown on a line:

$$\rho_{Poisson}(s) = e^{-s}$$

Bohigas-Giannoni-Schmit (BGS) conjecture: for chaotic systems, the distribution is the same as for eigenvalues of a random real symmetric matrix with independendent identically distributed entries:

$$\rho_{Wigner}(s) = \frac{\pi s}{2}\, e^{-\pi s^2/4}$$

In accord with such expectations, resonant systems with special analytic structures immediately stand out in eigenvalue spacing analysis.


ρ(s)
(a)
ρ(s)
(b)
ρ(s)
(c)
ρ(s)
(d)
s
s
s
s

Conclusions / Adverts

- A class of exceptionally simple interacting quantum field theories with very rich dynamics

- A wealth of structures begging for analytic validation for special cases – including a perfectly integer spectrum for the integrable Szegő case (C=1), which makes it the “harmonic oscillator” of interacting quantum field theories!

- Very neat level spacing statistics, which in accordance with the standard quantum chaos lore, suggest classical integrability of some physically relevant resonant systems

- Potential applications to Bose-Einstein condensates (=cold atomic gases)

- A promising arena for further exploration of quantum chaos (quantum thermalization, etc)

- A lot remains to be explored!