Mass dependence of the heavy quark potential
and its effects on quarkonium states

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XIV International Conference on Hadron Spectroscopy
June 14, 2011
Heavy quark-antiquark potential

- **History:** phenomenological potential models
  - Fitted to low lying charmonium and bottomonium states
  - Typical shape: “Coulomb-plus-linear”

- **Today:** heavy quark-antiquark potential from QCD
  - Characteristic scales of non-relativistic bound states
    - $m$: heavy quark mass
    - $mv$: heavy quark momentum
    - $mv^2$: heavy quark energy
  - Effective field theory (EFT) methods
    - QCD $\Rightarrow$ non-relativistic QCD (NRQCD, pNRQCD, vNRQCD)

**Topics:** Extended range of validity of perturbative potential
- Spectroscopy at order $1/m$
- Detailed analysis of the role of quark masses
1. Static quark-antiquark potential

2. Heavy quark potential at order $1/m$
The static potential

Non-perturbative sector: lattice studies of quenched and full QCD

- Static QCD potential (from static Wilson loop)


- Sea quark effects important at small distances

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Perturbative sector: static potential is known at three-loop order


Momentum space

\[ \tilde{V}^{(0)}(|\vec{q}|) = -\frac{4\pi C_F}{\vec{q}^2} \alpha_s(|\vec{q}|) \left[ 1 + \frac{\alpha_s(|\vec{q}|)}{4\pi} a_1 + \left( \frac{\alpha_s(|\vec{q}|)}{4\pi} \right)^2 a_2 \right. \]

\[ \left. + \left( \frac{\alpha_s(|\vec{q}|)}{4\pi} \right)^3 \left( a_3 + 8\pi^2 C_A^3 \ln \frac{\mu_{IR}^2}{\vec{q}^2} \right) + \ldots \right] \]

where \( C_F = 4/3, \ C_A = 3, \)
\( a_1 = 7, \ a_2 \approx 268.8, \ a_3 \approx 5199.8 \) \((n_f = 3)\)

- At N\(^3\)LO (three-loop order):
  - infrared divergences \( \mu_{IR}^2 \) from ultrasoft gluons

- Avoid expansion of \( \alpha_s(|\vec{q}|) \) about a fixed scale \( \mu \)

- Reliable potential from extremely small distances up to \( r \approx 0.15 \) fm needed
Potential subtracted (PS) scheme

PS scheme with numerical Fourier transform

Evaluate numerically (with a low-momentum cutoff $\mu_f$)

$$V^{(0)}(\vec{r}, \mu_f) = -4\pi C_F \int_{|\vec{q}| > \mu_f} \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{\alpha_s(|\vec{q}|)}{\vec{q}^2} \left[ 1 + \frac{\alpha_s(|\vec{q}|)}{4\pi} a_1 + \left(\frac{\alpha_s(|\vec{q}|)}{4\pi}\right)^2 a_2 + \ldots \right]$$

- No free scale parameter $\mu$
- Unknown constant is moved into the definition of $m_{PS}$:

$$2m_{pole} + V^{(0)}(r) = 2m_{PS}(\mu_f) + V^{(0)}(r, \mu_f)$$
Matching and uncertainty estimate

- Perturbative potential (here NNLO) and lattice potential matched

\[ V(0)(r) - V(0)(0.5 \text{ fm}) \text{ [GeV]} \]

\[ r \text{ [fm]} \]

- Differentiable quark-antiquark potential for distances up to \( \sim 1 \) fm
- Matching at 0.14 fm gives \( \mu_f = 0.9^{+0.3}_{-0.2} \) GeV
  
  (for charmonium and bottomonium)

- Grey band: uncertainty of lattice calculation and uncertainty of \( \alpha_s \)
- Dot-dashed curve: continuation of the “Coulomb-plus-linear” fit
Bottomonium spectrum

Solve the Schrödinger equation with this matched potential

![Graph showing the spectrum of bottomonium states with masses in GeV, single parameter $m_{PS}(0.908 \text{ GeV}) = 4.78 \text{ GeV}$, and PDG 2010 masses for bottom and charm quarks.]

- Single parameter $m_{PS}(0.908 \text{ GeV}) = 4.78 \text{ GeV}$
- Can be converted to the $\overline{\text{MS}}$ scheme

<table>
<thead>
<tr>
<th>$\overline{\text{MS}}$ masses [GeV]</th>
<th>$m_{\overline{\text{MS}}}$</th>
<th>PDG 2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>bottom quark</td>
<td>4.20±0.04</td>
<td>4.19$^{+0.18}_{-0.06}$</td>
</tr>
<tr>
<td>charm quark</td>
<td>1.23±0.04</td>
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1. Static quark-antiquark potential

2. Heavy quark potential at order $1/m$
Quark-antiquark potential at order $1/m$

- Expansion in inverse powers of the heavy quark mass $m$

$$V(r) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m/2} + \frac{V^{(2)}(r)}{(m/2)^2} + \ldots$$

- Non-perturbative expression for $1/m$ potential is known
  

- Lattice simulations

  Efficient method from M. & Y. Koma and H. Wittig
  
  Quenched simulation, renormalization issues ($\approx 15\%$ error estimated)

  Contains a non-perturbative contribution

  Fit function

  $$V_{ln}^{(1)}(r) = -\frac{A_2}{r^2} + B_2 \ln r + C_2$$

  Effective string theory suggests
  
  logarithmic shape: $V^{(1)} \propto \ln r + C$

Perturbative potential at order $1/m$ \((C_F = \frac{4}{3}, C_A = 3)\)

\[
\tilde{V}^{(1)}(|\vec{q}|) = \frac{C_F \pi^2 \alpha_s^2(|\vec{q}|)}{2|\vec{q}|} \left[ (-C_A) + \mathcal{O}(\alpha_s) \right]
\]

Restricted numerical Fourier transform

Differentiable quark-antiquark potential for distances up to $\sim 1$ fm

Matching at 0.14 fm gives $\mu'_f = 1.6^{+0.5}_{-0.8}$ GeV (for charmonium)

$\mu'_f = 1.9^{+0.4}_{-0.6}$ GeV (for bottomonium)

Grey band: uncertainty of lattice calculation and uncertainty of $\alpha_s$
PS mass needs redefinition \( m_{\overline{\text{PS}}} (\mu_f) \rightarrow m_{\overline{\text{PS}}} (\mu_f, \mu'_f) \)

\[ m_{\overline{\text{PS}}} (\mu_f, \mu'_f) \equiv m_{\overline{\text{PS}}} (\mu_f) - \frac{1}{8m} C_F C_A \alpha_s^2 \mu_f'^2 \]

- Quark masses from comparison with empirical quarkonium states

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<tr>
<th>( \overline{\text{MS}} ) masses [GeV]</th>
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<th>static + 1/m</th>
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- Error estimates include:
  - uncertainties in the potentials (static and order 1/m)
  - uncertainties from matching to experimental spectra
Bottomonium spectrum

- Tightly bound $\eta_b(1S)$ and $\Upsilon(1S)$ states are most sensitive to $1/m$-effects.

- Hyperfine effects (h.f.) added phenomenologically (one-gluon exchange) with $\alpha_s^{\text{eff}} = 0.3$.

  ... (work in progress) to be substituted by the full $1/m^2$ potential.

- String tension $\sigma = 1.01$ GeV/fm.

- Different strategies needed above $B\bar{B}$ threshold.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{spectrum_graph.png}
\caption{Mass dependence of the heavy quark potential.}
\end{figure}
### Spectroscopy

#### Charmonium spectrum

- **Downward shift from** $V^{(1)}$ **in the** 1S states ($\eta_c$ and $J/\psi$) **to large** $1/m^2$ **effects significant**

- Hyperfine effects (h.f.) added phenomenologically (one-gluon exchange) with $\alpha_s^{\text{eff}} = 0.3$

  ... (work in progress) to be substituted by the full $1/m^2$ potential

- String tension $\sigma = 1.01$ GeV/fm

- Different strategies needed above $D\bar{D}$ threshold

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<th>$\psi(2S)$</th>
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<tr>
<td></td>
<td>$2S$</td>
<td>$\psi(1S)$</td>
</tr>
<tr>
<td></td>
<td>$3S_1$</td>
<td>$\eta_c(1S)$</td>
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<th>+h.f.</th>
<th>experiment</th>
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<tr>
<td>$1S_0$</td>
<td>3.1</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
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<tr>
<td>$1S_1$</td>
<td>3.3</td>
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**Graphs:**

- Mass dependence of the heavy quark potential

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**Author:** Alexander Laschka

**Presentation:** Mass dependence of the heavy quark potential 

**Conference:** Hadron 2011
**Summary**

- Heavy quark-antiquark potential from QCD (perturbative QCD ↔ lattice QCD)
- Excellent matching in $r$-space up to order $1/m$
- Spectroscopy at order $1/m$
  - Works well for bottomonium
  - Less successful for charmonium ($1/m^2$ effects sizeable: work in progress)
- Quark masses can be extracted

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Thank you for your attention!