Heavy-Quark Masses and Heavy-Meson Decay Constants from Borel Sum Rules in QCD

Dmitri Melikhov[1,a,b,c], Wolfgang Lucha[a], and Silvano Simula[d]
[a]HEPHY, Austrian Academy of Sciences, Nikolsdorfergasse 18, A-1050 Vienna, Austria
[b]Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria
[c]SINP, Moscow State University, 119991 Moscow, Russia
[d]INFN, Sezione di Roma Tre, Via della Vasca Navale 84, I-00146 Roma, Italy

Slight sophistications of the QCD sum-rule formalism may have great impact on the reliability of predicted hadron observables, as exemplified for the case of heavy-meson decay constants.

Quark–Hadron Duality. The extraction of the decay constant f_P of any ground-state heavy pseudoscalar meson P from QCD sum rules [1–3] is a two-phase process: First, the operator product expansion (OPE) for the Borel-transformed correlation function of the two relevant pseudoscalar heavy-light currents has to be derived. Second, even if all parameters of this OPE were known exactly, the knowledge of merely *truncated* OPEs for correlators allows to extract bound-state features with only a limited accuracy, reflecting an intrinsic uncertainty of the QCD sum-rule formalism. Controlling this uncertainty poses a delicate challenge [4]. We consider mesons $P \equiv (Q\,\bar{q})$ of mass M_P composed of heavy quarks Q and light quarks $\bar{q}$. The *assumption* of *quark–hadron duality* entails a relation between the hadronic ground-state contribution and the QCD correlator truncated at a certain *effective continuum threshold* $s_{\rm eff}$:

$$(1)\qquad f_P^2\,M_P^4\exp(-M_P^2\,\tau)=\Pi_{\rm dual}(\tau,s_{\rm eff})\equiv\int_{(m_Q+m_q)^2}^{s_{\rm eff}}{\rm d}s\exp(-s\,\tau)\,\rho_{\rm pert}(s)+\Pi_{\rm power}(\tau)\ .$$

Obviously, in order to be able to extract f_P one has to develop a procedure determining $s_{\rm eff}$. Borel transformations introduce a mass parameter $\widetilde{M}$, included here in the form $\tau\equiv 1/\widetilde{M}^2$. A crucial, albeit rather trivial, observation is that $s_{\rm eff}$ must be a function of τ. Otherwise, the two members of (1) exhibit different τ-behaviour. The *exact* effective continuum threshold, which would reproduce the *true* values of hadron mass and decay constant on the left-hand side of (1), is, clearly, not known. Therefore, our ideas of *extracting* hadron parameters from sum rules consist in attempting to obtain a reliable approximation to the exact threshold $s_{\rm eff}$ and to control the accuracy of this approximation. In a recent series of publications [5], we have constructed all procedures, techniques, and algorithms required to achieve this goal: With our concept of $s_{\rm eff}(\tau)$, we define dual mass $M_{\rm dual}$ and dual decay constant $f_{\rm dual}$ of P by

$$M_{\rm dual}^2(\tau)\equiv-\frac{\rm d}{{\rm d}\tau}\log\Pi_{\rm dual}(\tau,s_{\rm eff}(\tau))\ ,\qquad f_{\rm dual}^2(\tau)\equiv M_P^{-4}\exp(M_P^2\,\tau)\,\Pi_{\rm dual}(\tau,s_{\rm eff}(\tau))\ .$$

[1]dmitri_melikhov@gmx.de

If the ground-state mass' actual value M_P is known, the deviation of our dual ground-state mass $M_{\rm dual}$ from this M_P indicates the amount of excited-state contributions picked up by our dual correlator $\Pi_{\rm dual}(\tau, s_{\rm eff}(\tau))$. Assuming specific Ansätze for our function $s_{\rm eff}(\tau)$ and requiring least deviation of our $M_{\rm dual}$ from the true M_P in the range of admissible values of the Borel parameter τ leads to a variational solution for the effective threshold. With $s_{\rm eff}(\tau)$ at hand, we find the P-meson's decay constant from the second of the above dual relations. The traditional *assumption* for the effective threshold is that it is a (τ-independent) constant. In addition to this very crude approximation, we consider for $s_{\rm eff}(\tau)$ also polynomials in τ. It is easy to imagine that a τ-dependent threshold greatly facilitates reproducing the true mass value M_P. This implies that a dual correlator with τ-dependent threshold isolates the ground state to much higher extent and is less plagued by excited-state contamination than a dual correlator with the conventional, but naïve, τ-independent threshold. Consequently, the accuracies of extracted hadron observables are drastically improved. Recent experience from various quantum-mechanical test grounds reveals that the band of results computed from linear, quadratic, and cubic Ansätze for $s_{\rm eff}(\tau)$ encompasses the exact f_P value [5] and that the extraction procedures in quantum mechanics and in QCD are (even quantitatively) very similar [6]. For all the details of our improved sum-rule approach, consult Refs. [2–7].

OPE and Heavy-Quark Mass Scheme. A close inspection shows that for both heavy-light correlators and resulting decay constants the choice of the precise mass scheme adopted for defining the heavy-quark mass is crucial. The OPE for the correlator (1) to three-loop order was derived in terms of the heavy-quark *pole mass* in [8]. An alternative is to reorganize the perturbative expansions in terms of the heavy-quark *running* $\overline{MS}$ *mass* [9]. Figure 1 presents the B-meson decay constant f_B resulting from both choices. In each case, a *constant* effective threshold [differing, of course, for pole (s_0) and $\overline{\rm MS}$ ($\overline{s_0}$) mass scheme] is fixed by requiring maximum stability of the f_B value obtained. From this exercise we gain important insights:

(a) In the pole-mass scheme, the perturbative series for the decay constant shows no sign of convergence. The separate contributions of LO, NLO, and NNLO terms are of similar size. Accordingly, the pole-mass-scheme result for f_B significantly underestimates its true value.

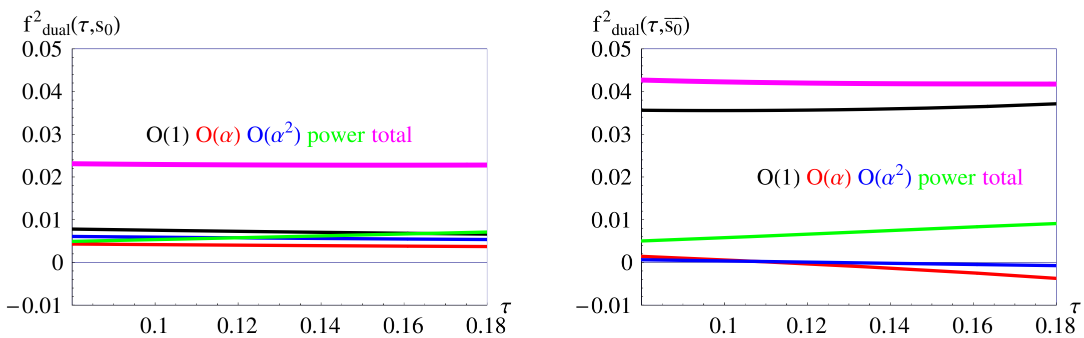

Figure 1: Dual decay constants $f_{\rm dual}$ of the B meson extracted, for constant thresholds $\overset{(-)}{s_0}$, from the correlator (1) expressed in terms of the b-quark's pole (left) and $\overline{\rm MS}$ (right) mass.

(b) Reorganizing the perturbative series in terms of the heavy-quark $\overline{\rm MS}$ mass generates an impressively distinct *hierarchy of the perturbative contributions*. Our dual decay constant $f_{\rm dual}$ obtained using the $\overline{\rm MS}$ scheme proves to be some 40% larger than in the pole-mass scheme.

(c) Interestingly, in *both* mass schemes the decay constant exhibits perfect stability in a wide range of the Borel parameter τ. This clearly tells us that mere Borel stability is not sufficient to ensure the reliability of a sum-rule extraction of bound-state features. Repeatedly [4], we tried to draw the attention of sum-rule practitioners to this observation; nevertheless, some authors seem to be content with Borel stability as a proof of the trustability of their findings.

In view of the above, we adopt the OPE expressed in terms of running $\overline{\rm MS}$ quark masses [9].

Decay Constants of D and D_s [2,3]. Straightforward application of our sum-rule algorithm yields, as our predictions for the decay constants of the charmed pseudoscalar mesons $D_{(s)}$,

$$
\begin{aligned}
f_D &= (206.2 \pm 7.3_{\rm (OPE)} \pm 5.1_{\rm (syst)})\ \text{MeV}\,,\\
f_{D_s} &= (245.3 \pm 15.7_{\rm (OPE)} \pm 4.5_{\rm (syst)})\ \text{MeV}\,.
\end{aligned}
$$

Herein, the OPE-related errors are computed by bootstrap studies allowing the parameters induced by QCD (i.e., quark masses, $\alpha_{\rm s}$, and condensates) to vary in their respective ranges. We observe perfect agreement of our results with the corresponding lattice QCD outcomes. Let us emphasize that the τ-dependent effective threshold constitutes *the* crucial ingredient for a successful prediction of decay constants of charmed heavy mesons by the sum rule (1). Standard τ-independent approximations entail a much lower value for the D-meson decay constant, f_D, that resides rather far from both the experimental data *and* the lattice findings.

Decay Constants of B and B_s [2]. Our QCD sum-rule results for the decay constants $f_{B_{(s)}}$ of the pseudoscalar beauty mesons $B_{(s)}$ turn out to be extremely sensitive to the input value of the b-quark mass; for instance, the b-quark's $\overline{\rm MS}$-mass range $\overline{m}_b(\overline{m}_b) = (4.163 \pm 0.016)$ GeV [10] gives results that are barely compatible with recent lattice computations of these decay constants. However, inverting the logic by requiring our sum-rule result for f_B to match the average of these lattice calculations provides the very precise value of the b-quark $\overline{\rm MS}$ mass

$$
\overline{m}_b(\overline{m}_b) = (4.245 \pm 0.025)\ \text{GeV}\,.
$$

The corresponding estimates for f_B and f_{B_s} emerging within our sum-rule prescriptions are

$$
\begin{aligned}
f_B &= (193.4 \pm 12.3_{\rm (OPE)} \pm 4.3_{\rm (syst)})\ \text{MeV}\,,\\
f_{B_s} &= (232.5 \pm 18.6_{\rm (OPE)} \pm 2.4_{\rm (syst)})\ \text{MeV}\,.
\end{aligned}
$$

Summary and Conclusions.

1. The τ-dependence of effective thresholds emerges naturally when one attempts to render the duality relation exact. Let us emphasize two facts: (a) In principle, this τ-dependence is *not* in conflict with the properties of quantum field theories. (b) Our analysis of $D_{(s)}$ mesons indicates that it will indeed raise the quality of the resulting sum-rule predictions *decisively*.

2. Our study of *charmed mesons* clearly demonstrates that using Borel-parameter-dependent thresholds leads to lots of essential improvements: (i) The accuracy of sum-rule predictions for decay constants is significantly increased. (ii) It has become possible to extract a realistic systematic error and to diminish it to the level of a few percent. (iii) Our prescription brings the QCD sum-rule approach into perfect agreement with both lattice QCD and experiment.

3. The *beauty-meson* decay constants $f_{B_{(s)}}$ are extremely sensitive to the choice of the b-quark mass: Regarding this as a kind of serendipity and matching our QCD sum-rule outcome for f_B to the corresponding average of lattice evaluations enables us to arrive at a rather precise estimate of $\overline{m}_b(\overline{m}_b)$ in good agreement with several lattice results but which, unfortunately, has no overlap with a recent, rather accurate determination [10]; for details, consult Ref. [2].

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