String Scattering Amplitudes in High Energy Limits

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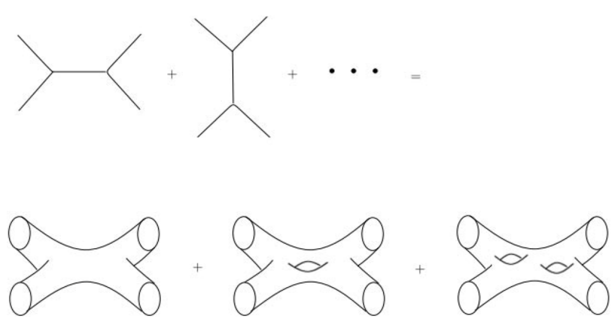
1 Introduction

Quantum Field Theory (QFT) is a powerful theory in modern physics. Based on QFT, standard model of particle physics successfully describes our microcosmic world. Most of the predictions by standard model have been observed in many experiments under rather precise level. However, the key technical procedure in QFT, i.e. renormalization, looks complicated and cumbersome. More seriously, the renormalization procedure does not work for gravity, which means that it is impossible to construct a consistent quantum gravity theory by using the conventional QFT. Many efforts have been made to understand the physical meaning of renormalization, while most of people believe that the infinity comes from the fundamental topological structure of QFT, which can not be cured without modifying its topological structure. In string theory, one extends a point particle to a small piece of a string. This simple modification dramatically changes the topological structure of the theory. The new "Feynman diagram" now is a smooth world-sheet instead of world-lines with singular intersection points, Fig.(1).



Feynman diagrams in quantum field theory and string theory.

Let us briefly look at the high energy behavior in QFT by a simple power counting. The tree amplitude of four spin-J particles behaves as $A_{tree}^{(J)} \sim E^{-2(1-J)}$, so that the one-loop amplitude behaves as

$$A_{1-loop}^{(J)} \sim \int d^4p \frac{\left(A_{tree}^{(J)}\right)^2}{\left(p^2\right)^2} \sim \int d^4E \frac{E^{-4(1-J)}}{E^4}, \tag{1}$$

which is finite for a scalar particle ($J = 0$) and renormalizable for a vector particle ($J = 1$), but is nonrenormalizable for the particle with $J \geq 2$, including graviton ($J = 2$). However, there is a loophole to bypass this argument. If we sum over all amplitudes with the different spins, the final amplitude will be

$$A = \sum_J A_{tree}^{(J)} \sim \sum_J a_J E^{-2(1-J)}, \tag{2}$$

which could be exponentially fall-off, i.e. $A \sim e^{-E}$, so that the all loop amplitudes would be finite, if the following two conditions are both satisfied:

1. there are infinite higher spin J particles
2. the coefficients a_J's should be precisely related to each other.

In string theory, the amplitude of four tachyons scattered, i.e. Veneziano amplitude, can be easily calculated as

$$\mathcal{T} = B\left(-\frac{s}{2} - 1, -\frac{t}{2} - 1\right), \tag{3}$$

where s, t are the Mandelstam variables. In the high energy and fixed angle limit (hard scattering where $s \to \infty$, $t \to \infty$ and s/t fixed), it is easy to verify that the Veneziano amplitude behaves as exponentially fall-off. This property applies to all four-point scattering amplitudes in string theory. Thus string theory is a finite theory! To explain the reason why string theory is finite, we borrow the previous argument of QFT and make our conjecture that string theory satisfies the above two conditions and is a loophole.

The first condition is trivially satisfied in string theory because a string has infinite oscillation modes which correspond to infinite higher spin particles. The second condition is highly nontrivial and will lead to the conjecture of the symmetries among the different string scattering amplitudes. We conjecture that these symmetries are the key point to "fine-tune" the string theory to be finite. These symmetries are usually complicated and not apparent so that we call them hidden symmetries. However, we expect that the hidden symmetries will reduce to some simpler relations in certain limits. In fact, Gross has conjectured that the string scattering amplitudes are linearly related each other in the high energy, fixed scattering angle limit [1, 2, 3]. Using the methods of Ward identities of zero norm states [4, 5, 6], Virasoro algebra and direct calculation of scattering amplitudes, we are able to prove the Gross conjecture and compute the linear ratios among the different string amplitudes [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. We also extend our study to the high energy, small angle limit, i.e. Regge scattering [18, 19, 20, 21, 22]. In the following sections, we will review our results in both limits.

2 Hard Scattering

In this section we consider the hard scattering, which is the limit of $E \to \infty$ with fixed scattered angle ϕ, or in Mandelstam variables $s \sim E^2 \to \infty$ with $t/s \sim \sin^2(\phi/2)$ fixed. It was shown [10, 11] that for the 26D open bosonic string the only four-point scattering amplitudes that will survive in the hard scattering limit at mass level $M^2 = (N-1)$ are of the form

$$\mathcal{T}^{(N,2m,q)} = \langle V_1 V_2^{(N,2m,q)}(k) V_3 V_4 \rangle, \tag{4}$$

where V_1, V_3 and V_4 are set to be tachyonic vertices for simplicity and the second vertex of the nontrivial higher spin state is

$$V_2^{(N,2m,q)}(k) \sim \left(\alpha_{-1}^T\right)^{N-2m-2q} \left(\alpha_{-1}^L\right)^{2m} \left(\alpha_{-2}^L\right)^q |0;k\rangle. \tag{5}$$

the polarizations of the second vertex with momentum k_2 on the scattering plane were defined to be $e^P = \frac{1}{M_2}(E_2, \mathrm{k}_2, 0) = \frac{k_2}{M_2}$ as the momentum polarization, $e^L = \frac{1}{M_2}(\mathrm{k}_2, E_2, 0)$ the longitudinal polarization and $e^T = (0,0,1)$ the transverse polarization. Note that e^P approaches to e^L in the hard scattering limit, and the scattering plane is defined by the spatial components of e^L and e^T. Polarizations perpendicular to the scattering plane are ignored because they are kinematically suppressed for four-point scatterings in the high-energy limit. One can then use three different methods, i.e. Ward identities of zero norm states, Virasoro algebra and saddle point approximation to calculate the ratios among the high energy scattering amplitudes. The final result of the ratios of the amplitude in hard scattering limit are [10, 11]

$$\frac{\mathcal{T}^{(N,2m,q)}}{\mathcal{T}^{(N,0,0)}} = \left(-\frac{1}{2M}\right)^q \left(\frac{1}{2M^2}\right)^m (2m-1)!! \tag{6}$$

The methods of Ward identities of zero norm states and Virasoro algebra are algebraic methods, which in principle can be applied to arbitrary higher loop level. The third method, i.e. saddle point method, is due to direct calculation of the string scattering amplitudes. To show the linear relations in the high energy limit, in most of cases, we only calculate the tree level scattering amplitudes. The direct calculation at higher loop levels is quite complicated, a simple case at 1-loop level has been showed in [23].

To see the meaning of the ratios in Eq.(6), let's consider a simple analogy from practical physics. The ratios of the nucleon-nucleon scattering processes

$$(a)\quad p+p\to d+\pi^{+}, \tag{7a}$$
$$(b)\quad p+n\to d+\pi^{0}, \tag{7b}$$
$$(c)\quad n+n\to d+\pi^{-} \tag{7c}$$

can be calculated to be

$$T_a : T_b : T_c = 1 : \frac{1}{\sqrt{2}} : 1 \tag{8}$$

from $SU(2)$ isospin symmetry. Similarly, as we will see in the rest of the paper, the ratios in Eq.(6) can be extracted from Kummer function. The key is to study high energy string scatterings in the Regge limit.

3 Regge Scattering

In this section we consider the Regge limit, which is the limit of $E\to\infty$ with small scattered angle ϕ, or in Mandelstam variables $s\sim E^2\to\infty$ with $t\sim E^2\sin^2(\phi/2)$ fixed. Now all four-point scattering amplitudes survive in the Regge limit at mass level $M^2=(N-1)$ with the vertices

$$V_2^{(N,k_n,q_m)}(k)\sim\prod_{n>0}(\alpha_{-n}^{T})^{k_n}\prod_{m>0}(\alpha_{-m}^{L})^{q_m}|0\rangle, \tag{9}$$

where the powers k_n's and q_m's satisfy the constraint,

$$\sum_{n,m} nk_n + mq_m = N. \tag{10}$$

The four-point string scattering amplitudes of the above vertex (9) in the Regge limit can be calculated to be [18]

$$\begin{aligned}\mathcal{T}^{(N,k_n,q_m)} =& \left(-\frac{i}{M_2}\right)^{q_1} U\left(-q_1,\frac{t}{2}+2-q_1,\frac{t+M^2+2}{2}\right)\\ &\cdot B\left(-1-\frac{s}{2},-1-\frac{t}{2}\right)\cdot\prod_{n=1}\left[i\sqrt{-t}(n-1)!\right]^{k_n}\\ &\cdot\prod_{m=2}\left[i\left(t+M^2+2\right)(m-1)!\left(-\frac{1}{2M_2}\right)\right]^{q_m},\end{aligned} \tag{11}$$

which is power-law behaved in the high energy limit as expected. The function $U(a,c,x)$ in Eq.(**??**) is the Kummer function of the second kind and is defined to be

$$U(a,c,x)=\frac{\pi}{\sin\pi c}\left[\frac{M(a,c,x)}{(a-c)!(c-1)!}-\frac{x^{1-c}M(a+1-c,2-c,x)}{(a-1)!(1-c)!}\right], \tag{12}$$

where

$$M(a,c,x)=\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}\frac{x^j}{j!}, \tag{13}$$

is the Kummer function of the first kind. $U(a,c,x)$ and $M(a,c,x)$ are the two solutions of the Kummer equation

$$xy^{''}(x)+(c-x)y'(x)-ay(x)=0. \tag{14}$$

At this point, it is crucial to note that, in our case of Eq.(11), $c=\frac{t}{2}+2-2m$ and is not a constant as in the usual definition, so the function $U\left(-q_1,\frac{t}{2}+2-q_1,\frac{t+M^2+2}{2}\right)$ in Eq.(**??**) is actually *not* a solution of the Kummer equation (14). This makes the function $U\left(-q_1,\frac{t}{2}+2-q_1,\frac{t+M^2+2}{2}\right)$ has some special properties which will lead to our expected result.

It is important to note that there is no linear relation among high energy string scattering amplitudes of different string states for each fixed mass level in the Regge limit as can be seen from Eq.(11). This is very different from the result in the hard scattering limit in Eq.(6). In other words, the ratios $\mathcal{T}^{(n,2m,q)}/\mathcal{T}^{(n,0,0)}$ are t-dependent functions. However, we believe that the hidden symmetry which we conjectured should be encoded in general string scattering amplitudes. Therefore the hidden symmetry should also hide in the scattering amplitudes of the Regge limit in some way. To see the hidden symmetry in the Regge limit, we take an additional limit $|t|\to\infty$ in the scattering amplitudes (11). Using the identity of our "special Kummer function",

$$\begin{aligned}&U\left(-2m,\frac{t}{2}+2-2m,\frac{t}{2}\right)\\&=0\cdot t^{2m}+0\cdot t^{2m-1}+\cdots\cdots+0\cdot t^{m+1}+\frac{(2m)!}{(-4)^m\, m!}t^m+O\left(t^{m-1}\right)\end{aligned} \tag{15}$$

We found that the ratios in Eq.(6) could be exactly reproduced among the leading terms of the scattering amplitudes (11) under the additional limit.

It can be shown that the identity (15) is satisfied order by order for arbitrary value m by computer, but it is highly non-trivial to prove it analytically. With the help of the mathematicians, we finally completed the proof two years after we conjectured [21].

4 Compactified Space

It is interesting to study that strings scattered in a compactified space. We calculated string scattering amplitudes in some compactified spaces for both hard scattering and Regge limits [24, 25]. Besides the two well known high energy behaviors of string scattering amplitudes, e.g. exponentially fall-off in hard scattering and power-law fall-off in Regge limit, we found two new behaviors under certain conditions:

1. Power-law fall-off in hard scattering.

2. Exponentially fall-off in Regge limit.

The two new high energy behaviors are completely not expected for the first sight. But when we carefully study the detail of the kinematic variables, by comparing the apparent scattering angle ϕ in the compactified space and the effective scattering angle ϕ' in the full space, we found that the high energy behaviors are actually determined by the effective scattering angle ϕ'. We summary our results in the following table:

ϕ	(s,t,N,K)	$E \to \infty$	ϕ'
ϕ fixed	$(s,t) \gg (N,K)$	e^{-E}	ϕ' fixed
ϕ fixed	$(s,t,K) \gg N$	E^{-c}	$\phi' \sim 0$
$\phi \sim 0$	$s \gg (t,N,K)$	E^{-c}	$\phi' \sim 0$
$\phi \sim 0$	$(s,K) \gg (t,N)$	e^{-E}	ϕ' fixed

If string theory showed to be correct and thus the extra dimensions exist, then our 3+1 space-time is compactified from a higher dimensional space-time. The scattering angles measured in our usual scattering experiments should be the apparent scattering angles. Therefore, under certain conditions as we showed before, the unexpected high energy behavior could be observed.

5 Conclusion

As a theory including quantum gravity, string theory has been showed to be finite to all loop level. But there are many higher spin modes in string theory and the scattering amplitudes of each of them are turn out to be infinite. So there must be a mysterious cancellation in string theory. We believe that the cancellation is due to the "hidden" symmetries in string theory. To explore the complicated hidden symmetries, we study the string scattering amplitudes in different high energy limits. In the hard scattering limit, we proved that the scattering amplitudes are linear related each other. We calculated the ratios by three different methods. In the Regge limit, we calculated the string scattering amplitudes, which is proportional to a "Kummer" function. By a nontrivial identity of the "Kummer" function, we are able to show that the string scattering amplitudes in the Regge limit reproduces to all the ratios in hard scattering limit when we take $t \to \infty$.

Our study is just the first step to explore the hidden symmetries in string theory. To understand the full symmetries, we need to study the string scattering amplitudes in general energy, but which is very complicated to deal with

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