The gauge invariant quark Green's function in QCD_2

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Abstract. The properties of the gauge invariant quark Green's function, defined with a path-ordered phase factor along a straight line, are studied in two-dimensional QCD and in the large-N_c limit. The analysis is done by means of an exact integrodifferential equation. The Green's function is found to be infrared finite, with singularities in the momentum squared variable represented by an infinite number of threshold type branch points with a power -3/2, starting at positive mass squared values, with cuts lying on the positive real axis. The expression of the Green's function is analytically determined.

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1 Introduction

Gauge invariant Green's functions are generally defined with the aid of path-ordered gauge field phase factors [1, 2]. Wilson loops [3] appear to be basic tools for the investigation of the properties of gauge invariant Green's functions [4, 5, 6, 7, 8]. We report in this talk results obtained in this area, concerning an integrodifferential equation satisfied by the two-point gauge invariant quark Green's function (2PGIQGF) and its resolution when QCD is considered in its two-dimensional version in the large-N_c limit [9, 10].

The 2PGIQGF is defined as

$$S_{\alpha\beta}(x,x';C_{x'x}) = -\frac{1}{N_c}\,\langle\overline{\psi}_\beta(x')\,U(C_{x'x};x',x)\,\psi_\alpha(x)\rangle, \tag{1}$$

where the averaging is considered in the path integral formalism; α and β are the Dirac spinor indices, while the color indices are implicitly summed, quarks being considered in the fundamental representation of the color gauge group $SU(N_c)$; U is a path-ordered gluon field phase factor along a line $C_{x'x}$ joining a point x to a point x', with an orientation defined from x to x':

$$U(C_{x'x};x',x) = Pe^{-ig\int_x^{x'} dz^\mu A_\mu(z)}. \tag{2}$$

2 Paths with polygonal lines

Green's functions with paths along skew-polygonal lines (polygonal lines in space) are of particular interest, since they can be decomposed into the succession of simpler straight line segments. On the other hand, the latter objects are Lorentz invariant in form and this leads to a simplification of the analysis of the spectral properties of the corresponding Green's function in momentum space. Furthermore, paths with polygonal lines can easily be classified according to the number of segments that are present.

For polygonal lines with n sides and $n-1$ junction points y_1, y_2, ..., y_{n-1} between the segments, we define:

$$S_{(n)}(x,x';y_{n-1},\ldots,y_1) = -\frac{1}{N_c}\,\langle\overline{\psi}(x')U(x',y_{n-1})U(y_{n-1},y_{n-2})\ldots U(y_1,x)\psi(x)\rangle, \quad (3)$$

where now each U is along a straight line segment. The simplest such function corresponds to $n=1$, for which the points x and x' are joined by a single straight line:

$$S_{(1)}(x,x') \equiv S(x,x') = -\frac{1}{N_c}\,\langle\overline{\psi}(x')\,U(x',x)\,\psi(x)\rangle. \quad (4)$$

(We shall generally omit the index 1 from that function.)

3 Integrodifferential equation

To quantize the theory one may proceed in two steps. First, one integrates with respect to the quark fields. This produces in various terms the quark propagator in the presence of the gluon field. Then one integrates with respect to the gluon field through Wilson loops. To accomplish the latter step, we use for the quark propagator in external field a representation which involves phase factors along straight lines together with the full gauge invariant quark Green's function [9, 11]. This representation is a generalization of the one introduced by Eichten and Feinberg when dealing with the heavy quark limit [12].

The quark propagator in the external gluon field is expanded around the following gauge covariant quantity:

$$\Big[\widetilde{S}(x,x')\Big]^a_{\ b} \equiv S(x,x')\Big[U(x,x')\Big]^a_{\ b}. \quad (5)$$

It is possible to set up an integral equation realizing iteratively the previous expansion. Its systematic use leads to the derivation of functional relations between the Green's functions $S_{(n)}$ (polygonal line with n segments) and S (one segment).

Use of the above representation allows the introduction of phase factors along an infinite sum of polygonal lines between the points x and x'. These lines, together with the phase factor lines already existing in the definition of the gauge invariant Green's functions, then form closed polygonal contours and produce Wilson loops. Using then the equations of motion relative to the Green's functions, one establishes the following equation for $S(x,x')$ [9]:

$$(i\gamma.\partial_{(x)} - m)S(x,x') = i\delta^4(x-x') + i\gamma^\mu\Big\{K_{2\mu}(x',x,y_1)\,S_{(2)}(y_1,x';x)$$
$$+\sum_{n=3}^{\infty} K_{n\mu}(x',x,y_1,\ldots,y_{n-1})\,S_{(n)}(y_{n-1},x';x,y_1,\ldots,y_{n-2})\Big\}, \tag{6}$$

where the kernel K_n $(n = 2,3,\ldots)$ contains globally n derivatives of the logarithm of the Wilson loop average along an $(n+1)$-sided polygonal contour and also the Green's function S and its derivative. The Green's functions $S_{(n)}$ being themselves related to the simplest Green's function S through series expansions resulting from functional relations, eq. (6) is ultimately an integrodifferential equation for S. One expects that the kernels with small number of derivatives will provide the most salient contributions. Therefore, the first kernel K_2 in eq. (6) would contain the leading effect of the interaction.

4 Physical interest of the quark Green's function

The physical interest of the 2PGIQGF is best exhibited from its spectral analysis. If the theory is confining, then it is not possible to cut the Green's function (1) and to saturate it with a complete set of physical states (hadrons), which are color singlets. Intermediate states are necessarily colored states, since single quark fields in the fundamental representation cannot produce with any number of gluon fields in the adjoint representation, present in the phase factor, color singlet objects. If one relates, as usual, the singularity structure of the Green's functions to the contribution of physical intermediate states [13, 14], then this would suggest that the Green's function above does not have any singularity. However, the equation it satisfies, eq. (6), derived from the QCD Lagrangian, contains singularities, generated by the free quark propagator (the inverse of the Dirac operator in the left-hand side of eq. (6)).

The above paradoxical situation is overcome with the acceptance that quarks and gluons, as the building blocks of the theory, continue forming a complete set of states with positive energies and could be used for any saturation scheme of intermediate states. It is the resolution of the equations of motion which should indicate to us how the related singularities combine to form the complete solutions.

Therefore, the knowledge of the 2PGIQGF provides a direct information about the effect of confinement in the colored sector of quarks.

5 Spectral functions

Green's functions with paths along straight lines are dependent only on the end points of the paths. This facilitates the passage to momentum space by Fourier transformation.

It is advantageous to consider for that purpose the path-ordered phase factor in its representation given by the formal series expansion in terms of the coupling constant g. Using for each term of the series, together with the quark fields, the spectral analysis with intermediate states and causality, one arrives at a generalized form of the Källén–Lehmann representation for the Green's function S in momentum space, in which the cut lies on the positive real axis starting from the quark mass squared m^2 and extending to infinity [15, 16, 17, 18, 19].

Taking into account translation invariance, we introduce the Fourier transform of the Green's function S into momentum space:

$$S(x,x') = S(x-x') = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip.(x-x')}\, S(p). \tag{7}$$

$S(p)$ has the following representation in terms of real spectral functions $\rho_1^{(n)}$ and $\rho_0^{(n)}$ $(n = 1, \ldots, \infty)$:

$$S(p) = i\int_0^\infty ds' \sum_{n=1}^{\infty} \frac{[\,\gamma.p\,\rho_1^{(n)}(s') + \rho_0^{(n)}(s')\,]}{(p^2 - s' + i\varepsilon)^n}. \tag{8}$$

Depending on the degrees of the singularities at threshold, simplifications may occur by integrations by parts, or otherwise by summation, reducing the series into more compact forms.

6 Two-dimensional QCD

Many simplifications occur in two-dimensional QCD at large N_c [20, 21, 22]. This theory is expected to have the essential features of confinement observed in four dimensions, with the additional simplification that asymptotic freedom is realized here in a trivial way, since the theory is superrenormalizable. For simple contours, Wilson loop averages in two dimensions are exponential functionals of the areas enclosed by the contours [23, 24, 25]. Furthermore, at large N_c, crossed diagrams and quark loop contributions disappear.

It turns out that in two dimensions and at large N_c, only the lowest-order kernel K_2 survives in eq. (6). The equation reduces then to the following form [10]:

$$\begin{aligned}(i\gamma.\partial - m)S(x) &= i\delta^2(x) - \sigma\gamma^\mu(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})x^\nu x^\beta \\ &\quad\times \Big[\int_0^1 d\lambda\,\lambda^2\, S((1-\lambda)x)\gamma^\alpha S(\lambda x) + \int_1^\infty d\xi\, S((1-\xi)x)\gamma^\alpha S(\xi x)\Big],\end{aligned} \tag{9}$$

where σ is the string tension.

The equation is solved by decomposing S into Lorentz invariant parts:

$$S(p) = \gamma.pF_1(p^2) + F_0(p^2), \tag{10}$$

or, in x-space:

$$S(x) = \frac{1}{2\pi}\Big(\frac{i\gamma.x}{r}\widetilde{F}_1(r) + \widetilde{F}_0(r)\Big), \qquad r = \sqrt{-x^2}. \tag{11}$$

One obtains, with the introduction of the Lorentz invariant functions, two coupled equations. Their resolution proceeds through several steps, mainly based on the analyticity properties resulting from the spectral representation (8). The solutions are obtained in explicit form for any value of the quark mass m.

The covariant functions $F_1(p^2)$ and $F_0(p^2)$ are, for complex p^2:

$$F_1(p^2) = -i\frac{\pi}{2\sigma}\sum_{n=1}^{\infty} b_n\,\frac{1}{(M_n^2 - p^2)^{3/2}}, \tag{12}$$

$$F_0(p^2) = i\frac{\pi}{2\sigma}\sum_{n=1}^{\infty} (-1)^n b_n\,\frac{M_n}{(M_n^2 - p^2)^{3/2}}. \tag{13}$$

The masses M_n $(n = 1, 2, \ldots)$ have positive values greater than the quark mass m and are labelled with increasing values with respect to n; their squares represent the locations of branch point singularities with power $-3/2$, with cuts lying on the positive real axis of the complex plane of p^2. The masses M_n and the coefficients b_n satisfy an infinite set of coupled algebraic equations that are solved numerically. Their asymptotic behaviors for large n, such that $\sigma\pi n \gg m^2$, are:

$$M_n^2 \simeq \sigma\pi n, \qquad b_n \simeq \frac{\sigma^2}{M_n}. \tag{14}$$

In x-space, the solutions are:

$$\widetilde{F}_1(r) = \frac{\pi}{2\sigma}\sum_{n=1}^{\infty} b_n\, e^{-M_n r}, \qquad \widetilde{F}_0(r) = \frac{\pi}{2\sigma}\sum_{n=1}^{\infty} (-1)^{n+1} b_n\, e^{-M_n r}. \tag{15}$$

[$r = \sqrt{-x^2}$.]

At high energies, the solutions satisfy asymptotic freedom [26]:

$$F_1(p^2) \underset{p^2\to-\infty}{=} \frac{i}{p^2}, \tag{16}$$

$$F_0(p^2) \underset{p^2\to-\infty}{=} \frac{im}{p^2}, \qquad m \neq 0, \tag{17}$$

$$F_0(p^2) \underset{p^2\to-\infty}{=} \frac{2i\sigma}{N_c}\frac{\langle\overline{\psi}\psi\rangle}{(p^2)^2}, \qquad m = 0, \tag{18}$$

where in the last equation we have introduced the one-flavor quark condensate.

We present in Fig. 1 the function iF_0 for spacelike p and in Fig. 2 its real part for timelike p, for the case $m = 0$.

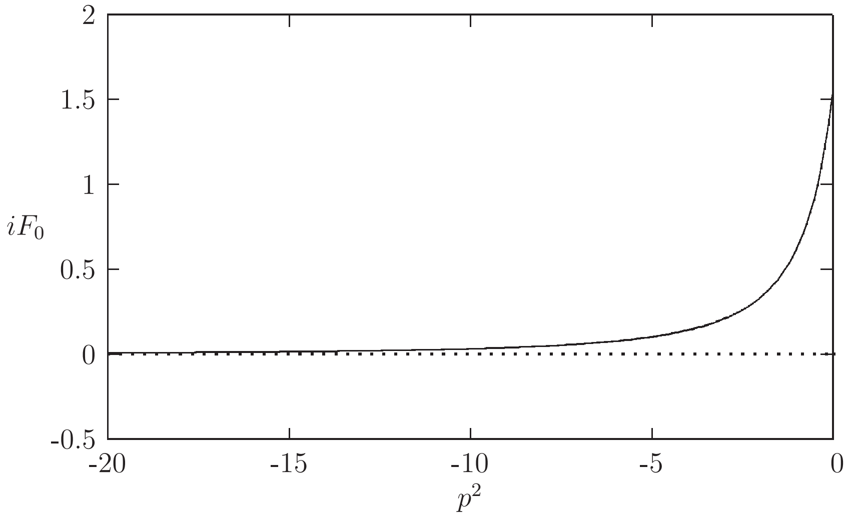

Figure 1: The function iF_0 for spacelike p, in mass unit of $\sqrt{\sigma/\pi}$, for $m = 0$.

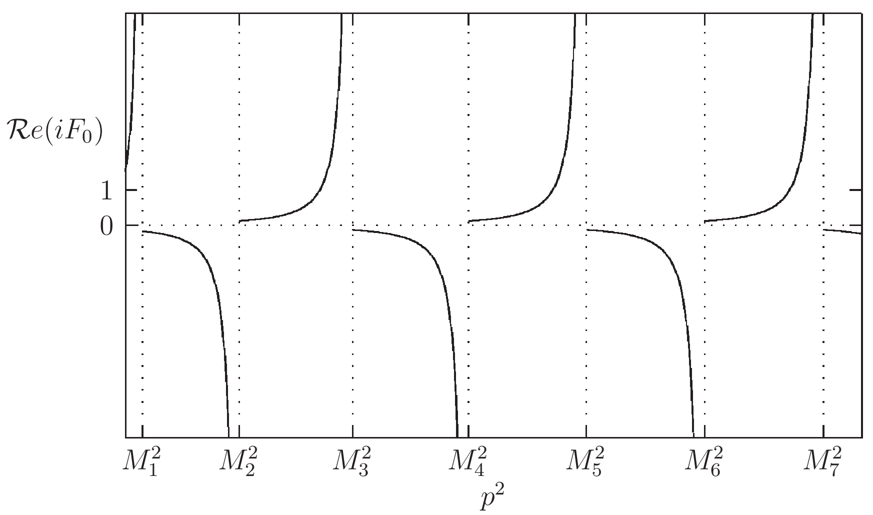

Figure 2: The real part of the function iF_0 for timelike p, in mass unit of $\sqrt{\sigma/\pi}$, for $m = 0$.

7 Conclusion

The spectral functions of the quark Green's function are infrared finite and lie on the positive real axis of p^2. No singularities in the complex plane or on the negative real axis have been found. This means that quarks contribute like physical particles with positive energies. (In two dimensions there are no physical gluons.)

The singularities of the Green's function are represented by an infinite number of threshold type singularities, characterized by a power of $-3/2$ and positive masses M_n $(n = 1, 2, \ldots)$. The corresponding singularities are stronger than simple poles and this feature might be at the origin of the unobservability of quarks as asymptotic states.

The threshold masses M_n represent dynamically generated masses, since they are not present in the QCD Lagrangian. They survive even when the quark mass is zero.

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References

[1] S. Mandelstam, Phys. Rev. **175**, 1580 (1968).

[2] Y. Nambu, Phys. Lett. **80B**, 372 (1979).

[3] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).

[4] A. M. Polyakov, Nucl. Phys. **B164**, 171 (1980).

[5] Yu. M. Makeenko and A. A. Migdal, Phys. Lett. **88B**, 135 (1979).

[6] Yu. M. Makeenko and A. A. Migdal, Nucl. Phys. **B188**, 269 (1981).

[7] A. A. Migdal, Phys. Rep. **102**, 199 (1983).

[8] Yu. Makeenko, arXiv: hep-th/0001047 (2000).

[9] H. Sazdjian, Phys. Rev. D **77**, 045028 (2008).

[10] H. Sazdjian, Phys. Rev. D **81**, 114008 (2010).

[11] F. Jugeau and H. Sazdjian, Nucl. Phys. **B670**, 221 (2003).

[12] E. Eichten and F. Feinberg, Phys. Rev. D **23**, 2724 (1981).

[13] R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960).

[14] C. Itzykson and J.-B. Zuber, *Quantum field theory* (McGraw-Hill, New York, 1980), pp. 301-317.

[15] G. Källén, Helv. Phys. Acta **25**, 417 (1952).

[16] H. Lehmann, Nuovo Cimento **11**, 342 (1954).

[17] A. S. Wightman, Phys. Rev. **101**, 860 (1956).

[18] S. S. Schweber, *An introduction to relativistic quantum field theory* (Row, Peterson and Co., Evanston, 1961), pp. 721-742.

[19] G. 't Hooft and M. Veltman, CERN Report 73-9 (1973).

[20] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).

[21] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).

[22] C. G. Callan, Jr., N. Coote, and D. J. Gross, Phys. Rev. D **13**, 1649 (1976).

[23] V. A. Kazakov and I. K. Kostov, Nucl. Phys. **B176**, 199 (1980).

[24] V. A. Kazakov, Nucl. Phys. **B179**, 283 (1981).

[25] N. E. Bralić, Phys. Rev. D **22**, 3090 (1980).

[26] H. D. Politzer, Nucl. Phys. **B117**, 397 (1976).