Anomalous dimensions at strong coupling

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1 Introduction

The computation of anomalous dimensions of gauge invariant operators is a central problem in non-abelian gauge theories. For example, they are related to the parton distribution function in the DIS regime of QCD. Typically, one can compute such quantities in perturbation theory at weak coupling. Solving this problem in the strong coupling regime or even at finite values of the coupling has been a formidable challenge and requires new insights. In the case of four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory, this seemingly hopeless task can be attacked using the methods of integrability and exploiting the holographic duality between gauge theories and superstring theory [1].

In the planar limit of the gauge theory, the interplay between these two approaches led eventually to the conjecture that the anomalous dimensions of all gauge invariant operators can be computed at any value of the coupling, using the so called Y-system [2]. By solving a complicated system of integral equations, one can numerically estimate the strong coupling expansion coefficients of such conformal dimensions. This conjecture has passed several checks, in particular in the sector of long gauge invariant operators, constructed using an asymptotically large number of elementary fields, both at weak and at strong coupling [1]. An extensive numerical study of the Y-system has been carried out and compared to semi-classical techniques. However, short non-protected operators, that constitute the most natural sector from the gauge theory point of view, are hard to study and existing methods are not reliable in this regime.

In this talk, we will report on a new method [3] to compute the anomalous dimensions of operators in $\mathcal{N} = 4$ super Yang-Mills theory at strong coupling, using the

AdS/CFT correspondence. We will describe this new method as applied to short operators, that consist of a small number of elementary fields. The simplest non-protected supermultiplet is the Konishi supermultiplet [4], whose anomalous dimension at weak coupling was studied in [5].

According to the AdS/CFT correspondence, short operators at strong coupling are dual to perturbative string states of type IIB superstring in the $AdS_5 \times S^5$ background [6]. Conformal dimensions of field theory operators are equal to energies of such string states. The conformal dimension of non-BPS operators, such as operators in the Konishi multiplet, receive quantum corrections, whose evaluation at strong coupling is the goal of this talk.

The way we will do this is "old-fashioned," *i.e.*by quantizing the string worldsheet theory. By imposing that the worldsheet string state satisfy the superstring physical state condition at the loop level, we derive an equation for its energy, which gives in turn the strong coupling expansion of the conformal dimension of the dual gauge theory operator. We apply our method to the simplest non-BPS operator, a member of the Konishi multiplet of string states, whose energy at strong coupling has been predicted numerically in [2] using the Y-system. Our worldsheet computation of the energy of a particular element gives the conformal dimension

$$\Delta - \Delta_0 = 2\sqrt[4]{\lambda} - 4 + \frac{2}{\sqrt[4]{\lambda}} + \mathcal{O}(1/\sqrt{\lambda}) \ , \tag{1}$$

where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling. In weak coupling language, the classical dimension Δ_0 of different members of the Konishi multiplet may take different integer values, but the anomalous dimension (1) is the same for all of them. Hence, we can pick our favourite member of the multiplet to perform the calculation. The leading term $2\sqrt[4]{\lambda}$ reproduces the expected leading behaviour for an operator dual to a string state at the first massive level. The last term $2/\sqrt[4]{\lambda}$ is the one-loop correction to the string energy of the Konishi multiplet of string states at strong coupling. The three terms in (1) confirm the numerical fit obtained using the Y-system in [2, 7] or semi-classical methods [8].

Our method can be used to solve for the whole energy spectrum of massive states of type IIB superstring in $AdS_5 \times S^5$ and can be expanded to any loop order at strong coupling. This will give an expansion of the conformal dimensions of short operators in super Yang-Mills theory in inverse powers of $\sqrt[4]{\lambda}$.

2 Motivations

Our goal is to compute anomalous dimensions in a strongly coupled gauge theory. We will be concerned with local gauge invariant operators $\mathcal{O}(x)$. Our favorite example is the simples four-dimensional gauge theory, namely $\mathcal{N} = 4$ super-Yang Mills theory

(SYM) with gauge group $SU(N)$. The field content of this theory is the $\mathcal{N}=4$ vector super-multiplet, consisting of a gauge field, four gauginos and six real scalars ϕ^i, for $i=1,\ldots,6$. An example of a local gauge invariant operator is $\mathcal{O}(x)=\mathrm{Tr}\,\phi^i\phi^i(x)$, where the trace is over the color indices. This SYM is a superconformal field theory, which includes among its global symmetries the invariance under dilations. The dilation operator D acts on local gauge invariant operators in the following way

$$[D,\mathcal{O}(x)]=i\Delta\mathcal{O}(x)\ .$$

The real dilation eigenvalue Δ is the *conformal dimension* of the operator $\mathcal{O}(x)$. The conformal dimension determines the two-point function of the operator with its adjoint. In a conformal field theory (CFT), a useful representation for the two-point correlation function is the OPE when two operators get close to each other

$$\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\rangle\sim\frac{1}{|x-y|^{2\Delta}}\ .$$

Therefore, knowledge of the conformal dimensions of all local operators is equivalent to solving for the spectrum of the CFT. The natural question arises: What do we know about conformal dimensions? We will answer this question in the planar limit $N\to\infty$ of very large number of colors. We will restrict ourselves to this regime for the rest of the talk.

At the classical level, when the gauge coupling $g_{YM}=0$, the conformal dimensions are equal to the engineering dimensions Δ_0, *i.e.*the canonical dimensions we can read from the lagrangian $\Delta=\Delta_0$. For example, the operator $\mathrm{Tr}\,\phi^i\phi^i$ has $\Delta_0=2$.

Let us turn on the 't Hooft coupling $\lambda=g_{YM}^2N$. Quantum corrections get into play and in general will shift the conformal dimensions from their classical values, generating non-zero *anomalous dimensions* $\Delta-\Delta_0\neq 0$. What do we know about anomalous dimensions? At small 't Hooft coupling $\lambda<<1$, known as the weak coupling regime, we can compute the anomalous dimensions perturbatively. This is going to give us the usual planar loop expansion

$$\Delta-\Delta_0=c_1\lambda+c_2\lambda^2+\ldots\ ,$$

where c_1 and c_2 are the one- and two-loop coefficients. This perturbative computation is a standard QFT technique and it has been carried out to very high orders in our favorite superconformal field theory. So far, so good.

What happens when the coupling becomes very large $\lambda>>1$, *i.e.* in the strong coupling regime? We can classify the behavior of local operators at strong coupling as follows. The BPS operators are protected and their conformal dimensions do not receive quantum corrections. The non-BPS operators receive corrections

$$\begin{array}{rll}\textit{protected (BPS)}: & \Delta-\Delta_0= & 0\\ \textit{short non-BPS}: & \Delta-\Delta_0\sim & \sqrt[4]{\lambda}+\ldots\\ \textit{long non-BPS}: & \Delta-\Delta_0\sim & \sqrt{\lambda}+\ldots\end{array}$$

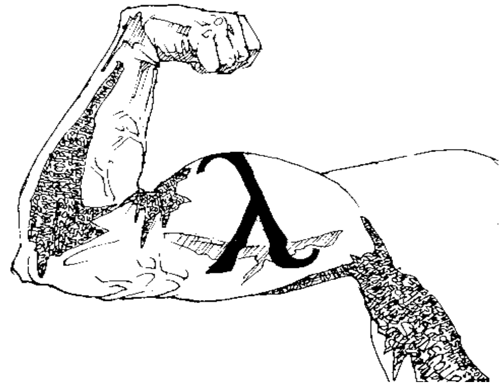

Figure 1: Artistic rendering of the strong coupling regime: $\lambda >> 1$.

Short operators are made out of traces of a small number of elementary fields, like $\mathrm{Tr}\,\phi^i\phi^i$. Long operators contain a very long trace, where the number of elementary fields scale like $\sqrt{\lambda}$. The "..." in the previous list stand for quantum corrections at strong coupling, that vanish in the limit $\lambda \to \infty$. How can we evaluate the quantum corrections "..." in this regime? We had better use holography.

According to the AdS/CFT correspondence [6], $\mathcal{N} = 4$ four-dimensional SYM theory is equivalent to type IIB superstring theory on the five-dimensional Anti-de Sitter space times the five-sphere. The basic dictionary of the correspondence is summarized in Table 1. According to the correspondence, BPS operators correspond on the gravity side to the supergravity modes. Non-protected operators, on the other hand, are genuinely stringy objects. Short non-BPS operators are described by quantum strings, what we would call perturbative massive string states in the usual flat ten-dimensional description of the string. Long non-BPS operators correspond to semi-classical (*i.e.* solitonic) strings, namely strings whose energy scales like $\sqrt{\lambda}$.

Since the field theory conformal dimensions Δ correspond to the energies E of strings propagating inside the AdS space, according to the correspondence in Table 1, we can obtain the quantum corrections to conformal dimensions at strong coupling by simply computing the spectrum of string states in AdS, in an perturbative expansion around the flat-space limit (the small curvature limit)

$$E = c_0\sqrt[4]{\lambda} + c_1 + \frac{c_2}{\sqrt[4]{\lambda}} + \mathcal{O}(1/\sqrt{\lambda}) \ .$$

3 String theory in Anti-de Sitter space

In order to carry out the program outlined above, we need to introduce the worldsheet theory of strings on an AdS background (for a comprehensive review, see [9]). The

gauge side		gravity side
strongly coupled CFT	$=$	perturbative string theory
$\mathcal{N}=4$ SYM	$=$	IIB string theory on $AdS_5\times S^5$
$\sqrt{\lambda}$	$=$	radius$^2/\alpha'$
$\lambda >> 1$	$=$	small curvature
CFT conformal dim. Δ	$=$	E energy in AdS

Table 1: The AdS/CFT dictionary.

$AdS_5\times S^5$ background is described by the supercoset $PSU(2,2|4)/SO(1,4)\times SO(5)$. A coset representative g transforms as $g\to g_0gh$ under global g_0 and local h transformations. Superstring propagation in this background is described in terms of a worldsheet non-linear sigma model with values in such a supercoset. Since we want to keep covariance manifest, we will make use of the pure spinor action for the superstring in the AdS background (we follow the notations in [9])

$$S=\frac{\sqrt{\lambda}}{2\pi}\int \mathrm{Str}[\frac{1}{2}J_2\overline{J}_2+\frac{3}{4}J_3\overline{J}_1+\frac{1}{4}J_1\overline{J}_3+w\overline{\nabla}l+\overline{w}\nabla\overline{l}-N\overline{N}]\ , \qquad (2)$$

where the worldsheet metric is in the conformal gauge. The matter sector of the pure spinor action is constructed in terms of the left-invariant currents $J=g^{-1}dg=\sum_{i=1}^{4}J_i$, which take values in the super-Lie algebra $psu(2,2|4)$. This algebra admits a $\mathbf{Z}_4$ grading, that we used to label the currents J_i. The grading zero part of the current describes the gauge degrees of freedom in the supercoset, namely the local Lorentz rotations, while the even J_2 and the odd J_1, J_3 describe respectively the ten bosonic and thirty-two fermionic directions of $AdS_5\times S^5$. In addition, the action contains a couple of pure spinor ghosts $l,\overline{l}$ of grading one and three respectively, which satisfy the pure spinor constraint $\{l,l\}=0=\{\overline{l},\overline{l}\}$, and their conjugate momenta $w,\overline{w}$. N and $\overline{N}$ are the Lorentz generators in the ghost sector. The pure spinor BRST charge is $Q=\oint d\sigma\,\mathrm{Str}[lJ_3+\overline{l}\overline{J}_1]$ and it is nilpotent on the pure spinor constraint, up to a gauge transformation. Physical states $|V\rangle$ are in the cohomology of the BRST charge and satisfy the Virasoro constraint $(T+\overline{T})|V\rangle=0$, where T and $\overline{T}$ are the left- and right-moving components of the worldsheet stress tensor.

The massless sector of type IIB superstring in AdS is described in terms of unintegrated vertex operators of ghost number $(1,1)$ and weight zero, that are in one-to-one correspondence with the type IIB supergravity spectrum in $AdS_5\times S^5$ [10]. In order to study the massive string spectrum, we will expand the sigma model around a classical string configuration, describing a point-like string sitting at the center of AdS in Fig. 2. This is analogous to the usual flat space bosonic string theory, where one expands around the vacuum with momentum k, with wavefunction e^{ikX}. Using the

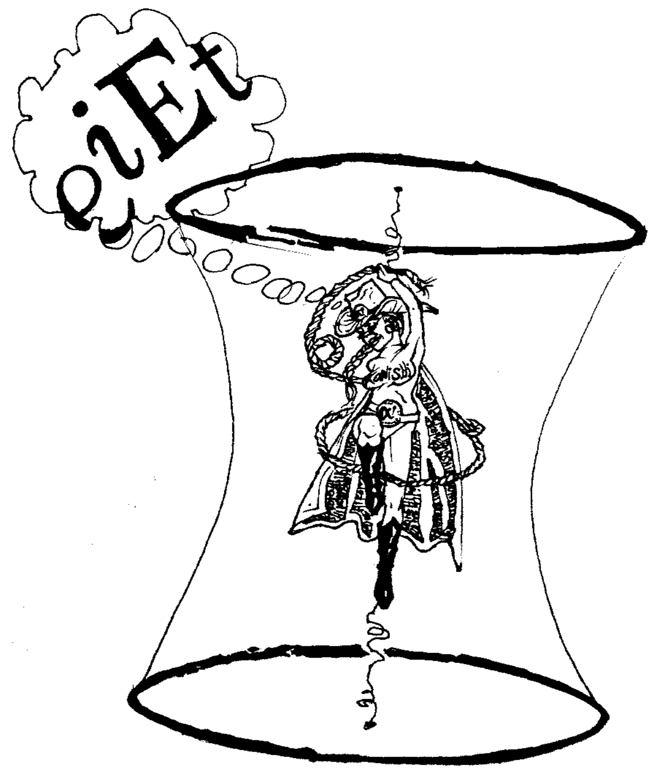

Figure 2: Artistic rendering of a pointlike string sitting at the center of AdS in global coordinates.

metric of AdS in Lorentzian global coordinates $ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, dS_3^2$, our string configuration sits at $\rho \sim 0$ and evolves in time as e^{iEt}. In the static gauge, this is described by the coset element

$$\tilde{g}(\sigma,\tau) = \exp[-\tau E\mathbf{T}/\sqrt{\lambda}] \ , \tag{3}$$

that solves the worldsheet equations of motion coming from the action (2), where $\mathbf{T}$ is the anti-hermitian $PSU(2,2|4)$ generator corresponding to the AdS time translations and τ is the worldsheet time. The only non-vanishing left-invariant current in this background is $\tilde{J}_\tau = \tilde{g}^{-1}\partial_\tau \tilde{g} = -E\mathbf{T}/\sqrt{\lambda}$. Hence, such classical configuration has vanishing BRST charge.

The Noether charge for the global $PSU(2,2|4)$ symmetry of the string sigma model is given by $Q_{PSU} = \oint d\sigma\, j_\tau$, where

$$j_\tau = \frac{\sqrt{\lambda}}{2\pi} g[J_1 + J_2 + J_3 + N + \overline{N}]_\tau g^{-1} \ . \tag{4}$$

In particular, the AdS energy operator E evaluated on the string configuration (3) gives

$$\mathsf{E} = \oint d\sigma \,\mathrm{Str}\mathbf{T}\tilde{j}_\tau = E \ , \tag{5}$$

where we used $\mathrm{Str}(\mathbf{TT}) = -1$. Since we consider positive energy configurations, we can take E to be positive in the following. The classical Virasoro constraint for such configuration reads

$$T + \overline{T} = \frac{\sqrt{\lambda}}{2}\mathrm{Str}\, \tilde{J}_\tau \tilde{J}_\tau = -\frac{E^2}{2\sqrt{\lambda}} \ . \tag{6}$$

The classical Virasoro constraint (6) will be modified by quantum effects, which are going to allow for a non-zero solution for E in the rest of the paper. Since for massive string states, such as the one we are considering, the energy scales as $E \sim \sqrt[4]{\lambda}$, the classical contribution to (6) is of order one and may be canceled against quantum effects.

4 Quantization

Let us quantize the action (2) around the classical configuration (3) using the background field method [9]. We parameterize the coset element by $g = \tilde{g}e^X$, where

$$X = t\mathbf{T} + X^A\mathbf{P_A} + \Phi\mathbf{J} + X^I\mathbf{P_I} + \Theta^a\mathbf{Q_a} + \Theta^{\dot{a}}\mathbf{Q_{\dot{a}}} + \widehat{\Theta}^a\widehat{\mathbf{Q}}_\mathbf{a} + \widehat{\Theta}^{\dot{a}}\widehat{\mathbf{Q}}_{\dot{\mathbf{a}}} \ , \tag{7}$$

are the quantum fluctuations and $\tilde{g}$ is given in (3). We chose a coset gauge in which the grading zero part of the fluctuations vanishes. The left invariant currents are

$$J_\tau = e^{-X}(\partial_\tau - E\mathbf{T}/\sqrt{\lambda})e^X, \qquad J_\sigma = e^{-X}\partial_\sigma e^X \ . \tag{8}$$

By expanding the action (2) up to quadratic order and using the $psu(2,2|4)$ structure constants, we can read the spectrum of fluctuations around the background (3). The quadratic part of the action for the fluctuations is

$$S = \frac{\sqrt{\lambda}}{2\pi}\int\Big[-\partial t\overline{\partial}t + \partial J\overline{\partial}J + \delta_{IJ}\partial X^I\overline{\partial}X^J + \delta_{AB}\left(\partial X^A\overline{\partial}X^B + \left(\frac{E}{2\sqrt{\lambda}}\right)^2 X^AX^B\right)$$

$$+\Pi_{ab}\partial\widehat{\Theta}^a\overline{\partial}\Theta^b + \Pi_{\dot{a}\dot{b}}\partial\widehat{\Theta}^{\dot{a}}\overline{\partial}\Theta^{\dot{b}} - \frac{E}{4\sqrt{\lambda}}\left[\delta_{ab}(\Theta^a\overline{\partial}\Theta^b + \widehat{\Theta}^a\partial\widehat{\Theta}^b) + \delta_{\dot{a}\dot{b}}(\Theta^{\dot{a}}\overline{\partial}\Theta^{\dot{b}} + \widehat{\Theta}^{\dot{a}}\partial\widehat{\Theta}^{\dot{b}})\right]$$

$$+w_a\overline{\partial}l^a + w_{\dot{a}}\overline{\partial}l^{\dot{a}} + \hat{w}_a\partial\hat{l}^a + \hat{w}_{\dot{a}}\partial\hat{l}^{\dot{a}}\Big]$$

The AdS_5 time direction as well as the five sphere directions remain massless, while the remaining four bosonic directions of AdS_5 acquire a mass squared $m_X^2 = (E/\sqrt{\lambda})^2$. The fermionic spectrum consists of sixteen massless fermions and sixteen massive fermions with mass squared $m_\Theta^2 = (E/2\sqrt{\lambda})^2$. The ghosts remain massless. There is no relation between the bosonic and fermionic spectrum, reflecting the fact that this background is not BPS.

We can canonically quantize the theory imposing the usual equal time commutation relations for the coordinates and their conjugate momenta. The equations of motion of some of the fluctuations are

$$\begin{aligned} \partial\overline{\partial}X^A + m_X^2 X^A &= 0 \ , \\ \partial\overline{\partial}\Theta + m_\Theta \Pi \partial\widehat{\Theta} &= 0 \ , \\ \partial\overline{\partial}\widehat{\Theta} - m_\Theta \Pi \overline{\partial}\Theta &= 0 \ . \end{aligned} \tag{9}$$

Their mode expansion is

$$\begin{aligned} \Theta = {} & \Theta_0 \sin m_\Theta\tau - \Pi\widehat{\Theta}_0 \cos m_\Theta\tau \\ & + \textstyle\sum_{n\neq 0} c_{\Theta,n} \left(\frac{im_\Theta}{\omega_{\Theta,n}+k_n} \varphi^1_{\Theta,n}\Theta_n - \varphi^2_{\Theta,n}\Pi\widehat{\Theta}_n \right) + \sum_n \vartheta_n e^{-in(\tau-\sigma)} \ , \\ \widehat{\Theta} = {} & \widehat{\Theta}_0 \sin m_\Theta\tau + \Pi\Theta_0 \cos m_\Theta\tau \\ & + \textstyle\sum_{n\neq 0} c_{\Theta,n} \left(\frac{im_\Theta}{\omega_{\Theta,n}+k_n} \varphi^2_{\Theta,n}\widehat{\Theta}_n + \varphi^1_{\Theta,n}\Pi\Theta_n \right) + \sum_n \widehat{\vartheta}_n e^{-in(\tau+\sigma)} \ , \\ X^A = {} & x_0^A \cos m_X\tau + p_0^A m_X^{-1} \sin m_X\tau + i \textstyle\sum_{n\neq 0} \frac{1}{\omega_{X,n}} \left(\varphi^1_{X,n}\alpha_n^{1A} + \varphi^2_{X,n}\alpha_n^{2A} \right) \ , \end{aligned}$$

where, for $s = (\Theta, X)$, we defined

$$\begin{aligned} \varphi^1_{s,n} &= \exp[-i(\omega_{s,n}\tau - k_n\widetilde{)}] \ , & \varphi^2_{s,n} &= \exp[-i(\omega_{s,n}\tau + k_n\widetilde{)}] \ , \\ \omega_{s,n} &= \sqrt{m_s^2 + k_n^2} \quad n > 0 \ ; & \omega_{s,n} &= -\sqrt{m_s^2 + k_n^2} \quad n < 0 \ , \\ k_n &= 2\pi n \ , & c_{s,n} &= \left(1 + (\omega_{s,n} - k_n)^2/m_s^2\right)^{-1/2} \ , \end{aligned}$$

We can now compute the conjugate momenta $P_\Theta = \frac{\delta S}{\delta\partial_\tau\Theta}$ and $P_X = \frac{\delta S}{\delta\partial_\tau X}$, and impose equal time commutation relations between coordinates and momenta $[P_X(\sigma), X(\sigma')] = -i\delta(\sigma - \sigma')$ and $\{P_\Theta(\sigma), \Theta(\sigma')\} = -i\delta(\sigma - \sigma')$, by which we derive the commutation relations of the modes. In particular, the zero modes of the fermions have commutation relations

$$\{\Theta_0, \Theta_0\} = -\frac{\mathbf{1}}{m_\Theta} = \{\widehat{\Theta}_0, \widehat{\Theta}_0\} \ , \qquad \{\Theta_0, \widehat{\Theta}_0\} = 0 \ . \tag{10}$$

Our vacuum state $|E\rangle$ is a scalar and is annihilated by all positive modes, including the zero modes of $w, \overline{w}$. This last requirement ensures that the Lorentz generators for the ghosts N and $\overline{N}$ annihilate the vacuum. We can choose sixteen fermionic zero modes as creation operators. Since we are using global AdS coordinates, these are linear combinations of supercharges and superconformal transformations. In the rest of the paper, we will evaluate the leading quantum contributions to the physical state condition $T + \overline{T} = 0$, applied to a particular vertex operator to be introduced below.

The first quantum correction to (6) comes from the central charge. Even if for $E = 0$, namely in empty AdS, the central charge vanishes [11], there is a normal ordering contribution coming from the quadratic part of the stress tensor when $E \neq 0$. This is given by the sum of the energies of the oscillator modes

$$\frac{2E}{\sqrt{\lambda}} + \frac{1}{2}\sum_{n=1}^{\infty}\Big(6\sqrt{n^2} + 4\sqrt{n^2 + (E/\sqrt{\lambda})^2} - 16\sqrt{n^2} - 16\sqrt{n^2 + (E/2\sqrt{\lambda})^2} + 22\sqrt{n^2}\Big) \ . \tag{11}$$

The first term is the contribution from the zero modes of the bosons; by looking at the commutation relations of the fermionic zero modes, one can easily see that they do not contribute to (11). Inside the sum, the first two terms come from the bosonic oscillators, the second two terms from the fermionic ones and the last term from the ghosts. We have not computed the precise value of E yet, but we want it to correspond to a stringy state, whose energy scales as $E \sim \sqrt[4]{\lambda}$, which gives $E/\sqrt{\lambda} \sim 1/\sqrt[4]{\lambda} \ll 1$, obtaining from (11) the total contribution $2\frac{E}{\sqrt{\lambda}} - \frac{3}{16}\zeta(3)(\frac{E}{\sqrt{\lambda}})^4$. We can drop the second term as it does not contribute to the energy at the order we are considering, so we are left with the contribution

$$2E/\sqrt{\lambda} \ . \tag{12}$$

Note that at order $(\frac{E}{\sqrt{\lambda}})^2$ the four massive bosonic modes cancel with the massive sixteen fermionic modes. The contribution $2\frac{E}{\sqrt{\lambda}}$, which would vanish in a BPS background, will affect the one-loop correction to the energy of the string, contributing to the last term in (1).

Let us consider now a specific worldsheet state. All of the members of the Konishi multiplet have the same anomalous dimension (in the weak coupling sense) and they are in one to one correspondence with the string states at the first massive level. Thus we will choose a particularly simple state in the first massive string level, that will simplify the computation. Physical states are given by unintegrated vertex operators of ghost number $(1,1)$. The simplest one is $\mathrm{Str}\, l\bar{l}$ and it corresponds to the radius modulus at zero momentum [12]. We will denote the corresponding state as $|l\bar{l}\rangle \equiv \mathrm{Str}(l\bar{l})|0\rangle$. We choose the simple state

$$|V\rangle = x^{+}_{-1}\overline{x}^{+}_{-1}|l\bar{l}\rangle \ , \tag{13}$$

where x^{+}_{-1} and $\overline{x}^{+}_{-1}$ are the first left- and right-moving oscillators coming from the fluctuations of the AdS "space-cone" coordinate $x^{+} = x^1 + ix^2$. Although it does not look covariant, we can interpret this state as being created by non-zero modes of the global symmetry right invariant currents. We should emphasize that this is *not* a global $PSU(2,2|4)$ transformation. We identify the state (13) as a particular member of the Konishi multiplet of string states with classical dimension $\Delta_0 = 6$ at weak coupling, Lorentz spin two and singlet of $SU(4)$. The operator $|l\bar{l}\rangle$, corresponding to the radius changing operator, is dual to the Yang-Mills lagrangian and it has $\Delta_0 = 4$, so it is natural to expect that (13) has two units more of classical dimension. The same type of state was discussed in [12], where it is argued that such states are physical, *i.e.* they are annihilated by the BRST charge. This state is created by acting on the vacuum with two oscillators of mass $E/\sqrt{\lambda}$, whose contribution to the physical state condition is

$$2\sqrt{1+(E/\sqrt{\lambda})^2}\ . \tag{14}$$

5 Quartic corrections

Other possible contributions to the physical state condition at this order may come from the terms in the stress tensor $T+\overline{T}$, expanded to quartic order in fluctuations around the classical background (3) and acting on the specific state (13). Let us analyze the possible terms.

The factor of 2 in front of the square root in (14) may get corrected by quartic terms in $T+\overline{T}$ of the form $(\partial X)^2X^2$, due to normal ordering. However, there is no normal ordering due to this term. This comes from the fact that any correction to this term has to be proportional to the one-loop beta function, which vanishes [13]. The same type of normal ordering contribution was discussed in [14], where they argued it should be zero using $PSU(2,2|4)$ symmetry (the same reason why the beta function vanishes). There are other corrections that are not protected by the beta function argument, but they are of the form $(E/\sqrt{\lambda})\partial XX^3$ and $(E/\sqrt{\lambda})^2X^4$. They give higher order contributions to the energy and we can safely neglect them.

The last possible contribution comes from the fact that the operator (13) might mix with other operators due to quartic terms in $T+\overline{T}$. In order to study the mixing, we have to compute the momenta conjugate to the fields up to quartic terms in the action, then plug these back in the stress tensor. The conjugate momenta are

$$P_i = \frac{\delta S}{\delta\partial_\tau X_i} = \partial_\tau X_i + \dots \tag{15}$$

where '...' are higher order terms in the fluctuations X_i. In this way we eliminate all the time derivatives in the stress tensor. For the particular state (13) we may only consider the terms with four bosonic or two bosonic and two fermionic fields. Terms quartic in bosons can both introduce mixing and also correct the energy of our state. Terms with two fermions and two bosons will only give mixing. For the particular choice of the "space-cone" polarization in (13), it is easy to see that there will be no mixing with other bosonic states, nor mixing with states created by two fermions, since the stress tensor will only have commutators and products of gamma matrices, which vanish for this choice of polarization. Another way to see this is to note that since $T+\overline{T}$ is a scalar only terms that match the spin of out initial state can mix with it. However, there is a non-vanishing correction to the physical state condition, proportional to the state itself, coming from the term $\frac{1}{6}(\mathrm{Str}\,[P_2, X_2]^2 - \mathrm{Str}\,[\partial_\sigma X_2, X_2]^2)$ in the stress tensor. Expanding $T + \overline{T}$ into modes and computing the relevant terms one finds that the corresponding correction is (it coincides with the result in [14])

$$-2/\sqrt{\lambda}\ . \tag{16}$$

6 Conformal dimension

Summing up the contributions (6), (12), (14), and (16) to the Virasoro constraint, we find that the physical state condition $(T + \overline{T})|V\rangle = 0$ reads

$$-\frac{E(E-4)}{2\sqrt{\lambda}} + 2\sqrt{1 + (\frac{E}{\sqrt{\lambda}})^2} - \frac{2}{\sqrt{\lambda}} = 0\ . \tag{17}$$

The positive energy solution of this equation gives the energy of our string state (13). According to the AdS/CFT dictionary, the energy operator on the string side of the correspondence is mapped to the dilatation operator on the field theory side, whose eigenvalues are the conformal dimensions of operators. Above, we identified the field theory dual to the string state (13) as a member of the Konishi multiplet with classical dimension $\Delta_0 = 6$, Lorentz spin two and singlet of $SU(4)$. Its conformal dimension at strong coupling is therefore

$$\Delta = E = 2\sqrt[4]{\lambda} + 2 + \frac{2}{\sqrt[4]{\lambda}} + \mathcal{O}(\lambda^{-1/2}). \tag{18}$$

Hence we derived (1) with $\Delta_0 = 6$.

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