From correlators to Wilson loops and super-amplitudes

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1 Introduction

The Maldacena conjecture (AdS/CFT correspondence) [1] states that $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensions and IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ capture the same physics. The field theory description is mainly useful at weak coupling; the string theory picture can be exploited to obtain strong coupling information. Anomalous dimensions of composite operators in the field theory correspond to the energy levels of the dual strings. The computation of the one-loop anomalous dimension of the so-called BMN operators can be interpreted as the problem of finding the energy eigenvalues of the Heisenberg $\mathrm{XXX}_{\frac{1}{2}}$ chain [2]. In the planar limit, higher loop effects define an integrable deformation of the spin chain Hamiltonian [3]. The corresponding all-loops Bethe equations have been generalised to other types of composite operators, too. A high spin continuum limit of the Bethe equations in the twist operator sector leads to an integral equation [4] from which the "cusp anomalous dimension" can be derived analytically as a weak and strong coupling expansion, or numerically for any value of the coupling. Since scattering amplitudes in strongly coupled $\mathcal{N} = 4$ SYM are also described by a type of integrable system [5] there is hope that such deeply non-perturbative sectors will also become accessible for higher-point objects in the model.

The strong coupling limit of these scattering amplitudes takes the form of polygonal Wilson loops with light-like edges [6]. Between MHV amplitudes and Wilson loops this relation continues to hold also at weak coupling [7, 8]. The present talk summarises recent work in which we elaborated that both the MHV amplitudes and the dual Wilson loops can be generated from correlation functions of gauge invariant composite operators in a light-cone limit [9, 10]. What is more, the correlators of stress-energy tensor multiplets can also produce N^kMHV amplitudes [11]. The integrand that this construction yields for the amplitudes identically reproduces the all-loops proposal of [12].

2 Definitions

The $\mathcal{N}=4$ SYM theory is the maximally supersymmetric non-abelian gauge theory in four dimensions. It has the elementary fields

$$\{\phi^{[AB]},\, \psi^{\alpha A},\, \overline{\psi}^{\dot\alpha}_A,\, A^\mu\}\,, \qquad A \in \{1,2,3,4\}\,, \qquad \phi^{AB} = \frac{1}{2}\,\epsilon^{ABCD}\,\overline{\phi}_{CD}\,, \tag{1}$$

which are related by the on-shell supersymmetry transformations[1]

$$\begin{aligned} Q^\alpha_A\,\phi^{BC} &= i\sqrt{2}\,(\delta^B_A\psi^{C\alpha} - \delta^C_A\psi^{B\alpha})\,, \qquad & Q^\alpha_A\,A_{\beta\dot\beta} &= -2\,i\,\delta^\alpha_\beta\,\overline{\psi}_{A\dot\beta}\,, \\ Q^\alpha_A\,\psi^B_\beta &= \delta^B_A F^\alpha_\beta + i\,g[\phi^{BC},\phi_{CA}]\,\delta^\alpha_\beta\,, \qquad & Q^\alpha_A\,\overline{\psi}^{\dot\beta}_B &= \sqrt{2}\,D^{\dot\beta\alpha}\phi_{AB} \end{aligned} \tag{2}$$

and the conjugate. The fields $\{\phi,\,\psi,F\}$ all transform in the adjoint representation of the gauge group, say $SU(N_c)$: Upon introducing Grassmann odd variables $\theta^{\alpha A}, \overline{\theta}^{\dot\alpha}_A$ we can build a field strength multiplet

$$W^{[AB]} \;=\; \phi^{[AB]}(x) + \theta^{\alpha[A}\psi(x)^{B]}_\alpha + \theta^{[A}_{(\alpha}\theta^{B]}_{\beta)}F^{\alpha\beta} + O(\overline{\theta})\,. \tag{3}$$

There is no quantum formalism with manifest $\mathcal{N}=4$ supersymmetry. Instead, one can use standard field theory methods and observe that results will be organised in multiplets of on-shell $\mathcal{N}=4$ symmetry, or one can use quantum formalisms in which one or two supersymmetries are made manifest by introducing superspace coordinates. A simple reduction to $\mathcal{N}=2$ superfields is achieved by breaking the $SU(4)$ internal symmetry in the following way:

$$W_{\mathcal{N}=2} \;=\; W^{23}_{\mathcal{N}=4}|_{\theta_1=\theta_4=0}\,, \qquad q \;=\; \{\phi^{1i},\psi^1,\overline{\psi}_4\}\,. \tag{4}$$

The single superfield incorporating the elementary fields of $\mathcal{N}=4$ SYM thus breaks into $\mathcal{N}=2$ Yang-Mills and matter fields. In this reduction supersymmetry still only closes on the fields of the matter multiplet q (the "hypermultiplet") if the equations of motion are satisfied. The introduction of an additional bosonic coordinate u solves this problem [13]. We shall work with the resulting "harmonic superspace" formalism. Hence there are two off-shell $\mathcal{N}=2$ supersymmetric quantum fields:

$$\begin{aligned} &q^+(x_A,\,\theta^+,\overline{\theta}^+,u)\,, \qquad && \theta^+ \;=\; \theta^i\,u^+_i\,,\, \overline{\theta}^+ \;=\; \overline{\theta}_i\,u^{+i}\,, \quad x_A = x - 4\,i\,\theta^{(i}\overline{\theta}^{j)}\,u^+_i u^-_j\,, \\ &V^{++}(x_A,\,\theta^+,\overline{\theta}^+,u)\,, \qquad && u = (u^+,u^-) \,\in\, SU(2)/U(1) \end{aligned} \tag{5}$$

The action of $\mathcal{N}=4$ SYM takes the form

$$S_{\mathcal{N}=4\,\mathrm{SYM}} \;=\; S_{\mathrm{HM/SYM}} + S_{\mathcal{N}=2\,\mathrm{SYM}} \tag{6}$$

[1]The transformations only close on the fields when the equations of motion are satisfied.

$$
\begin{aligned}
S_{\mathrm{HM/SYM}} &= -2\int du\, d^4x_A\, d^2\theta^+\, d^2\overline{\theta}^+\ \mathrm{Tr}(\tilde{q}^+ D^{++} q^+ + i\,\sqrt{2}\,\tilde{q}^+[V^{++}, q^+])\,, &(7)\\
S_{\mathcal{N}=2\,\mathrm{SYM}} &= -\frac{1}{4\,g^2}\int d^4x_L\, d^4\theta\ \mathrm{Tr}\, W^2 = -\frac{1}{4\,g^2}\int d^4x_R\, d^4\overline{\theta}\ \mathrm{Tr}\,\overline{W}^2\,, &(8)\\
W &= \frac{i}{4} u_i^+ u_j^+ \overline{D}^i_{\dot\alpha} \overline{D}^{j\dot\alpha} \sum_{r=1}^{\infty} \int du_1 \ldots du_r\, \frac{(-i\sqrt{2})^r\, V^{++}(u_1)\ldots V^{++}(u_r)}{(u^+u_1^+)(u_1^+u_2^+)\ldots(u_r^+u^+)}\,.
\end{aligned}
$$

Here $x_L^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} - 2i\theta^{i\alpha}\overline{\theta}_i^{\dot\alpha}$ and x_A indicates a similar change of basis in case of the hypermultiplet. Of the corresponding Feynman rules we will only need the $\tilde{q}Vq$ vertex, the matter propagator (in the diagrams below depicted by a solid line) and a mixed WV propagator (a curly line with a dot on the W end). The vertex does not carry covariant derivatives; it is simply given by an integration over space-time, the plus projected Grassmann coordinates and the u's. The $SU(2)$ integration can always be accomplished by a set of algebraic rules.

In configuration space the conformal symmetry of the model is transparent. The propagators take a simple form also in x space because the theory is massless:

$$
\begin{aligned}
\langle \tilde{q}_a^+(1) q_b^+(2)\rangle &= \frac{(12)}{4\pi^2\,\hat{x}_{12}^2}\,\delta_{ab}\,, \qquad (12) = -(21) = u_1^{+i}\epsilon_{ij}u_2^{+j}\,, &(9)\\
\langle W_a(1) V_b^{++}(2)\rangle &= -\frac{g^2\delta_{ab}}{4\pi^2\,\tilde{x}_{12}^2}\,(\theta_{12})^2
\end{aligned}
$$

with the supersymmetric coordinate differences

$$
\begin{aligned}
\hat{x}_{12}^\mu &= x_{A1}^\mu - x_{A2}^\mu + \frac{2i}{(12)}[(1^-2)\theta_1^+\sigma^\mu\overline{\theta}_1^+ + (2^-1)\theta_2^+\sigma^\mu\overline{\theta}_2^+ + \theta_1^+\sigma^\mu\overline{\theta}_2^+ + \theta_2^+\sigma^\mu\overline{\theta}_1^+]\,,\\
\tilde{x}_{12}^{\alpha\dot\alpha} &= x_{L1}^{\alpha\dot\alpha} - x_{A2}^{\alpha\dot\alpha} - 4iu_{2i}^-\theta_1^{i\alpha}\,\overline{\theta}_2^{+\dot\alpha}\,, \qquad \theta_{12}^\alpha = u_{2i}^+\theta_1^{i\alpha} - \theta_2^{+\alpha}\,. &(10)
\end{aligned}
$$

We will study correlation functions of

$$
O\ =\ \mathrm{Tr}(q^+q^+)\,, \qquad \tilde{O}\ =\ \mathrm{Tr}(\tilde{q}^+\tilde{q}^+)\,, \qquad \hat{O}\ =\ 2\,\mathrm{Tr}(\tilde{q}^+q^+)\,, \tag{11}
$$

i.e.

$$
G_n = \int \mathcal{D}\Phi\ e^{iS_{\mathcal{N}=4\,\mathrm{SYM}}}\ \tilde{O}(x_1)\,O(x_2)\,\tilde{O}(x_3)\ldots O(x_n)\,. \tag{12}
$$

Correlators of $\mathcal{N}=4$ stress-tensor multiplets can be reconstructed from several such $SU(2)$ projections.

3 Wilson loops from correlators

To begin with consider only scalar fields. The connected tree level graphs yield

$$
G_n^{(0)} = N_c^2 \sum_{\{i_1,\ldots,i_n\}} S^{(0)}(x_{i_1,i_2})S^{(0)}(x_{i_2,i_3})\ldots S^{(0)}(x_{i_n,i_1})\,, \qquad S^{(0)}(x) = \frac{1}{4\pi^2\,x^2}\,. \tag{13}
$$

The dominant power singularity in the limit $x_{i,i+1}^2 \to 0$ is:

$$G_n^{(0)} \stackrel{x_{i,i+1}^2 \to 0}{\longrightarrow} N_c^2\, S^{(0)}(x_{12})S^{(0)}(x_{23})\ldots S^{(0)}(x_{n1}) = \frac{(2\pi)^{-2n} N_c^2}{x_{12}^2 x_{23}^2 \ldots x_{n1}^2} \tag{14}$$

which is depicted in diagram (a). Diagram (b) is suppressed by $x_{34}^2 x_{1n}^2/(x_{3n}^2 x_{14}^2)$. In

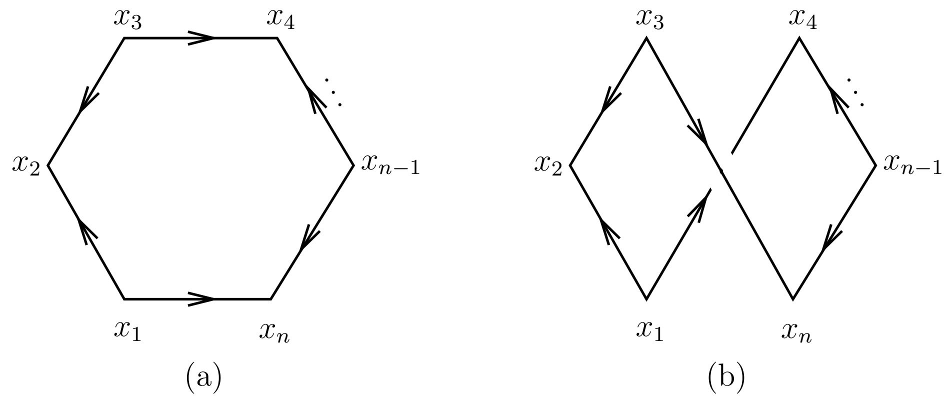


Figure 1: Singularities of tree level diagrams.

[9] we present a detailed analysis of the interactions of a scalar propagating from one cusp to another in the full $\mathcal{N} = 4$ theory . The light-cone condition $x_{i,i+1}^2 = 0$ is Lorentz invariant. In a frame in which the scalar is boosted to a very high velocity essentially all interactions are seen to be suppressed. The gauge field behaves as a classical background, leading to a dressed scalar propagator:

$$\lim_{x_{i,i+1}^2 \to 0} G_n^{(0)} = \langle 0|\mathrm{Tr}[S(x_1,x_2;A)S(x_2,x_3;A)\ldots S(x_n,x_1;A)]|0\rangle \tag{15}$$

with

$$S(x_i,x_{i+1};A) = S^{(0)}(x_{i,i+1})P\exp\left(ig\int_{x_i}^{x_{i+1}} dz\cdot\tilde{A}(z)\right) G(x_i,x_{i+1};A)\,. \tag{16}$$

The factor G should have an OPE expansion (Δ is the twist)

$$G(x,0;A) = \sum (x^2)^{\Delta} C_{\Delta,N}(x^2\mu^2)\; x_{\mu_1}\ldots x_{\mu_N}\mathcal{O}_{\Delta}^{\mu_1\ldots\mu_N}(0) \tag{17}$$

implying $\lim_{x^2\to 0} G = 1$. It follows

$$\frac{G_n}{G_n^{(0)}} \to W^{\mathrm{adj}}[C_n] = \frac{1}{N_c^2-1}\langle 0|\,\mathrm{Tr}_A\left[P\exp\left(ig\int_{C_n} dz\cdot\tilde{A}(z)\right)\right]|0\rangle \tag{18}$$

where

$$[\tilde{A}_\mu(z)]_{ab} = -if_{abc}A_\mu^c(z)\,, \qquad C_n = [x_1,x_2]\cup[x_2,x_3]\cup\ldots\cup[x_n,x_1]. \tag{19}$$

Last, the Wilson loop in the adjoint representation is actually equal to the square of the same Wilson loop with the generators in the fundamental representation.

In the following we present explicit calculations at one- and two-loop level in the quantum theory in order to illustrate that this simple argument correctly describes the underlying physics. Let us first consider $\langle \tilde{\mathbf{O}}(\mathbf{1})\, \mathbf{O}(\mathbf{2})\, \tilde{\mathbf{O}}(\mathbf{3})\, \mathbf{O}(\mathbf{4})\rangle$ at tree-level and one loop.

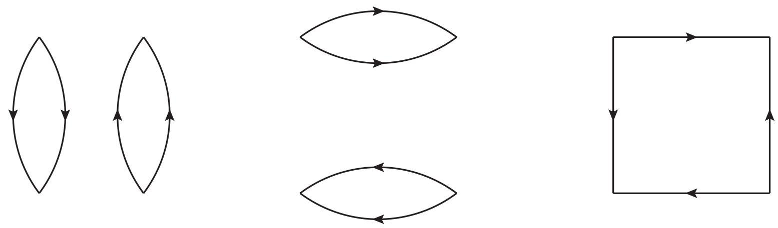


Figure 2: Tree level

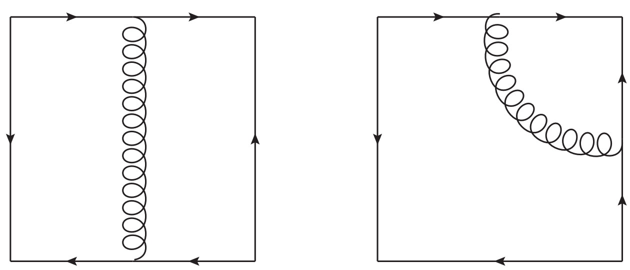


Figure 3: One loop

After the superspace and $SU(2)$ integrations the one-loop correction becomes [14]:

$$\langle O(1)\, \tilde{O}(2)\, O(3)\, \tilde{O}(4)\rangle_{\frac{g^2 N_c}{4\,\pi^2}} \;\propto\; (14)^2(23)^2 A_1 + (12)^2(34)^2 A_2 + (12)(23)(34)(41) A_3 \quad (20)$$

Here A_1 and A_2 are simple

$$A_1 = \frac{1}{x_{14}^2\, x_{23}^2}\, g(1,2,3,4)\,, \quad A_2 = \frac{1}{x_{12}^2\, x_{34}^2}\, g(1,2,3,4)\,, \tag{21}$$

but

$$\begin{aligned} A_3 \;\; = \;\; & \frac{2\,\partial_2.\partial_4\, f(1,2,1,4)}{x_{23}^2\, x_{34}^2} + \frac{2\,\partial_1.\partial_3\, f(2,1,2,3)}{x_{14}^2\, x_{34}^2} + \frac{2\,\partial_2.\partial_4\, f(3,2,3,4)}{x_{12}^2\, x_{14}^2} \\ + \;\; & \frac{2\,\partial_1.\partial_3\, f(4,1,4,3)}{x_{12}^2\, x_{23}^2} + \frac{(\partial_2+\partial_3)^2 f(1,2,3,4)}{x_{14}^2 x_{23}^2} + \frac{(\partial_1+\partial_2)^2 f(1,4,2,3)}{x_{12}^2 x_{34}^2}\,, \end{aligned} \tag{22}$$

where

$$g(1,2,3,4) \;=\; \frac{1}{4\pi^2}\int \frac{d^4x_0}{x_{10}^2\,x_{20}^2\,x_{30}^2\,x_{40}^2}\,,\; f(1,2;3,4) \;=\; \frac{1}{16\pi^4}\int \frac{d^4x_0\,d^4x_{0'}}{x_{10}^2\,x_{20}^2\,x_{00'}^2\,x_{30'}^2\,x_{40'}^2}\,. \tag{23}$$

We are ultimately interested in the $x_{12}^2 = x_{23}^2 = x_{34}^2 = x_{41}^2 \to 0$ limit of A_3. Short of differentiating the explicitly known but rather complicated integral f it is not obvious how this limit could be taken. For simplicity, let us first study the light-cone limit of $g(1,2,3,4)$ in dimensional regularisation (the mass-scale μ is suppressed):

$$g_\epsilon \;=\; \int \frac{d^{4-2\epsilon}x_0}{(x_{10}^2\,x_{20}^2\,x_{30}^2\,x_{40}^2)^{1-\epsilon}} \;=\; \frac{\Gamma(2-3\epsilon)}{(x_{24}^2)^{2-3\epsilon}}\int_0^1 \frac{dz_1\dots dz_4\,\delta(1-\sum_i z_i)\,(z_1\dots z_4)^{-\epsilon}}{(z_2z_4+z_1z_3\,X)^{2-3\epsilon}} \tag{24}$$

using the Feynman parameters z_1, z_2, z_3, z_4. Due to the on-shell conditions the denominator contains a sum of only two terms; the shorthand denotes $X = x_{13}^2/x_{24}^2$. We can separate the denominator using a single Mellin-Barnes (MB) parameter

$$\frac{1}{(A+B)^\nu} \;=\; \frac{1}{2\pi i\,\Gamma(\nu)\,B^\nu}\int_{-i\infty}^{i\infty} dz\,A^z\,B^{-z}\,\Gamma(-z)\,\Gamma(\nu+z) \tag{25}$$

and integrate out the z_i employing

$$\int_0^1 \left(\Pi_i\,dz_i\,z_i^{q_i-1}\right)\delta(1-\sum_i z_i) \;=\; \frac{\Pi_i\,\Gamma(q_i)}{\Gamma(\sum_i q_i)}\,. \tag{26}$$

It follows

$$g_\epsilon \;=\; \int_{-i\infty}^{i\infty} dz \frac{X^{-z}\,\Gamma(-z)\,\Gamma(2-3\epsilon+z)\Gamma(1-\epsilon+z)^2\,\Gamma(-1+2\epsilon-z)^2}{\pi i\;(x_{24}^2)^{2-3\epsilon}\;X^{2-3\epsilon}\,\Gamma(2\epsilon)} \tag{27}$$

and finally:

$$\lim_{x_{i,i+1}^2\to 0}\left(x_{13}^2\,x_{24}^2\right)^{1-\epsilon} g_\epsilon(1,2,3,4) \;=\; -2\left[-\frac{(x_{13}^2)^\epsilon}{\epsilon^2} - \frac{(x_{24}^2)^\epsilon}{\epsilon^2} + \log^2(X) + O(\epsilon)\right] \tag{28}$$

This is exactly the functional form of the square light-like Wilson loop at one loop!

4 Amplitudes via Lagrangian insertions

The computation of loop corrections to the correlators by the method of Lagrangian insertions [14] yields a simple way of taking the desired light-cone limit [9, 10]. It is a so far unexplained observation that the use of $\mathcal{N}=2$ superfields in combination with this technique exactly reproduces the all-loops integrand of [12].

Differentiating a correlator w.r.t. the coupling constant brings down the Yang-Mills action $S_{\mathcal{N}=2\,\mathrm{SYM}}$ from the exponential weight factor in (12):

$$g^2 \frac{\partial}{\partial g^2} G_n = -\int d^D x_0 \, G^{(0)}_{n+1;1}(x_0; x_1, u_1, \ldots, x_n, u_n) \, + \, O(g^4) \tag{29}$$

with the Born-level $(n+1)$-point correlator

$$G^{(0)}_{n+1;1}(x_0; x_1, u_1; \ldots; x_n, u_n) = \int d^4\theta_0 \, \langle L_{\mathcal{N}=2\,SYM}(x_0, \theta_0) \, O(x_1, u_1) \ldots O(x_n, u_n) \rangle + O(g^4) \, . \tag{30}$$

The relevant one-loop Feynman diagrams are as before, but with the Yang-Mills action "inserted" into the gluon lines: The integrals in the "building blocks" become

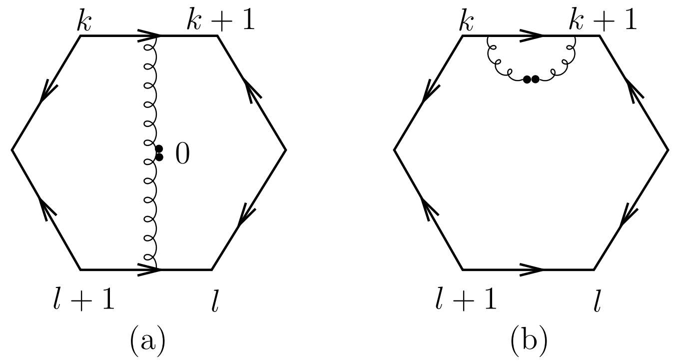


Figure 4: One-loop graphs with insertions

rational due to differentiation:

$$\langle \tilde{q}^+_a(1) \, W_b(0) \, q^+_c(2) \rangle \;=\; -\frac{2ig^2 f_{abc}}{(2\pi)^4} \, \frac{(12)}{x^2_{12}} \, i_{12} \tag{31}$$

$$\langle \tilde{q}^+_a(1) \, \mathrm{Tr}(W^2)(0) \, q^+_b(2) \rangle \;=\; \frac{g^4 N_c \, \delta_{ab}}{(2\pi)^6} \, \frac{(12)}{x^2_{12}} \, i^2_{12} \tag{32}$$

with

$$i_{12} \;=\; \frac{x^2_{12}}{(12)} \, \frac{\theta^+_{0/1} \cdot \theta^+_{0/2}}{x^2_{10} \, x^2_{20}} - \frac{\theta^+_{0/1} \cdot \theta^-_{0/1}}{x^2_{10}} + \frac{\theta^+_{0/2} \cdot \theta^-_{0/2}}{x^2_{20}} - \frac{(\theta^+_{0/1}[x_{10}, x_{20}]\theta^+_{0/2})}{(12) \, x^2_{10} \, x^2_{20}} \, . \tag{33}$$

Notice that i_{12} is rational and proportional to $(12)/x^2_{12}$ so that it is elementary to take the light-cone limit. Summing up all graphs:

$$\begin{aligned} G^{(0)}_{n+1;1} \;&=\; \frac{a}{4\pi^2} \, G^{(0)}_n \, \int d^4\theta_0 \, \Big(\sum_{k=1}^{n} i_{k,k+1} \Big)^2 \, , \qquad\qquad a = \frac{g^2 N_c}{4\,\pi^2} \\ &\to\; \frac{a}{8\pi^2} \, G^{(0)}_n \, \sum_{k,l} \frac{x^2_{k,l+1} x^2_{k+1,l} - x^2_{kl} x^2_{k+1,l+1} \, + \, 4\, i \, \epsilon_{\mu\nu\lambda\rho} \, x^\mu_{k,0} x^\nu_{k+1,0} x^\lambda_{l,0} x^\rho_{l+1,0}}{x^2_{k,0} x^2_{k+1,0} x^2_{l,0} x^2_{l+1,0}} \end{aligned} \tag{34}$$

The last formula exactly reproduces the parity-even part of dimensionally regularised n-point one-loop MHV amplitudes if the algebra is kept in four dimensions and only the dimension of the measure at the insertion point is changed! Note that the parity-odd sector in the last formula vanishes upon integration. Summarising:

- *Amplitudes*:

$$\langle\overline{\varphi}(x_i)\,\varphi(x_{i+1})\rangle \;=\; \frac{1}{4\pi^2\,p_i^2}\,, \quad p_i \;=\; x_i - x_{i+1}\,, \qquad\qquad \int d^{4-2\epsilon}l\,, \quad l \;=\; x_{0k} \tag{35}$$

  The p_i are dual momenta. In the light-like limit the $g(k, k+1, l, l+1)$ become two-mass easy box integrals and

$$\lim_{x^2_{i,i+1}\to 0} \frac{G_n^{(1)}}{G_n^{(0)}} \;=\; 2\,\frac{A^{(1)}_{\mathrm{MHV}n}}{A^{(0)}_{\mathrm{MHV}n}}\,. \tag{36}$$

- By contrast, *Wilson loops* arise in standard dimensional regularisation

$$\langle\overline{\varphi}(x_i)\,\varphi(x_{i+1})\rangle \;=\; \frac{1}{4\pi^2\left(x^2_{i,i+1}\right)^{1-\epsilon}}\,, \qquad\qquad \int d^{4-2\epsilon}x_0\,. \tag{37}$$

In [9] the argument yielding (34) is repeated in standard dimensional regularisation. It is shown that the light-cone limit yields an n-gon one-loop Wilson loop in the adjoint representation; the integration over the insertion point puts the Yang-Mills line from any gauge into (supersymmetric) Landau gauge. This constitutes a proof of the conjecture of the preceding section at one-loop level, because the vacuum expectation value of any closed Wilson loop is gauge invariant.

5 Two loops

The four-point two-loop correlation functions of hypermultiplet bilinears were studied in [14].

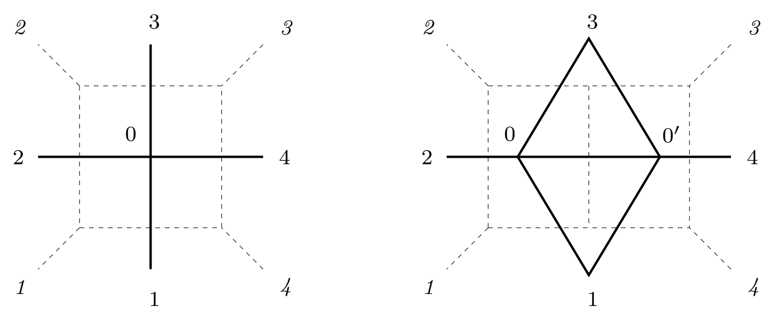
$$\begin{aligned}\lim_{x^2_{i,i+1}\to 0} G_4/G_4^{(0)} \;&=\; 1 + 2\,a\,x^2_{13}x^2_{24}\,g(1,2,3,4) + a^2\Big[(x^2_{13}x^2_{24}\,g(1,2,3,4))^2 \\ &+\; 2\,x^2_{13}x^2_{24}\Big(x^2_{13}\,h(1,2,3;1,3,4) + x^2_{24}\,h(1,2,4;2,3,4)\Big)\Big] + O(a^3)\end{aligned} \tag{38}$$

The four-point two-loop MHV amplitude was published in [15]:

$$A_4/A_4^{(0)} = 1 + a\,x^2_{13}x^2_{24}\,g(1,2,3,4) + a^2\,x^2_{13}x^2_{24}\Big[x^2_{13}\,h(1,2,3;1,3,4) + x^2_{24}\,h(1,2,4;2,3,4)\Big] \tag{39}$$

The new integral in these formulae is the "double-box"

$$h(1,2,3;1,2,4) \;=\; \frac{1}{16\pi^4}\int \frac{d^Dx_0 d^Dx_{0'}}{(x_{10}^2 x_{20}^2 x_{30}^2)x_{00'}^2(x_{10'}^2 x_{20'}^2 x_{40'}^2)}\,. \tag{40}$$



$g(1,2,3,4)$ $h(1,2,3;1,3,4)$

Figure 5: One- and two-loop boxes, momentum representation and dual graphs

Comparing (38) with (39) we see

$$\lim_{x_{i,i+1}^2\to 0} G_4/G_4^{(0)} \;=\; \left(A_4/A_4^{(0)}\right)^2 + \,O(a^3) \tag{41}$$

not only w.r.t. the overall coefficient but also the functional form. The same holds at five points, although this involves an identity between integrals.

What is more, the equivalence between the light-like limit of the correlators and MHV amplitudes holds on the level of the *integrand*, for both the parity-even and the parity odd-parts [10]. Suppose that we derive an l-loop correlator G_n by l Lagrangian insertions:

$$G_n^{(l)}(x_1,\ldots,x_n) \propto \int \prod_{i=1}^{l} d^4z_i\, d^4\rho_i\; G_{n+l;l}^{(0)}(z_1,\ldots,z_l;x_1,\ldots,x_n) \tag{42}$$

The integrand of the off-shell correlator is

$$G_{n+l;l}^{(0)} = \langle L(z_1)\ldots L(z_l)\,O(x_1)\,O(x_2)\ldots O(x_n)\rangle^{\rm tree}\,. \tag{43}$$

Let us denote its light-cone limit as $\mathcal{I}_{n+l} = \lim_{x_{i,i+1}^2\to 0}\,(G_{n+l;l}^{(0)}/G_n^{(0)})$ and the integrand of the l-loops n-point MHV amplitude as $\widehat{\mathcal{A}}_{n+l}$. We conjecture:

$$1 + \sum_{l\geq 1} a^l\, \mathcal{I}_{n+l} = (1 + \sum_{l\geq 1} a^l\, \widehat{\mathcal{A}}_{n+l})^2 \tag{44}$$

Integrands are not uniquely defined. By way of example the parity-odd part in (34) is non-vanishing as a rational function but it integrates to zero due to its antisymmetry. Remarkably, our procedure leads to exactly the same integrand as postulated in the all-loops proposal of [12].

In order to state an example we need to introduce some details of the momentum super-twistor formalism in terms of which the conjecture of [12] is formulated. The outer points (so the vertices of the n-gon) are defined by n momentum twistors Z^a, $a \in \{1,2,3,4\}$, a pair $\{A,B\},\{C,D\},\ldots$ is needed for each integration point.

$$Z^a \,=\, (\lambda^\alpha, \mu^{\dot\alpha})\,, \qquad x_i^{\alpha\dot\alpha} \,=\, \frac{\lambda_i^\alpha \mu_{i+1}^{\dot\alpha} - \lambda_{i+1}^\alpha \mu_i^{\dot\alpha}}{\lambda_i^\beta \lambda_{i+1\,\beta}} \tag{45}$$

and the two integration variables $x_0, x_{0'}$ depend on $\{A,B\},\{C,D\}$. We further need

$$\begin{aligned} \langle AB\rangle &= \lambda_A^\beta \lambda_{B\,\beta}\,, \\ \langle ijkl\rangle &= \det(Z_i, Z_j, Z_k, Z_l)\,, \\ \langle AB\overline{ij}\rangle &= \langle A, i, i-1, i, i+1\rangle\langle B, j, j-1, j, j+1\rangle - (A \leftrightarrow B)\,. \end{aligned} \tag{46}$$

Note that

$$\langle i, i+1, j, j+1\rangle \,=\, \langle i, i+1\rangle\,\langle j, j+1\rangle\, x_{ij}^2 \tag{47}$$

while the determinant is more complicated if the twistors belong to more than two points. The last formula shows clearly that adjacent x_i are in fact light-like separated for any choice of Z's because $x_{i,i+1}^2 \propto \langle i, i+1, i+1, i+2\rangle$ which trivially vanishes because of the repeated argument of the determinant. Hence we can use random complex integers for the $Z_i = (\lambda_i, \mu_i)$ for numerical tests of the conjectured integrand identity between correlators and amplitudes. All expressions remain rational so that *Mathematica* can calculate exactly. Any disagreement would immediately show.

$$\begin{aligned} \widehat{\mathcal{A}}_{5+1} &= \frac{2}{5}\frac{\langle 1234\rangle\langle 2345\rangle\langle AB\rangle^4}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB45\rangle} \\ &+ \frac{\langle AB\overline{25}\rangle\langle 2534\rangle\langle AB\rangle^4}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB45\rangle\langle AB51\rangle} + (\text{cyclic}) \end{aligned} \tag{48}$$

$$\begin{aligned} \widehat{\mathcal{A}}_{5+2} &= \frac{1}{2}\,\frac{\langle 1234\rangle\langle 2345\rangle\langle 5123\rangle\langle AB\rangle^4\langle CD\rangle^4}{\langle AB12\rangle\langle AB23\rangle\langle AB51\rangle\langle ABCD\rangle\langle CD23\rangle\langle CD34\rangle\langle CD45\rangle} \\ &+ \frac{1}{2}\,\frac{\langle 1345\rangle\langle 3451\rangle\langle AB\overline{13}\rangle\langle AB\rangle^4\langle CD\rangle^4}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB51\rangle\langle ABCD\rangle\langle CD34\rangle\langle CD45\rangle\langle CD51\rangle} \\ &+ (AB \leftrightarrow CD) + (\text{cyclic}) \end{aligned} \tag{49}$$

according to [12]. We confirm

$$\mathcal{I}_{5+1} \,=\, 2\,\widehat{\mathcal{A}}_{5+1}\,, \qquad \mathcal{I}_{5+2} \,=\, 2\,\widehat{\mathcal{A}}_{5+2} \,+\, (\widehat{\mathcal{A}}_{5+1})^2\,. \tag{50}$$

The same holds at six points. For these tests, the correlators have been derived using $\mathcal{N}=2$ superfields and two insertions as in [14].

6 Conclusions and Outlook

In $\mathcal{N} = 4$ SYM, a light-like limit sends n-point functions of gauge invariant composite operators to Wilson loops or scattering amplitudes. Which object is obtained depends on the regularisation prescription. The connection to Wilson loops is "manifest" while that to amplitudes needs to be understood. The correlator/amplitude duality works for non-MHV amplitudes, too [11]. In all cases that we studied the higher-loop amplitude integrands of Arkani-Hamed et al. were exactly matched. Last, we remark that the usual x variables were not worse than the momentum twistors when deriving the six-point one-loop NMHV amplitude from a correlator.

In future work we would like to find a field theory based proof for the triangle relation between correlators, Wilson loops and amplitudes. In order to sidestep the difficulties in handling the complicated rational functions we have performed numerical (though exact) tests of the integrand relation between amplitudes and correlators. It would be good to show the exact equality of the integrands by analytic means. It is certainly also worthwhile to analyse the factorisation properties of the residues of the higher-loop correlation functions in some additional limits; such techniques have yielded the recursion relations for amplitude integrands that form the basis of the all-loops conjecture [12].

Our work concerns correlators related to the stress-tensor multiplet of $\mathcal{N} = 4$ SYM. A proof of the correlator/Wilson loop duality was suggested for n-point functions of the Konishi operator in [16]. The duality is in fact quite universal; we may wonder for what other correlators it will work. Further, it remains to explore the strong-coupling regime of the construction.

Last, it would be fascinating to study to what extent off-shell correlators can be determined from amplitudes.

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