Quark Matter with a Chiral Chemical Potential

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1 Introduction

In this talk, we report the results discussed in [1], related to the phase structure of hot quark matter in presence of a background of chiral charge density, n_5. The latter is introduced in the simplest way possible, namely by virtue of a chemical potential, μ_5, conjugated to n_5. In more detail, after a brief introduction and list of motivations of this kind of study, we discuss the interplay between chiral symmetry restoration and deconfinement at finite μ_5, as well as the critical endpoint in the phase diagram and its possible relationship with the critical endpoint of the phase diagram od Quantum Chromodynamics (QCD). In the talk, due to time limitation, we can emphasize few results related to the latter topic. Therefore we need to leave apart several applications of the ideas, as well as of the formalism, developed here to the physics of heavy ion collisions, with particular reference to the Chiral Magnetic Effect [2, 3, 4]. The latter might be relevant for the phenomenology of heavy ion collisions, because a copious production of gluon configurations, the QCD sphalerons, with a finite winding number is expected in the quark-gluon-plasma phase of QCD, see [5] and references therein. Because of the chiral Ward identity, the interaction of the sphalerons with the quarks causes a chirality change of the latters. As a consequence, a copious production of local domains in which chirality is imbalanced, is expected in the quark-gluon-plasma.

The critical endpoint, CP, of QCD [6] is the cornerstone of the phase diagram of strongly interacting matter. At CP, a crossover line and a first order line are supposed to intercept. It is thus not surprising that an intense experimental activity is nowadays dedicated to the detection of such a point, which involves the large facilities at RHIC and LHC; moreover, further experiments are expected after the development of FAIR at GSI. Several theoretical signatures of CP have been suggested [7, 8]. Despite the importance of CP, a firm theoretical evidence of its existence is still missing. In fact, the sign problem makes the Lattice Monte Carlo simulations difficult, if not impossible, in the large baryon-chemical potential (μ) region for $N_c = 3$ [9], see [10] for a recent review. Therefore, it has not yet been possible to prove unambiguously the existence and the location of CP starting from first principles simulations of grand-canonical ensembles. Moreover, the predictions of effective models are spread in the $T - \mu$ plane, see for example [11, 12].

Interesting overcomings of the sign problem for the quest of CP are: analytic continuation of data obtained at imaginary chemical potential, μ_I [13, 14, 15, 16]; simulations at finite isospin chemical potential, see for example [17, 18, 19]; simulations in canonical, rather then grand-canonical, ensembles [20]; strong coupling expansion of Lattice QCD [21, 22]. On the purely theoretical side, it has been suggested [23] that the use of orbifold equivalence in the large N_c approximation of QCD can lead to relations between the coordinates of CP at finite chemical potential, with those at finite isospin chemical potential.

In this talk, we present the idea suggested in [1] about a new theoretical way to detect the CP, by means of Lattice simulations with $N_c = 3$. In order to accomplish this important program, we suggest to simulate QCD with a chiral chemical potential, μ_5, conjugated to the chiral charge density, $n_5 = n_R - n_L$, see [3, 24, 25, 26, 27] for previous studies. Our idea, supported by concrete calculations within acmicroscopic effective model, is that CP can be continued to a critical endpoint at $\mu_5 \neq 0$ and $\mu = 0$, that we denote by CP_5, the latter being accessible to $N_c = 3$ Lattice QCD simulations of grand-canonical ensembles [3, 28]. Therefore, the detection of the former endpoint via Lattice simulations, can be considered as a signal of the existence of the latter.

The model used in the calculation, namely the Nambu-Jona-Lasinio model with the Polyakov loop [29] (PNJL model in the following) with tree level coupling among chiral condensate and Polyakov loop [30], gives numerical relations among the coordinates of CP_5 and those of CP. In particular, the critical temperature turns out to be almost unaffected by the process of continuation; the critical value of the chemical potential, μ_c, on the other hand turns out to be almost half of the critical chiral chemical potential, μ_{5c}.

Before discussing our results, it is important to spend some word more about the chiral chemical potential. In particular, we are aware that world at finite μ_5 should be considered as a fictional one. As a matter of fact, μ_5 cannot be considered as a true chemical potential because, in the confinement phase, the chiral condensate $\langle \overline{q}q \rangle$ mixes left- and right-handed components of the quark field, leading to non-conservation of n_5. Moreover, the quantum chiral anomaly leads to fluctuations of the topological charge, which in turn causes the changes of the chiral density because of the Ward identity. Therefore, the point of view that we adopt is to consider μ_5 as a mere mathematical artifice. However, the world at finite μ_5 with $N_c = 3$ can be simulated on the Lattice. Therefore, it is worth to study it by grand-canonical ensemble simulations: it might furnish an evidence of the existence of the critical endpoint in the real world.

2 The Model with the Polyakov loop

Because of its non-perturbative nature, we cannot make first principles calculations within QCD in the regimes to which we are interested in, namely moderate T, μ and μ_5. Hence we need to rely on some effective model, which is built in order to respect (at least some of) the symmetries of the QCD action. To this end, we make use of the Nambu-Jona-Lasinio model [31] (see [32] for reviews) improved with the Polyakov loop [29], dubbed PNJL model, which has been used many times in recent years to describe successfully the thermodynamics of QCD with two and two-plus-one flavors, see [30, 33, 34, 35, 36, 37, 38, 39, 40] and references therein. The model is interesting because it allows for a self-consistent description of spontaneous chiral symmetry breaking; even more, it allows for a simultaneous computation of quantities sensible to confinement and chiral symmetry breaking. We restrict here to a brief summary of the main equations; we refer to [1] for a more detailed discussion.

In the PNJL model, quark propagation in the medium is described by the following lagrangian density:

$$\mathcal{L} = \overline{q}\left(i\gamma^\mu D_\mu - m + \mu_5\gamma^0\gamma^5 + \mu\gamma^0\right)q + G\left[\left(\overline{q}q\right)^2 + \left(i\overline{q}\gamma_5\boldsymbol{\tau}q\right)^2\right] \ ; \tag{1}$$

In the above equation, q corresponds to a quark field in the fundamental representation of color group $SU(3)$ and flavor group $SU(2)$. We have a introduced chemical potential for the quark number density, μ, and a pseudo-chemical potential conjugated to chirality imbalance, μ_5. The chiral charge density, $n_5 = n_R - n_L$, represents the difference in densities of the right- and left-handed quarks. The imbalance of chiral density can be created by instanton/sphaleron transition in QCD, see [3] and references therein. At finite μ_5, a chirality imbalance is created, namely $n_5 \neq 0$. For example, in the massless limit and at zero baryon chemical potential one has [3]

$$n_5 = \frac{\mu_5^3}{3\pi^2} + \frac{\mu_5 T^2}{3} \ . \tag{2}$$

If quark mass (bare or constituent) is taken into account, the relation $n_5(\mu_5)$ cannot be found analytically in the general case, and a numerical investigation is needed, see for example [24].

In our computation we follow the idea implemented in [30], which brings to a Polyakov-loop-dependent coupling constant:

$$G = g\left[1 - \alpha_1 LL^\dagger - \alpha_2(L^3 + (L^\dagger)^3)\right] \ , \tag{3}$$

The ansatz in the above equation was inspired by [42, 43] in which it was shown explicitly that the NJL vertex can be derived in the infrared limit of QCD, it has a non-local structure, and it acquires a non-trivial dependence on the phase of the Polyakov loop. We refer to [30] for a more detailed discussion. This idea has been

analyzed recently in [44], where the effect of the confinement order parameter on the four-fermion interactions and their renormalization-group fixed-point structure has been investigated. The numerical values of α_1 and α_2 have been fixed in [30] by a best fit of the available Lattice data at zero and imaginary chemical potential of Ref. [45, 46]. In particular, the fitted data are the critical temperature at zero chemical potential, and the dependence of the Roberge-Weiss endpoint on the bare quark mass. The best fit procedure leads to $\alpha_1 = \alpha_2 \equiv \alpha = 0.2 \pm 0.05$.

In the one-loop approximation, the effective potential of this model is given by

$$\begin{aligned} V &= \mathcal{U}(L, L^\dagger, T) + \sigma^2 G - N_c N_f \sum_{s=\pm 1} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \omega_s \\ &\quad - \frac{N_f}{\beta} \sum_{s=\pm 1} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \log\left(F_+ F_-\right) \end{aligned} \tag{4}$$

where

$$\omega_s = \sqrt{(|\boldsymbol{p}|s - \mu_5)^2 + m_q^2}\,, \tag{5}$$

corresponds to the pole of the quark propagator, and

$$F_- = 1 + 3Le^{-\beta(\omega_s - \mu)} + 3L^\dagger e^{-2\beta(\omega_s - \mu)} + e^{-3\beta(\omega_s - \mu)}\,, \tag{6}$$

$$F_+ = 1 + 3L^\dagger e^{-\beta(\omega_s + \mu)} + 3Le^{-2\beta(\omega_s + \mu)} + e^{-3\beta(\omega_s + \mu)}\,, \tag{7}$$

denote the statistical confining thermal contributions to the effective potential; ω_s is given by Equation (5), with $m_q = m - 2G\sigma$. Once again the vacuum fluctuation term is regularized by means of a ultraviolet cutoff, that we denote by M. The relation between the chiral condensate and σ in the PNJL model is $\sigma = \langle \overline{q} q \rangle$.

We notice that the PNJL model considered here, which is dubbed Extended-PNJL in [30], has been tuned in order to reproduce quantitatively the Lattice QCD thermodynamics at zero and imaginary quark chemical potential. Hence, it represents a faithful description of QCD, in terms of collective degrees of freedom related to chiral symmetry breaking and deconfinement.

The potential term $\mathcal{U}$ in Eq. (4) is built by hand in order to reproduce the pure gluonic lattice data with $N_c = 3$ [33]. We adopt the following logarithmic form,

$$\mathcal{U}[L, \overline{L}, T] = T^4 \left\{ -\frac{a(T)}{2} \overline{L} L + b(T) \ln\left[1 - 6\overline{L}L + 4(\overline{L}^3 + L^3) - 3(\overline{L}L)^2\right] \right\}. \tag{8}$$

We refer to [33, 1] for the numerical values of the parameters used in this study.

3 Critical endpoint at zero chemical potential

In Figure 1 we plot the phase diagram of the model in the $\mu_5 - T$ plane, for the case $\mu = 0$. At any value of T and μ_5, the chiral condensate and the Polyakov loop

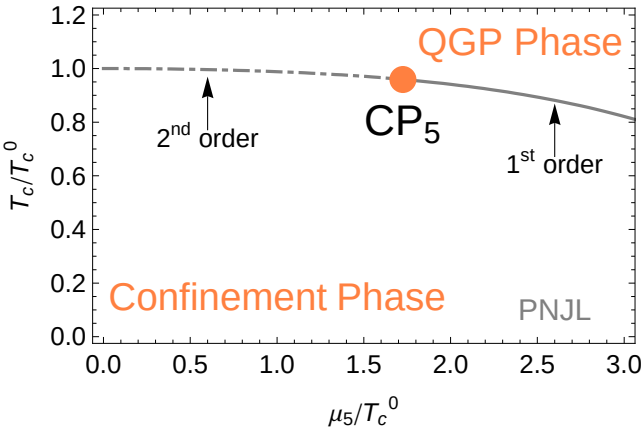

Figure 1: (*Color online*). Phase diagram of the PNJL model. The scale $T_c^0 = 173.9$ MeV corresponds to the critical temperature at $\mu_5 = 0$.

expectation value are computed by a minimization procedure of the full potential (4). The structure of our phase diagram is in agreement with previous model studies, see [24, 25]. Since chiral symmetry is broken explicitly by the quark mass and the phase transitions are actually crossovers, we identify the critical temperature with that at which dL/dT is maximum. We have checked that the latter deviates from that at which $|d\sigma/dT|$ is maximum only of a few MeV, in the whole range of parameters studied. With an abuse of nomenclature, we dub the pseudo-critical lines as second order and first order. It is clear from the context that the term second order transition has to be taken as a synonym of smooth crossover; similarly, the term first order transition is a synonym of discontinuous jump of the order parameters.

In the Figure 1, the grey dashed line corresponds to a smooth crossover. The solid line, on the other hand, denotes the first order transition. The dot corresponds to CP_5. In the PNJL model we have access to the chiral condensate and to the Polyakov loop expectation value. As a consequence, we can label the phases of the model in terms both of confining properties, and of chiral symmetry. In the model at hand, because of the entanglement in Equation (3), the deconfinement and chiral symmetry restoration crossovers take place simultaneously. The region below the pseudo-critical line is characterized by confinement and spontaneous breaking of chiral symmetry; we label this phase as the confinement phase. On the other hand, the phase above the critical line is identified with the Quark-Gluon-Plasma phase. In this case, CP_5 is both *chiral* and *deconfinement* critical endpoint. For what concerns the coordinates of CP_5 we find, for the PNJL model,

$$\left(\frac{\mu_{5c}}{T_c^0}, \frac{T_c}{T_c^0}\right) = (1.73, 0.96)\ , \qquad \mathrm{CP}_5\ (\mathrm{PNJL})\ , \tag{9}$$

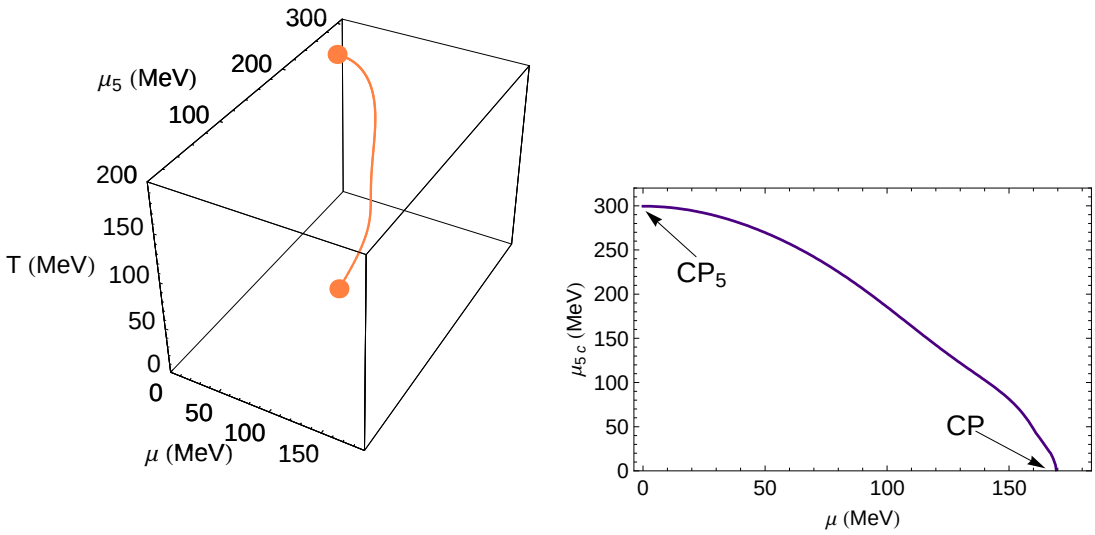

Figure 2: (*Color online*). Evolution of the critical endpoint in the $\mu - \mu_5 - T$ space, for the PNJL model.

where $T_c^0 = 173.9$ MeV is the deconfinement temperature at $\mu = \mu_5 = 0$.

Next we turn to discuss the more general case with both μ_5 and μ different from zero. Our scope is to show that, at least within the models, CP naturally evolves into CP$_5$. In particular the PNJL model, which is in quantitative agreement with the Lattice at zero chemical potential, gives a numerical relation among the coordinates of CP and CP$_5$, which might be taken as a guide to estimate the coordinates of CP in QCD, once CP$_5$ is detected.

In Figure 2 we collect our data on the critical point of the phase diagram in the $\mu - \mu_5 - T$ space, in the case of the PNJL model. The orange solid line is the union of the critical points computed self-consistently at several values of μ: at any value of μ, a point on the line corresponds to the critical point of the phase diagram in the $\mu_5 - T$ plane. Thus the line pictorially describes the evolution of the critical point of the chiral model at hand, from CP to CP$_5$. In the same Figure we plot a projection of the critical endpoint evolution curve onto the $\mu - \mu_5$ plane, for the PNJL model. The indigo solid line corresponds to the μ_5-coordinate of the critical endpoint. The critical temperature is not so much affected when we continue CP$_5$ to CP (we measure a change approximately equal to the 3%), therefore the projection in the $\mu - T$ plane is redundant. Our numerical results suggest the following relationships between the

critical coordinates at $\mu = 0$ and $\mu_5 = 0$:

$$\frac{\mu_c}{\mu_{5c}} \approx 0.53 \ , \quad \frac{T_c}{T_{5c}} \approx 0.97 \ , \qquad \text{(PNJL)} \ . \tag{10}$$

The model predictions (10) relate the coordinates of CP to those of CP_5. In particular, it is interesting that the critical temperature is almost unchanged in the continuation of CP to CP_5. Of course, since these results are deduced by a model, it is extremely interesting and important to study how Equation (10) is affected by the value of the bare quark mass, as well as by further interactions in the vector and axial-vector channels. These topics will be the subject of a forthcoming publication [47]. It is worth to anticipate some of the results of [47], namely that a larger value of the quark mass, as well as the vector interaction, move CP_5 to larger values of μ_5. The combination of these two factor, together with the finite size of the lattice cell, might explain the absence of CP_5 in the Lattice simulations [28].

4 Conclusions and Outlook

In this talk, we have reported on our results about the phase structure of hot quark matter in presence of a background of chiral charge density. Such a background is introduced by virtue of a chemical potential, μ_5, conjugated to the chirality imbalance, $n_5 = n_R - n_L$. Because of the fluctuations of the topological charge, which is connected with chirality imbalance in QCD via the quantum anomaly, μ_5 should be treated as a pure mathematical artifice, and cannot be considered as a true chemical potential.

This study is partly motivated by the potential applications to the Chiral Magnetic Effect [2, 3, 4]. The latter might be relevant for the phenomenology of heavy ion collisions, because a copious production of gluon configurations, the QCD sphalerons, with a finite winding number is expected in the quark-gluon-plasma phase of QCD, see [5] and references therein. Because of the chiral Ward identity, the interaction of the sphalerons with the quarks causes a chirality change of the latters. As a consequence, a copious production of local domains in which chirality is imbalanced, is expected in the quark-gluon-plasma.

After an overview on the phase diagram of hot quark matter at finite μ_5, obtained within an effective model, we have suggested the possibility of continuation of the critical endpoint of the phase diagram of $N_c = 3$ QCD, CP, to a critical endpoint dubbed CP_5 at finite μ_5 and $\mu = 0$. The worldsheet $\mathcal{W}_5 \equiv \{\mu = 0, \mu_5 = 0\}$ has the merit that it can be simulated on the Lattice [3, 28] for $N_c = 3$. Even if Lattice results have been already published [28], more care should be taken in the derivation of the relation between n_5 and μ_5, since n_5 is a nonconserved quantity, hence it suffers renormalization effects which should be taken into account.

The phase structure that we have discussed here is based on the PNJL model with entanglement vertex, introduced in [30], which offers a description of the QCD

thermodynamics in terms of collective degrees of freedom, which is in quantitative agreement with Lattice data at zero and imaginary chemical potential.

One of our ideas is that simulations in the worldsheet $\mathcal{W}_5$ might reveal the existence of a critical endpoint, CP_5, in the phase diagram. Then, this critical point might be interpreted as the continuation of the critical point which is expected to belong to the phase diagram of real QCD, because of the continuity summarized in Fig. 2. Hence it would be an indirect evidence of the existence of the critical point in real QCD.

In our calculations there are some factors that we have not included for simplicity, and that affect the location of CP_5. For example, the bare quark mass and the vector interaction move CP_5 to higher values of μ_5. These observations might be helpful to understand why in the Lattice simulations of [28], no critical endpoint is detected. We plan to report on the aforementioned topics in the next future [47].

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References

[1] M. Ruggieri, Phys. Rev. **D84**, 014011 (2011).

[2] D. E. Kharzeev, L. D. McLerran, H. J. Warringa, Nucl. Phys. **A803**, 227-253 (2008).

[3] K. Fukushima, D. E. Kharzeev and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008).

[4] P. V. Buividovich, M. N. Chernodub, E. V. Luschevskaya, M. I. Polikarpov, Phys. Rev. **D80**, 054503 (2009).

[5] G. D. Moore, M. Tassler, JHEP **1102**, 105 (2011).

[6] M. Asakawa and K. Yazaki, Nucl. Phys. A **504** (1989) 668.

[7] M. A. Stephanov, K. Rajagopal and E. V. Shuryak, Phys. Rev. Lett. **81** (1998) 4816; D. T. Son and M. A. Stephanov, Phys. Rev. D **70** (2004) 056001; H. Fujii,

Phys. Rev. D **67** (2003) 094018; M. A. Stephanov, Phys. Rev. Lett. **102** (2009) 032301; Y. Minami and T. Kunihiro, Prog. Theor. Phys. **122** (2010) 881.

[8] M. A. Stephanov, K. Rajagopal and E. V. Shuryak, Phys. Rev. D **60** (1999) 114028.

[9] Z. Fodor and S. D. Katz, JHEP **0203** (2002) 014; S. Ejiri *et al.*,Prog. Theor. Phys. Suppl. **153** (2004) 118; R. V. Gavai and S. Gupta, Phys. Rev. D **71** (2005) 114014; P. de Forcrand, S. Kim and O. Philipsen, PoS **LAT2007** (2007) 178; P. de Forcrand and O. Philipsen, JHEP **0811** (2008) 012.

[10] P. de Forcrand, PoS **LAT2009**, 010 (2009).

[11] M. A. Stephanov, PoS **LAT2006** (2006) 024.

[12] A. Ohnishi, H. Ueda, T. Z. Nakano, M. Ruggieri, K. Sumiyoshi, [arXiv:1102.3753 [nucl-th]].

[13] M. G. Alford, A. Kapustin, F. Wilczek, Phys. Rev. **D59**, 054502 (1999).

[14] P. de Forcrand, O. Philipsen, Nucl. Phys. **B642**, 290-306 (2002).

[15] M. D'Elia, M. P. Lombardo, Phys. Rev. **D67**, 014505 (2003).

[16] P. de Forcrand, O. Philipsen, JHEP **0811**, 012 (2008); Phys. Rev. Lett. **105**, 152001 (2010).

[17] J. B. Kogut, D. K. Sinclair, Phys. Rev. **D66**, 034505 (2002).

[18] P. de Forcrand, M. A. Stephanov, U. Wenger, PoS **LAT2007**, 237 (2007).

[19] P. Cea, L. Cosmai, M. D'Elia, C. Manneschi, A. Papa, Phys. Rev. **D80**, 034501 (2009).

[20] A. Li, A. Alexandru, K. -F. Liu, [arXiv:1103.3045 [hep-ph]].

[21] T. Z. Nakano, K. Miura and A. Ohnishi, Phys. Rev. D **83** (2011) 016014.

[22] P. de Forcrand, M. Fromm, Phys. Rev. Lett. **104**, 112005 (2010).

[23] M. Hanada, N. Yamamoto, [arXiv:1103.5480 [hep-ph]].

[24] K. Fukushima, M. Ruggieri and R. Gatto, Phys. Rev. D **81**, 114031 (2010).

[25] M. N. Chernodub and A. S. Nedelin, arXiv:1102.0188 [hep-ph].

[26] L. D. McLerran, E. Mottola, M. E. Shaposhnikov, Phys. Rev. **D43**, 2027-2035 (1991); H. B. Nielsen, M. Ninomiya, Phys. Lett. **B130**, 389 (1983); A. N. Sisakian, O. Y. .Shevchenko, S. B. Solganik, [hep-th/9806047]; M. Joyce, T. Prokopec, N. Turok, Phys. Rev. **D53**, 2958-2980 (1996).

[27] C. A. B. Bayona, K. Peeters, M. Zamaklar, JHEP **1106**, 092 (2011).

[28] A. Yamamoto, Phys. Rev. Lett. **107**, 031601 (2011); [arXiv:1108.0937 [hep-lat]].

[29] K. Fukushima, Phys. Lett. **B591**, 277-284 (2004).

[30] Y. Sakai, T. Sasaki, H. Kouno and M. Yahiro, Phys. Rev. D **82**, 076003 (2010).

[31] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); Y. Nambu and G. Jona-Lasinio, Phys. Rev. **124**, 246 (1961).

[32] U. Vogl and W. Weise, Prog. Part. Nucl. Phys. **27**, 195 (1991); S. P. Klevansky, Rev. Mod. Phys. **64**, 649 (1992); T. Hatsuda and T. Kunihiro, Phys. Rept. **247**, 221 (1994); M. Buballa, Phys. Rept. **407**, 205 (2005).

[33] S. Roessner, C. Ratti and W. Weise, Phys. Rev. D **75**, 034007 (2007);

[34] C. Sasaki, B. Friman and K. Redlich, Phys. Rev. D **75**, 074013 (2007); S. K. Ghosh, T. K. Mukherjee, M. G. Mustafa and R. Ray, Phys. Rev. D **77**, 094024 (2008); A. Bhattacharyya, P. Deb, S. K. Ghosh and R. Ray, Phys. Rev. D **82**, 014021 (2010).

[35] H. Abuki, R. Anglani, R. Gatto, G. Nardulli and M. Ruggieri, Phys. Rev. D **78**, 034034 (2008).

[36] K. Kashiwa, H. Kouno, M. Matsuzaki and M. Yahiro, Phys. Lett. B **662**, 26 (2008); Y. Sakai, H. Kouno, T. Sasaki, M. Yahiro, [arXiv:1105.0413 [hep-ph]]; H. Kouno, Y. Sakai, T. Sasaki, K. Kashiwa, M. Yahiro, Phys. Rev. **D83**, 076009 (2011).

[37] T. K. Herbst, J. M. Pawlowski and B. J. Schaefer, arXiv:1008.0081 [hep-ph].

[38] T. Kahara and K. Tuominen, Phys. Rev. D **78**, 034015 (2008); Phys. Rev. D **80**, 114022 (2009); arXiv:1006.3931 [hep-ph].

[39] V. Skokov, B. Friman, K. Redlich, [arXiv:1008.4570 [hep-ph]]; V. Skokov, B. Friman, E. Nakano, K. Redlich, B. -J. Schaefer, Phys. Rev. **D82**, 034029 (2010).

[40] J. O. Andersen, R. Khan, L. T. Kyllingstad, [arXiv:1102.2779 [hep-ph]].

[41] H. D. Politzer, Nucl. Phys. B **117**, 397 (1976); Phys. Lett. B **116**, 171 (1982).

[42] K. I. Kondo, Phys. Rev. D **82**, 065024 (2010).

[43] M. Frasca, arXiv:0803.0319 [hep-th]; arXiv:1002.4600 [hep-ph]; arXiv:1105.5274 [hep-ph].

[44] J. Braun, A. Janot, [arXiv:1102.4841 [hep-ph]].

[45] M. D'Elia and F. Sanfilippo, Phys. Rev. D **80**, 111501 (2009).

[46] C. Bonati, G. Cossu, M. D'Elia and F. Sanfilippo, arXiv:1011.4515 [hep-lat].

[47] R. Gatto and M. Ruggieri, in preparation.