Two-loop Remainder Functions in $\mathcal{N} = 4$ SYM

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1 Introduction

Over the last few years a lot of progress has been made in understanding the structure multi-loop multi-leg scattering amplitudes in the planar $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory, both at weak and at strong coupling. At the heart of this progress are a set of dualities which state that, loosely speaking, scattering amplitudes in $\mathcal{N} = 4$ SYM are equal to Wilson loops and correlations functions of certain gauge invariant operators [2, 3, 4, 5].

The duality between scattering amplitudes and Wilson loops was first uncovered at strong coupling, where it was shown that n-point color-ordered amplitudes with a special helicity assignment for the external gluons, the so called maximally-helicity violating (MHV) amplitudes, are equal to Wilson loops computed along a lightlike polygonal contour [1]. A similar duality was then also found to hold on the weak coupling side [6], and has by now been proven to hold for all one-loop MHV amplitudes [7], as well as for the four, five, and six-point two-loop amplitudes [8, 9, 10, 11, 12]. Recently, a supersymmetric version of the Wilson loop has been proposed [13, 14] which reproduces correctly the integrand of loop amplitudes in $\mathcal{N} = 4$ SYM, hence extending the duality to non-MHV cases.

The lightlike polygonal Wilson loops dual to scattering amplitudes possess a (dual) conformal symmetry. This symmetry is, however, broken by the cusp singularities of the Wilson loops, themselves dual to the infrared divergences of the scattering amplitude. The (logarithms of the) Wilson loops then satisfy an anomaly equation, whose solution is given by the cusp anomalous dimension multiplied by the one-loop correction to the Wilson loop, augmented by an arbitrary finite function of conformal ratios, termed the remainder function. For four and five edges, the conformal symmetry forces the remainder function to be zero, whereas it is known to be non-zero starting from six edges. Improving our understanding of multi-loop multi-leg scattering amplitudes in $\mathcal{N} = 4$ SYM is thus equivalent to increasing our knowledge of Wilson loop remainder functions.

In this paper we give a review of the currently available results for remainder functions at two-loops and beyond. The paper is organized as follows: In Section 2 we give a review of Wilson loops in planar $\mathcal{N}=4$ SYM. In Section 3 we review the available analytic results at two-loops for the hexagon remainder function in general kinematics and for all remainder functions in special two-dimensional kinematics, before commenting on recent development beyond two-loops in Section 4. Finally, in Section 5 we draw our conclusions.

2 Wilson loops in $N = 4$ SYM

Wilson loops are defined through the path-ordered exponential,

$$W[\mathcal{C}_n] = \mathrm{Tr}\; P\; \exp\left[ig \oint_{\mathcal{C}_n} \mathrm{d}x^\mu A_\mu(x)\right] , \tag{1}$$

where $\mathcal{C}_n$ denotes an n-edged polygon. The cusps of the polygon are denoted by x_i, $i = 1, \ldots, n$, and the edges correspond to the momenta $p_i = x_i - x_{i+1}$ in the original scattering amplitude. Momentum conservation, $\sum_{i=1}^n p_i = 0$, then forces the polygon to close, provided that we make the identification $x_{n+1} = x_1$.

Through the non-abelian exponentiation theorem [15, 16], the vacuum expectation value of a Wilson loop can be written as an exponential,

$$\langle W[\mathcal{C}_n]\rangle = 1 + \sum_{L=1}^{\infty} a^L W_n^{(L)} = \exp \sum_{L=1}^{\infty} a^L w_n^{(L)} , \tag{2}$$

where the coupling is defined as $a = \frac{g^2 N}{8\pi^2}$. At one-loop level, the duality between scattering amplitudes and Wilson loops states that these two quantities are equal up to a constant [6, 7],

$$w_n^{(1)} = m_n^{(1)} - n\frac{\zeta_2}{2} + \mathcal{O}(\epsilon) , \tag{3}$$

where $m_n^{(1)}$ denotes the one-loop MHV amplitude rescaled by the tree amplitude, given as a sum of two-mass easy box functions [17]. The conformal Ward identities imply the following expression for $w_n^{(L)}$ [9],

$$w_n^{(L)}(\epsilon) = f^{(L)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C^{(L)} + R_n^{(L)} + \mathcal{O}(\epsilon) , \tag{4}$$

where $f^{(L)}(\epsilon)$ is related to the L-loop correction to the cusp anomalous dimension, $C^{(L)}$ is a constant and $R_n^{(2)}$, termed the remainder function, is a finite function of conformal cross ratios,

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2} , \tag{5}$$

where $x_{ij}^2 = (x_i - x_j)^2$, but is not constraint by conformal symmetry otherwise and must hence be computed.

3 Analytic results for two-loop remainder functions

As already mentioned in the introduction, the first non-trivial remainder function $R_6^{(2)}$ appears in the two-loop correction to the polygonal Wilson loop with six lightlike edges. $R_6^{(2)}$ is a totally symmetric function of the three conformally invariant cross ratios

$$u_1 = u_{13} = \frac{x_{13}^2\, x_{46}^2}{x_{36}^2\, x_{41}^2}\,, \quad u_2 = u_{14} = \frac{x_{15}^2\, x_{24}^2}{x_{14}^2\, x_{25}^2}\,, \quad u_3 = u_{25} = \frac{x_{26}^2\, x_{35}^2}{x_{25}^2\, x_{36}^2}\,, \tag{6}$$

but its functional form is not fixed otherwise. The first fully analytic result for this quantity was obtained by Del Duca, Smirnov and the author in Ref. [18, 19], where all the integrals that contribute to the two-loop correction to the hexagonal Wilson loop were computed explicitly. The computation was made possible by exploiting the fact that lightlike polygonal Wilson loops in $\mathcal{N}=4$ SYM are Regge exact [18], *i.e.*, it is enough to compute the analytic expression for the Wilson loop in some restricted kinematical regime (the so-called Regge limit) where the computation is simpler, and the expression obtained in this way is valid in general kinematics. The result of Ref. [18, 19] was rather lengthy and expressed through a complicated combination of multiple polylogarithms [20] with algebraic arguments involving square roots of the kinematic invariants. It was later rewritten in a much more compact form by Goncharov, Spradlin, Vergu and Volovich in Ref. [21] by using the symbol map, a linear map $\mathcal{S}$ that associates a certain tensor to an iterated integral, and thus to a multiple polylogarithm. In the following we give a very brief summary of the symbol technique, referring to Ref. [21] for further details. As an example, the tensor associated to the classical polylogarithm $\text{Li}_n(x)$ is,

$$\mathcal{S}(\text{Li}_n(x)) = -(1-x)\otimes \underbrace{x\otimes\ldots\otimes x}_{(n-1)\text{ times}}\,. \tag{7}$$

Furthermore, the tensor maps products that appear inside the tensor product to a sum of tensors,

$$\ldots\otimes(x\cdot y)\otimes\ldots = \ldots\otimes x\otimes\ldots + \ldots\otimes y\otimes\ldots\,. \tag{8}$$

It is conjectured that all the functional identities among (multiple) polylogarithms are mapped under the symbol map $\mathcal{S}$ to algebraic relations among the tensors. The result obtained in Ref. [21] takes the very simple form

$$\begin{aligned} R_6^{(2)}(u_1,u_2,u_3) &= \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\text{Li}_4\left(1-1/u_i\right)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\text{Li}_2\left(1-1/u_i\right)\right)^2 \\ &\quad + \frac{1}{24}J^4 + \chi\frac{\pi^2}{12}J^2 + \chi\frac{\pi^4}{72}\,, \end{aligned} \tag{9}$$

with $x_i^{\pm} = u_i\, x^{\pm}$ and

$$x^{\pm} = \frac{u_1 + u_2 + u_3 - 1 \pm \sqrt{\Delta}}{2u_1u_2u_3}\,, \tag{10}$$

where $\Delta = (1 - u_1 - u_2 - u_3)^2 - 4u_1u_2u_3$. Furthermore, the functions appearing in Eq. (9) are defined by

$$L_4(x^+, x^-) = \sum_{k=0}^{3} \frac{(-1)^k}{(2k)!!} \ln^k(x^+x^-)\,(\ell_{4-k}(x^+) + \ell_{4-k}(x^-)) + \frac{1}{8!!}\ln^4(x^+x^-)\,, \tag{11}$$

with

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n \mathrm{Li}_n(1/x)\right) \quad \text{and} \quad J = \sum_{i=1}^{3}\Big(\ell_1(x_i^+) + \ell_1(x_i^-)\Big)\,, \tag{12}$$

and

$$\chi = \begin{cases} -2\,, & \Delta < 0 \text{ and } u_1 + u_2 + u_3 > 1\,, \\ 1\,, & \text{otherwise}\,. \end{cases} \tag{13}$$

Finally, it was shown in Ref. [21] that the square roots in Eq. (10) can be interpreted as cross ratios in momentum twistor space, *e.g.*,

$$x_1^+ = -\frac{\langle 3456\rangle\langle 1245\rangle}{\langle 1456\rangle\langle 2345\rangle}\,. \tag{14}$$

Momentum twistors were introduced by Hodges in Ref. [22] and are four-component objects Z_i living in a three-dimensional complex projective space. They provide a way to encode the kinematics of a massless scattering, the kinematic invariants being related to the determinants formed out of four twistors,

$$x_{ij}^2 \sim \langle (i-1)i(j-1)i\rangle\,, \tag{15}$$

with

$$\langle ijkl\rangle = \det(Z_i\, Z_j\, Z_k\, Z_l) = \begin{pmatrix} Z_i^1 & Z_j^1 & Z_k^1 & Z_l^1 \\ Z_i^2 & Z_j^2 & Z_k^2 & Z_l^2 \\ Z_i^3 & Z_j^3 & Z_k^3 & Z_l^3 \\ Z_i^4 & Z_j^4 & Z_k^4 & Z_l^4 \end{pmatrix}\,. \tag{16}$$

For an introduction to momentum twistors we refer to Ref. [23].

Beyond six points at two-loops, no analytic expression for two-loop remainder functions similar to Eq. (9) is currently known. However, if the kinematics is restricted to two-dimensions (*i.e.*, all the external momenta lie inside a common two-dimensional subspace), it is possible to write down a closed form for all two-loop remainder functions with an even number of edges[1] [24, 25],

$$R_n^{(2)} = -\frac{1}{2}\sum_S \ln u_{i_1i_5} \ln u_{i_2i_6} \ln u_{i_3i_7} \ln u_{i_4i_8} - (n-4)\frac{\pi^4}{72}\,, \tag{17}$$

[1] Note that there are no lightlike polygonal Wilson loops with an odd number of edges in two dimensional kinematics

where

$$S = \{i_1, \dots, i_8 : 1 \le i_1 < i_2 < \dots < i_8 \le n, i_k - i_{k-1} = \text{odd}\}\,. \tag{18}$$

4 Remainder functions with more loops and legs

While the results presented in the previous section are the only fully analytic results for remainder functions available at the moment, there is a vast activity in trying to compute remainder functions with more loops and legs. In this context, the symbols of all two-loop remainder functions have recently been computed [26], but the question of finding the functions associated to these symbols is still open. Furthermore, the symbols do not fix the function completely. Indeed, the symbol map has a non-trivial kernel, and in particular all terms proportional to zeta values vanish under the symbol map. Other approaches, based on an operator product expansion around the collinear limit of scattering amplitudes [27], have also been considered and were shown to reproduce the known analytic results for two-loop remainder functions presented in the previous section [28, 29]. However, none of these approaches was able so far to fix the functional form of the two-loop remainder functions in general kinematics beyond six points.

Beyond two-loops, conjectures have been made for the symbols of the three-loop six-point remainder function in general kinematics [30] and for the three-loop octagon in two-dimensional kinematics [31]. Making some assumptions on the entries that should appear in the symbol, the authors of Ref. [30, 31] constrained the form of the symbol by imposing physical constraints. However, in both cases the constraints were not enough to fix the form of the symbol completely, but it could only be fixed up to some free parameters that cannot be determined solely from some purely general considerations.

5 Conclusion

In this paper we provided a review of two-loop remainder functions in $\mathcal{N} = 4$ SYM, a necessary building block to improve our understanding of multi-loop multi-leg scattering amplitudes. We started by reviewing the available analytic results for two-loop remainder functions at six-points in general kinematics and for all two-loop remainder functions in two-dimensional kinematics. Beyond these cases, however, no explicit analytic results are currently known. While it was possible to determine the symbol of the answer in some cases, we are still lacking the functional form of remainder functions with more loops and / or legs. In the quest of gaining a deeper understanding of what kind of functions could appear in higher-point two-loop cases, it was noted that the scalar massless one-loop hexagon integral in $D = 6$ dimensions [32, 33] admits a very simple and compact analytic expression, which is very close to the result (9) for the two-loop hexagon remainder function, albeit only involving polylogarithms of lesser weight. This observation spurred the computation of scalar one-loop hexagon integrals in $D = 6$ dimensions with massive external lines [34, 35], in the hope that these functions should be related to higher-point two-loop remainder functions in a similar way as the corresponding massless integral is related to the six-point remainder function. The conclusion was that all these hexagon integrals admit a very compact analytic expression very similar to the massless case, fostering our hope that similar simple results must exist for all two-loop remainder functions in $\mathcal{N} = 4$ SYM.

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