

Classical Particles Having Complex Energy Exhibit Quantum-Like Behavior

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1 Introduction

Because the energy of a quantum particle cannot be determined exactly unless there is an infinite amount of time in which to perform the measurement, there will always be some uncertainty ΔE in the measured energy of a particle. We consider here the possibility that ΔE may be complex. To understand the effect of a particle having a complex energy, we study the behavior of a *classical* particle in a one-dimensional periodic potential $V(x) = -\cos(x)$. Detailed numerical simulations suggest that if the energy of such a particle is allowed to be complex, the classical motion of the particle exhibits quantum-like behavior. A particle in such a periodic potential displays two qualitatively different behaviors:

- *Localized tunneling behavior* for which the particle hops from well to adjacent well, either to the right or to the left. This hopping resembles the tunneling of a quantum particle from site to site and is a deterministic random walk. This kind of behavior is what is usually observed for an arbitrary choice of complex energy. The motion resembles that of a quantum particle in a crystal, where the energy of the particle is not in a conduction band.
- *Delocalized conduction* for which the particle drifts through the potential in one direction. This classical motion resembles the resonant tunneling behavior of a quantum particle in a crystal, where the energy of the particle lies in a conduction band.

2 Complex Classical Mechanics

It is possible to extend the Hamiltonian that describes a quantum system into the complex domain while still retaining the fundamental properties of a quantum theory

[1, 2]. Complex quantum mechanics has proved to be so interesting that research activity in this area has motivated studies of complex classical mechanics. Early work on particle trajectories in complex classical mechanics is reported in Refs. [3, 4]. Subsequently, studies of the complex extensions of conventional classical-mechanical systems were undertaken: The remarkable properties of complex classical trajectories were examined in Refs. [5, 6, 7, 8, 9]; the complex behavior of the pendulum, the Lotka-Volterra equations for population dynamics and the Euler equations for rigid body rotation were studied in Refs. [10, 11]; the complex Korteweg-de Vries equation was examined in Refs. [12, 13, 14, 15]; the complex Riemann equation was examined in Ref. [16]; complex Calogero models were examined in Ref. [17]; the complex extension of chaotic behavior was recently discussed in Ref. [18].

In this paper we describe a recently discovered aspect of complex classical mechanics; namely, that there are similarities between complex classical mechanics and quantum mechanics: Bender, Brody, and Hook performed numerical studies to show that standard quantum effects such as tunneling could be displayed by classical systems having complex energy [19]. In Ref. [19] it is shown that when the energy is real, the classical trajectories in the complex plane are closed and periodic but that when the energy is complex, the classical trajectories are open and nonperiodic [11].

In Ref. [19] these features are illustrated by using the the classical anharmonic oscillator, whose Hamiltonian is $H = \frac{1}{2}p^2 + x^4$. When the classical energy is real, the classical orbits in the complex plane are all closed and periodic (see Fig. 1), but if the energy is complex, the classical orbits are no longer closed and they spiral outward to infinity (see Fig. 2). We emphasize that closed orbits are associated with real energy and open orbits are associated with complex energy.

To understand heuristically the connection between real energy and closed orbits and the connection between complex energy and open orbits we recall the Bohr-Sommerfeld quantization formula. This quantization formula has the form

$$\oint p dx \sim \left(n + \frac{1}{2}\right) \pi \quad (n \gg 1). \quad (1)$$

This integral is performed along a closed classical orbit and gives a semiclassical approximation to the energies of a Hamiltonian for large quantum number. This formula uses particle-wave duality and assumes that there are an integral number of wavelengths along the classical orbit. For the anharmonic-oscillator Hamiltonian considered in Figs. 1 and 2 this formula gives the (real) spectrum of this Hamiltonian, whether or not the classical orbit lies on the real axis in the complex- x plane. However, the formula cannot be applied if the orbit is open and the energy is complex.

In Ref. [19] it was seen that it is possible to observe tunneling-like behavior in classical mechanics when the classical energy of a particle is taken to be complex. The double-well anharmonic oscillator $H = p^2 + x^4 - 5x^2$ was used to illustrate this observation. If the energy of a classical particle is taken to be real, the classical

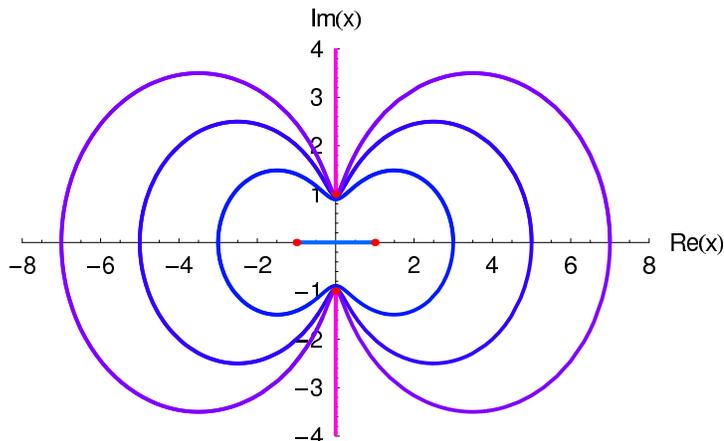


Figure 1: Classical trajectories $x(t)$ in the complex- x plane for the anharmonic-oscillator Hamiltonian $H = \frac{1}{2}p^2 + x^4$. All trajectories represent a particle of energy $E = 1$. One real trajectory oscillates between the turning points at $x = \pm 1$. In addition, an infinite family of nested complex trajectories, which enclose the real turning points but lie inside the imaginary turning points at $\pm i$, are shown. The turning points are indicated by dots. Two other trajectories begin at the imaginary turning points and go off to infinity along the imaginary- x axis. Apart from the trajectories beginning at $\pm i$, all trajectories are closed and periodic. All orbits in this figure have the same period $\sqrt{\pi/2} \Gamma\left(\frac{1}{4}\right) / \Gamma\left(\frac{3}{4}\right) = 3.70815 \dots$

motion is periodic and the orbits lie either on the left side or on the right side of the imaginary axis, which separates the two wells (see Fig. 3).

However, if the energy of a classical particle whose motion is determined by the Hamiltonian $H = p^2 + x^4 - 5x^2$ is taken to be complex, the classical trajectory is no longer closed. Rather, the trajectory spirals outward, crosses the imaginary axis, and then spirals *inward* into the companion well. It then repeats this process, alternately visiting one well and then the other. This motion is shown in Fig. 4. Note that the trajectory never crosses itself. This strange-attractor-like behavior resembles quantum-mechanical tunneling. Also, it provides a heuristic answer to the question, How did the particle get from one well to the other? In a tunneling experiment the particle disappears from one well and then appears in the other well; what path did the particle follow to get from one well to the other? Figure 4 is that the particle can travel smoothly from one well to the other if the trajectory lies in the complex plane.

The heuristic picture proposed in Ref. [19] is that there is some quantum uncertainty in the measurement of the energy of a particle, $(\Delta E)(\Delta t) \sim \hbar$, because an energy measurement is performed in a finite, rather than an infinite, amount of time. If this uncertainty in the energy has an imaginary component, then the classical pic-

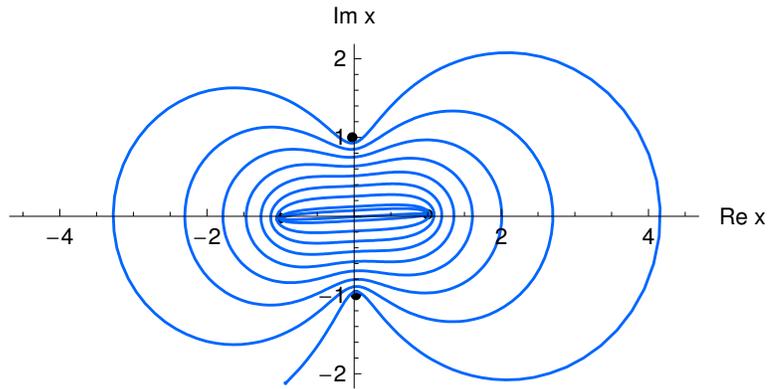


Figure 2: A single classical trajectory in the complex- x plane for a particle governed by the anharmonic-oscillator Hamiltonian $H = \frac{1}{2}p^2 + x^4$. This trajectory begins at $x = 1$ and represents the complex path of a particle whose energy $E = 1 + 0.1i$ is complex. The trajectory is not closed or periodic. The four turning points are indicated by dots. The trajectory cannot cross itself.

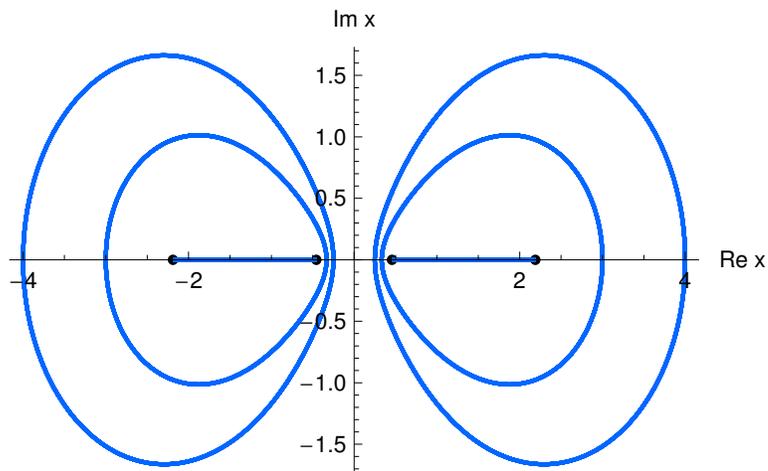


Figure 3: Six classical trajectories in the complex- x plane for a particle of energy $E = -1$ in the potential $x^4 - 5x^2$. The turning points are located at $x = \pm 2.19$ and $x = \pm 0.46$ and are indicated by dots. Because the energy is real, the trajectories are all closed. The classical particle stays in either the right-half or the left-half plane and cannot cross the imaginary axis. Thus, when the energy is real, there is no effect analogous to tunneling.

ture of particle motion and the quantum concept of particle motion come closer to one another.

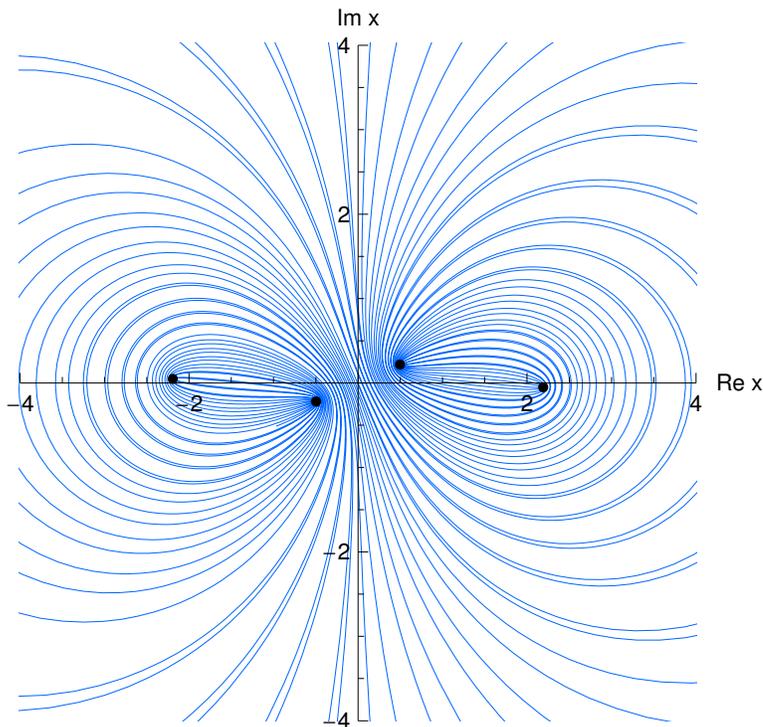


Figure 4: Classical trajectory of a particle moving under the influence of a double-well $x^4 - 5x^2$ potential in the complex- x plane. The particle has complex energy $E = -1 - i$. Its trajectory does not close; rather it spirals outward around one pair of turning points, crosses the imaginary axis, and then spirals inward around the other pair of turning points. It then spirals outward again, crosses the imaginary axis, and goes back to the original pair of turning points. The particle repeats this behavior endlessly but at no point does the trajectory cross itself. This classical-particle motion is analogous to the behavior of a quantum particle that repeatedly tunnels between two classically allowed regions. Here, the particle does not disappear into the classically forbidden region during the tunneling process; rather, it moves along a path in the complex- x plane from one well to the other.

This paper explains how a classical particle whose energy is complex behaves in a *periodic* potential. In Ref. [19] numerical experiments were performed for the potential $V(x) = -\cos(x)$ which suggested that depending on the real and the imaginary parts of the energy of a particle one can observe two kinds of classical motion: (i) First, the particle may hop from classically-allowed site to nearest-neighbor classically-allowed site in the potential, behaving as if it were a quantum particle in an energy gap and undergoing repeated tunneling processes. (ii) Second, the particle may behave as a quantum particle in a conduction band and drift at a constant average velocity

through the potential as if it were undergoing resonant tunneling. Here, the classical conduction bands for this potential are determined numerically with great precision.

These two kinds of behaviors were conjectured in Ref. [19] on the basis of limited numerical experiments. An extremely detailed numerical study of the behavior of a classical particle in a periodic potential well was then reported in Ref. [20]. This study shows that the types of behavior that were conjectured in Ref. [19] do indeed occur. More importantly, there are well-defined and narrow energy bands with precise edges in which “conduction” takes place.

3 Complex motion in a periodic potential

We consider here the Hamiltonian

$$H(x, p) = \frac{1}{2}p^2 - \cos(x) \quad (2)$$

and solve the associated Hamilton’s equations of motion,

$$\frac{dx}{dt} = p \quad \text{and} \quad \frac{dp}{dt} = -\sin(x), \quad (3)$$

to determine the kinds of classical behaviors that arise. If we choose a value of the classical energy at random, say $E = 0.1 - 0.15i$, we find that the classical particle usually tunnels from well to well following a deterministic random walk (see Fig. 5). On the other hand, if we search carefully for a conduction band, we see the classical particle drift through the lattice in one direction only (see Fig. 6).

After running continuously on a computer for several months to determine which complex energies give rise to tunneling (hopping) behavior and which complex energies produce conduction-like behavior, we have found that conduction behavior occurs when the energy lies in narrow bands with crisp well-defined edges (see Fig. 7). To distinguish between hopping behavior and conduction behavior we allowed the particle to undergo ten tunneling incidents, and if the particle always tunneled in the same direction we classified the energy as lying in a conduction band. (In Ref. [19], only seven tunneling incidents were used.) Figure 7 reports the major result in this paper.

Two detailed blow-up views of the complex-energy plane shown in Fig. 7 are given in Figs. 8 and 9. In Fig. 8 a region of the complex- E plane is shown for which $0.28 < \text{Re } E < 0.31$ and $-0.72 < \text{Im } E < -0.71$, and in Fig. 9 a portion of the complex- E plane is shown for which $.69 < \text{Re } E < 0.72$ and $-0.26 < \text{Im } E < -0.25$. These figures indicate that the border between hopping behavior and conduction behavior is clean and crisp.

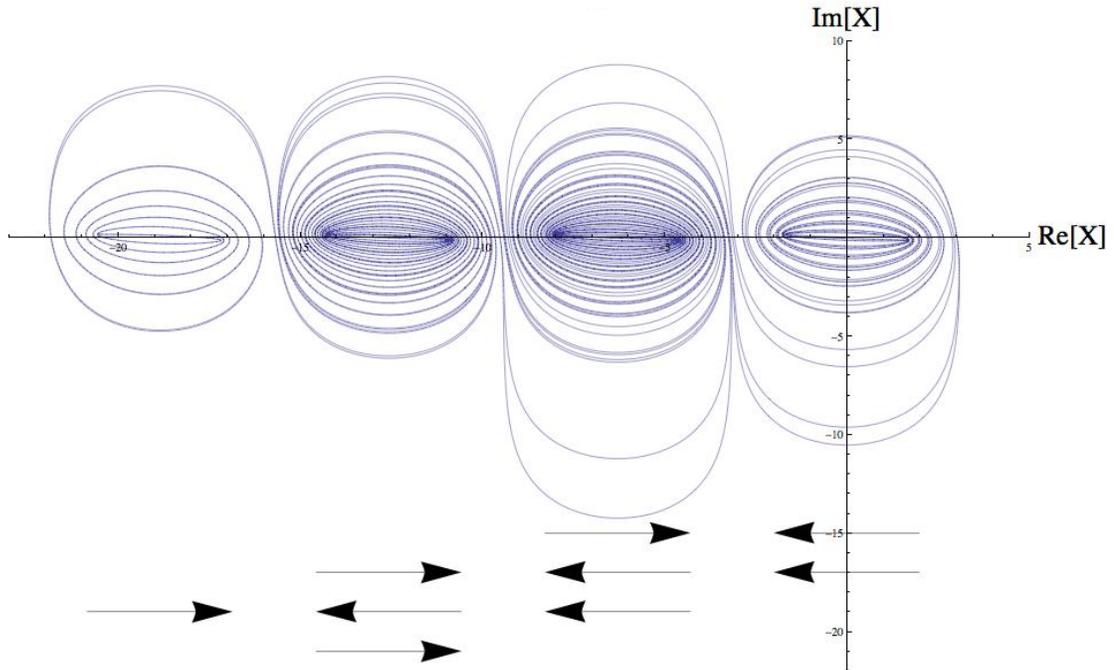


Figure 5: A tunneling trajectory for the Hamiltonian (2) with $E = 0.1 - 0.15i$. The classical particle hops at random from well to well in a random-walk fashion. The particle starts at the origin and then hops left, right, left, left, right, left, left, right, right. This is the sort of behavior normally associated with a particle in a crystal at an energy that is not in a conduction band. At the end of this simulation the particle is situated one site to the left of its initial position. The trajectory never crosses itself.

4 Concluding remarks

We know that quantum-mechanical amplitudes are determined by performing infinite sums over classical configurations. However, tunneling is not thought of as a possible classical behavior. Thus, the usual view is that quantum phenomena can appear only when an infinite sum over classical behaviors is performed. However, in this paper we have argued that some aspects of quantum behavior (such as bands and gaps in periodic potentials) can already be seen in the context of classical mechanics. As an example, we can already see in classical mechanics a version of the time-energy uncertainty principle. In Fig. 10 we have plotted the tunneling time versus the size of the imaginary part of the energy. The graph is a hyperbola, and thus the product of these two quantities is a constant.

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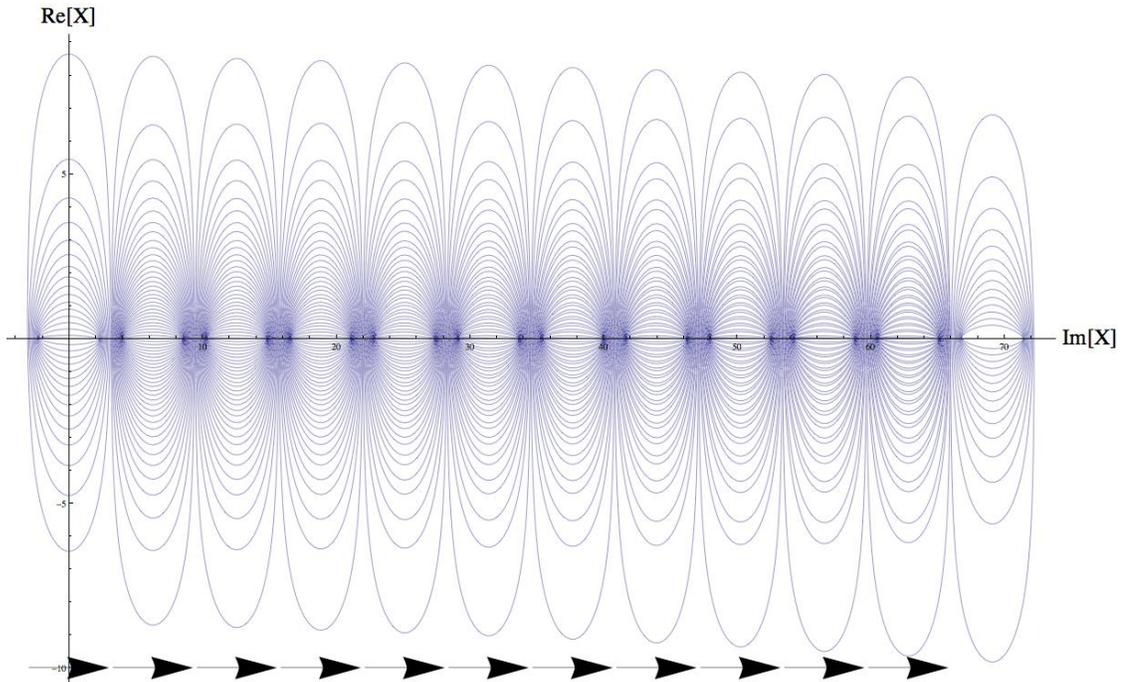


Figure 6: A classical particle exhibiting a behavior analogous to that of a quantum particle in a conduction band. The classical particle behaves as if it is undergoing resonant tunneling. Unlike the particle in Fig. 5, this classical particle tunnels in one direction only and drifts at a constant average velocity through the potential.

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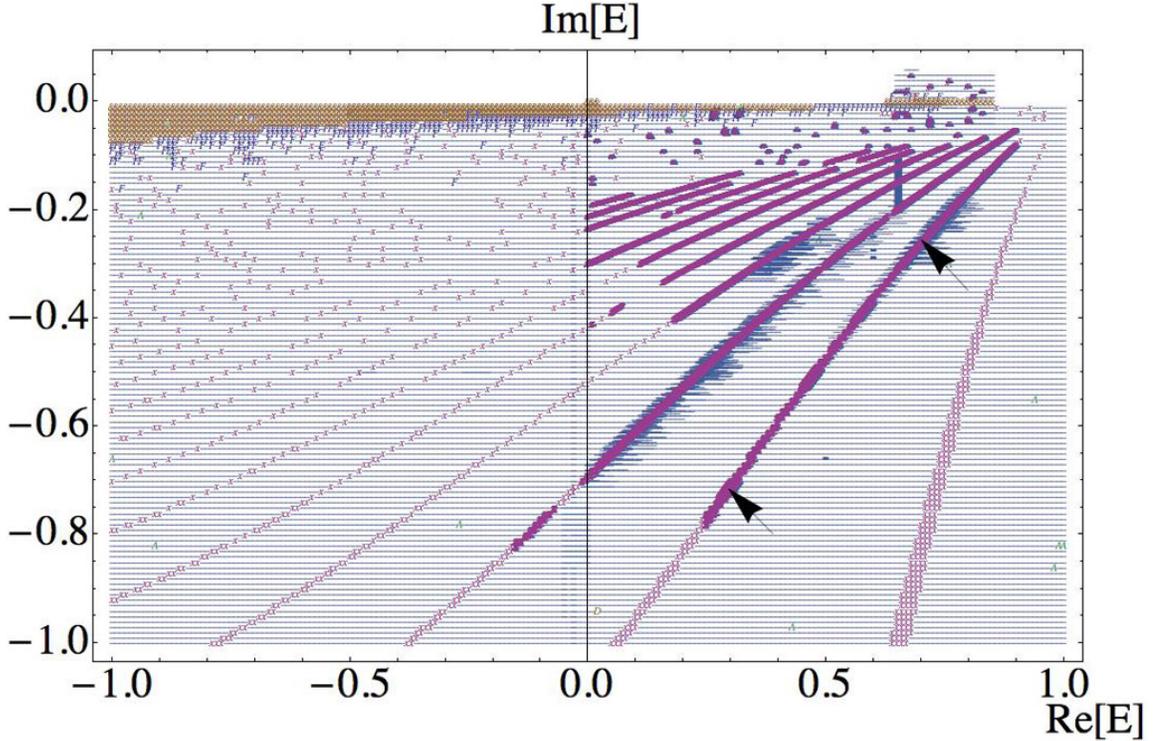


Figure 7: Complex-energy plane showing those energies that lead to tunneling (hopping) behavior and those energies that give rise to conduction. Hopping behavior is indicated by a hyphen - and conduction is indicated by an X. The symbol & indicates that no tunneling takes place; tunneling does not occur for energies whose imaginary part is close to 0. In some regions of the energy plane we have done very intensive studies and the X's and -'s are densely packed. This picture suggests the features of band theory: If the imaginary part of the energy is taken to be -0.9 , then as the real part of the energy increases from -1 to $+1$, five narrow conduction bands are encountered. These bands are located near $\text{Re } E = -0.95, -0.7, -0.25, 0.15, 0.7$. This picture is symmetric about $\text{Im } E = 0$ and the bands get thicker as $|\text{Im } E|$ increases. A total of 68689 points were classified to make this plot. In most places the resolution (distance between points) is 0.01, but in several regions the distance between points is shortened to 0.001. The regions indicated by arrows are blown up in Figs. 8 and 9.

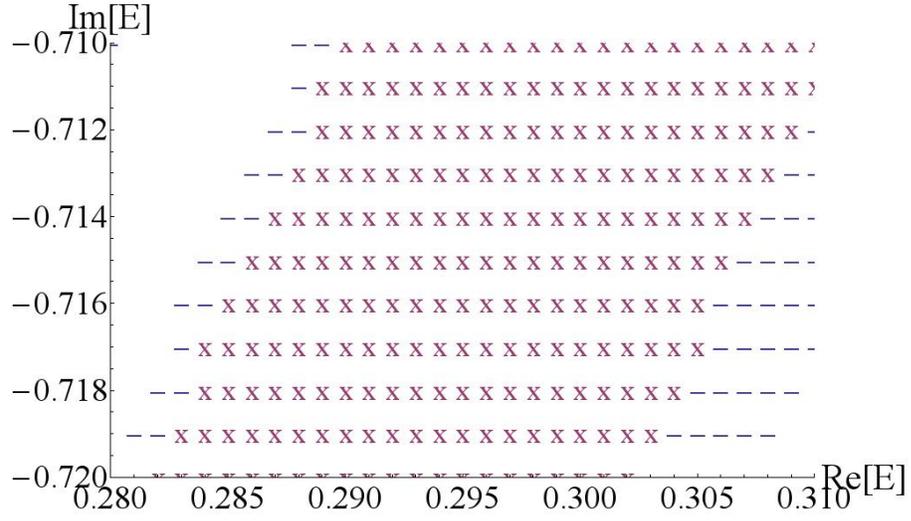


Figure 8: Detailed portion of the complex-energy plane shown in Fig. 8 containing a conduction band. Note that the edge of the conduction band, where tunneling (hopping) behavior changes over to conducting behavior, is very sharp.

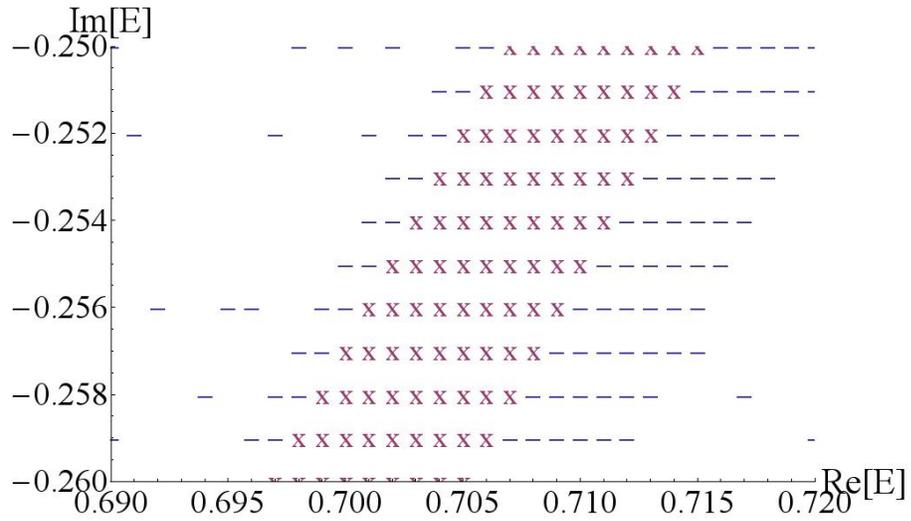


Figure 9: Like Fig. 8, a detailed portion of the complex-energy plane in Fig. 8 containing a conduction band. Note that the edges of the conduction band are sharp.

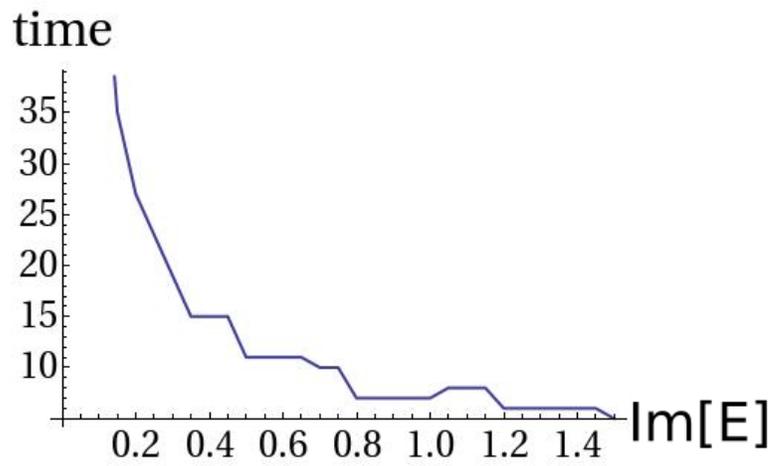


Figure 10: Tunneling time as a function of the imaginary part of the energy. The curve is a hyperbola; that is, the product of the imaginary part of the energy and the tunneling time is a constant. This result is strongly reminiscent of the time-energy uncertainty principle.