EXPRESSING A TENSOR PERMUTATION MATRIX $p^{\otimes n}$ IN TERMS OF THE GENERALIZED GELL-MANN MATRICES

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Abstract

We have shown how to express a tensor permutation matrix $p^{\otimes n}$ as a linear combination of the tensor products of $p \times p$-Gell-Mann matrices. We have given the expression of a tensor permutation matrix $2 \otimes 2 \otimes 2$ as a linear combination of the tensor products of the Pauli matrices.

Introduction

The expression of the tensor commutation matrix $2 \otimes 2$ as a linear combination of the tensor products of the Pauli matrices

$$U_{2\otimes 2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{2} I_2 \otimes I_2 + \frac{1}{2} \sum_{i=1}^{3} \sigma_i \otimes \sigma_i$$

with I_2 the 2×2 unit matrix, [1], [2] are frequently found in quantum theory. The tensor commutation matrix $3 \otimes 3$ is expressed as a linear combination of the tensor products of the 3×3-Gell-Mann matrices [3]

$$U_{3\otimes 3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{3} I_3 \otimes I_3 + \frac{1}{2} \sum_{i=1}^{8} \lambda_i \otimes \lambda_i$$

and as generalization of these two formulae the tensor commutation matrix $p \otimes p$ is expressed as a linear combination of the tensor products of the $p \times p$-Gell-Mann matrices[4]

$$U_{p\otimes p} \;=\; \frac{1}{p} I_p \otimes I_p + \frac{1}{2}\sum_{a=1}^{p^2-1} \Lambda_a \otimes \Lambda_a \tag{1}$$

where I_p the $p \times p$-unit matrix.

It is natural to think to more general formulae in the direction from tensor commutation matrix to tensor permutation matrix. However, the aim of this paper is not to construct a more general formula but to show only how to express a tensor permutation matrix $p^{\otimes n}$ as a linear combination of the tensor products of the $p \times p$-Gell-Mann matrices, with for $p = 2$ the expression is in terms of the Pauli matrices.

In the firt four sections we will construct the tools which we will need for the examples in the last section. The main idea is to decompose the tensor permutation matrix $p^{\otimes n}$ as a product of some tensor transposition matrices.

1 Tensor product of matrices

Theorem 1 *Consider* $(A_i)_{1\leq i\leq n\times m}$ *a basis of* $\mathcal{M}_{n\times m}(\mathbb{C})$, $(B_j)_{1\leq j\leq p\times r}$ *a basis of* $\mathcal{M}_{p\times r}(\mathbb{C})$. *Then,* $(A_i \otimes B_j)_{1\leq i\leq n\times m, 1\leq j\leq p\times r}$ *is a basis of* $\mathcal{M}_{np\times mr}(\mathbb{C})$.

Proposition 2 *Consider* $(A_i)_{1\leq i\leq n}$ *a set of elements of* $\mathcal{M}_{p\times r}(\mathbb{C})$, $(B_i)_{1\leq i\leq n}$ *a set of elements of* $\mathcal{M}_{l\times m}(\mathbb{C})$, $A \in \mathcal{M}_{p\times r}(\mathbb{C})$ *and* $B \in \mathcal{M}_{l\times m}(\mathbb{C})$. *If*

$$A \otimes B = \sum_{i=1}^{n} A_i \otimes B_i \tag{1.2}$$

then, for any matrix K,

$$A \otimes K \otimes B = \sum_{i=1}^{n} A_i \otimes K \otimes B_i$$

2 Tensor permutation matrices

Definition 3 *For* $p,\, q \in \mathbb{N}$, $p \geq 2$, $q \geq 2$, *we call tensor commutation matrix* $p \otimes q$ *the permutation matrix* $U_{p\otimes q} \in \mathcal{M}_{pq\times pq}(\mathbb{C})$ *formed by 0 and 1, verifying the property*

$$U_{p\otimes q}.(a \otimes b) \;=\; b \otimes a$$

for all $a \in \mathcal{M}_{p\times 1}(\mathbb{C})$, $b \in \mathcal{M}_{q\times 1}(\mathbb{C})$.
More generally, for $k \in \mathbb{N}$, $k > 2$ *and for* σ *permutation on* $\{1, 2, \ldots, k\}$, *we call* σ*-tensor permutation matrix* $n_1 \otimes n_2 \otimes \ldots \otimes n_k$ *the permutation matrix* $U_{n_1\otimes n_2\otimes\ldots\otimes n_k}(\sigma) \in \mathcal{M}_{n_1 n_2\ldots n_k\times n_1 n_2\ldots n_k}(\mathbb{C})$ *formed by 0 and 1, verifying the property*

$$U_{n_1\otimes n_2\otimes\ldots\otimes n_k}(\sigma)\cdot(a_1 \otimes a_2 \otimes \ldots \otimes a_k) = a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \ldots \otimes a_{\sigma(k)}$$

for all $a_i \in \mathcal{M}_{n_i\times 1}(\mathbb{C})$, $(i \in \{1, 2, \ldots, k\})$.

Definition 4 *For $k \in \mathbb{N}$, $k > 2$ and for σ permutation on $\{1, 2, \ldots, k\}$, we call σ-tensor transposition matrix $n_1 \otimes n_2 \otimes \ldots \otimes n_k$ a σ-tensor permutation matrix $n_1 \otimes n_2 \otimes \ldots \otimes n_k$ with σ is a transposition.*

Consider the σ-tensor transposition matrix $n_1 \otimes n_2 \otimes \ldots \otimes n_k$, $U_{n_1 \otimes n_2 \otimes \ldots \otimes n_k}(\sigma)$ with σ the transposition $(i \;\; j)$.

$$\begin{aligned} & U_{n_1 \otimes n_2 \otimes \ldots \otimes n_k}(\sigma) \cdot (a_1 \otimes \ldots \otimes a_i \otimes a_{i+1} \otimes \ldots \otimes a_j \otimes a_{j+1} \otimes \ldots \otimes a_k) \\ & \qquad = a_1 \otimes \ldots \otimes a_{i-1} \otimes a_j \otimes a_{i+1} \otimes \ldots \otimes a_{j-1} \otimes a_i \otimes a_{j+1} \otimes \ldots \otimes a_k \end{aligned}$$

for any $a_l \in \mathcal{M}_{n_l \times 1}(\mathbb{C})$.
If $(B_{l_i})_{1 \leq l_i \leq n_j n_i}$, $(B_{l_j})_{1 \leq l_j \leq n_i n_j}$ are respectively bases of $\mathcal{M}_{n_j \times n_i}(\mathbb{C})$ and $\mathcal{M}_{n_i \times n_j}(\mathbb{C})$, then the tensor commutation matrix $U_{n_i \otimes n_j}$ can be decomposed as a linear combination of the basis $(B_{l_i} \otimes B_{l_j})_{1 \leq l_i \leq n_j n_i, 1 \leq l_j \leq n_i n_j}$ of $\mathcal{M}_{n_i n_j \times n_i n_j}(\mathbb{C})$. We want to prove that $U_{n_1 \otimes n_2 \otimes \ldots \otimes n_k}(\sigma)$ is a linear combination of

$$\left(I_{n_1 n_2 \ldots n_{i-1}} \otimes B_{l_i} \otimes I_{n_{i+1} n_{i+2} \ldots n_{j-1}} \otimes B_{l_j} \otimes I_{n_{j+1} n_{j+2} \ldots n_k}\right)_{1 \leq l_i \leq n_j n_i, 1 \leq l_j \leq n_i n_j}.$$

For doing so, it suffices to prove the following theorem by using the proposition 2.

Theorem 5 *Suppose $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \end{pmatrix}$ permutation on $\{1, 2, 3\}$, $(B_{i_1})_{1 \leq i_1 \leq N_3 N_1}$, $(B_{i_3})_{1 \leq i_3 \leq N_1 N_3}$ bases respectively of $\mathcal{M}_{N_3 \times N_1}(\mathbb{C})$ and $\mathcal{M}_{N_1 \times N_3}(\mathbb{C})$. If $U_{N_1 \otimes N_3} = \sum_{i_1=1}^{N_1 N_3} \sum_{i_3=1}^{N_1 N_3} \alpha^{i_1 i_3} B_{i_1} \otimes B_{i_3}$, $\alpha^{i_1 i_3} \in \mathbb{C}$, then*
$U_{N_1 \otimes N_2 \otimes N_3}(\sigma) = \sum_{i_1=1}^{N_1 N_3} \sum_{i_3=1}^{N_1 N_3} \alpha^{i_1 i_3} B_{i_1} \otimes I_{N_2} \otimes B_{i_3}$.

3 Decomposition of a tensor permutation matrix

Notation 6 *Let $\sigma \in S_n$, that is σ a permutation on $\{1, 2, \ldots, n\}$, $p \in \mathbb{N}$, $p \geq 2$, we denote the tensor permutation matrix $U_{\underbrace{p \otimes p \otimes \ldots \otimes p}_{n-times}}(\sigma)$ by $U_{p^{\otimes n}}(\sigma)$,[5].*

We use the following lemma for demonstrating the theorem below.

Lemma 7 *Let $\sigma \in S_n$, whose cycle is*

$$\mathcal{C}_\sigma = (i_1 \; i_2 \; i_3 \; \ldots \; i_{n-1} \; i_n)$$

then,

$$\mathcal{C}_\sigma = (i_1 \; i_2 \; i_3 \; \ldots \; i_{n-1}) \, (i_{n-1} \; i_n)$$

Theorem 8 *For $n \in \mathbb{N}^*$, $n > 1$, $\sigma \in S_n$ whose cycle is*

$$\mathcal{C}_\sigma = (i_1\ i_2\ i_3\ \ldots\ i_{k-1}\ i_k)$$

with $k \in \mathbb{N}^*$, $k > 2$. Then

$$U_{p^{\otimes n}}(\sigma) = U_{p^{\otimes n}}((i_1\ i_2\ \ldots\ i_{k-1})) \cdot U_{p^{\otimes n}}((i_{k-1}\ i_k))$$

or

$$U_{p^{\otimes n}}(\sigma) = U_{p^{\otimes n}}((i_1\ i_2)) \cdot U_{p^{\otimes n}}((i_2\ i_3)) \cdot \ldots \cdot U_{p^{\otimes n}}((i_{k-1}\ i_k))$$

with $p \in \mathbb{N}^*$, $p > 2$.

So, a tensor permutation matrix can be expressed as a product of tensor transposition matrices.

Corollary 9 *For $n \in \mathbb{N}^*$, $n > 2$, $\sigma \in S_n$ whose cycle is*

$$\mathcal{C}_\sigma = (i_1\ i_2\ i_3\ \ldots\ i_{n-1}\ n)$$

Then,

$$U_{p^{\otimes n}}(\sigma) = \left[U_{p^{\otimes(n-1)}}((i_1\ i_2\ \ldots\ i_{n-2}\ i_{n-1})) \otimes I_p\right] \cdot U_{p^{\otimes n}}((i_{n-1}\ n))$$

4 $n \times n$-Gell-Mann matrices

The $n \times n$-Gell-Mann matrices are hermitian, traceless matrices Λ_1,Λ_2,...,Λ_{n^2-1} satisfying the commutation relations (Cf. for example [6], [7])

$$[\Lambda_a, \Lambda_b] = 2i \sum_{a=1}^{n^2-1} f_{abc}\Lambda_c \tag{4.1}$$

where f_{abc} are the structure constants which are reals and totally antisymmetric, and

$$Tr(\Lambda_a \Lambda_b) = 2\delta_{ab}$$

where δ_{ab} the Kronecker symbol.
For $n = 2$, the 2×2-Gell-Mann matrices are the usual Pauli matrices.

They satisfy also the anticommutation relation (Cf. for example[6], [7])

$$\{\Lambda_a, \Lambda_b\} = \frac{4}{n}\delta_{ab} I_n + 2 \sum_{c=1}^{n^2-1} d_{abc}\Lambda_c \tag{4.2}$$

where I_n denotes the n-dimensional unit matrix and the constants d_{abc} are reals and totally symmetric in the three indices, and by using the relations (4.1) and (4.2), we have

$$\Lambda_a \Lambda_b = \frac{2}{n}\delta_{ab} + \sum_{c=1}^{n^2-1} d_{abc}\Lambda_c + i \sum_{c=1}^{n^2-1} f_{abc}\Lambda_c \tag{4.3}$$

The structure constants satisfy the relation (Cf. for example[6])

$$\sum_{e=1}^{n^2-1} f_{abe}f_{cde} = \frac{2}{n}\left(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\right) + \sum_{e=1}^{n^2-1} d_{ace}d_{dbe} - \sum_{e=1}^{n^2-1} d_{ade}d_{bce} \tag{4.4}$$

5 Examples

Now, we have some theorems and relations on the generalized Gell-Mann matrices which we need for expressing a tensor permutation matrix in terms of the generalized Gell-Mann matrices. In this section, we treat some examples.

5.1 $U_{n^{\otimes 3}}(\sigma)$

$\sigma = (1\ 2\ 3)$
By employing the Lemma 7

$$\sigma = (1\ 2)(2\ 3)$$

and by using the Theorem 8

$$U_{n^{\otimes 3}}\left((1\ 2\ 3)\right) = U_{n^{\otimes 3}}\left((1\ 2)\right) \cdot U_{n^{\otimes 3}}\left((2\ 3)\right) \tag{5.1}$$

However, using (1)

$$U_{n^{\otimes 3}}\left((1\ 2)\right) = \frac{1}{n} I_n \otimes I_n \otimes I_n + \frac{1}{2}\sum_{a=1}^{n^2-1} \Lambda_a \otimes \Lambda_a \otimes I_n$$

and

$$U_{n^{\otimes 3}}\left((2\ 3)\right) = \frac{1}{n} I_n \otimes I_n \otimes I_n + \frac{1}{2}\sum_{a=1}^{n^2-1} I_n \otimes \Lambda_a \otimes \Lambda_a$$

So, by using (5.1) we have

$$U_{n^{\otimes 3}}\left((1\ 2\ 3)\right) = \frac{1}{n^2} I_n \otimes I_n \otimes I_n + \frac{1}{2n}\sum_{a=1}^{n^2-1} I_n \otimes \Lambda_a \otimes \Lambda_a$$
$$+ \frac{1}{2n}\sum_{a=1}^{n^2-1} \Lambda_a \otimes \Lambda_a \otimes I_n + \frac{1}{4}\sum_{a=1}^{n^2-1}\sum_{b=1}^{n^2-1} \Lambda_a \otimes \Lambda_a\Lambda_b \otimes \Lambda_b$$

Hence, employing the relation (4.3)

$$U_{n^{\otimes 3}}\left((1\;2\;3)\right) = \frac{1}{n^2} I_n \otimes I_n \otimes I_n + \frac{1}{2n}\sum_{a=1}^{n^2-1} I_n \otimes \Lambda_a \otimes \Lambda_a$$
$$+ \frac{1}{2n}\sum_{a=1}^{n^2-1} \Lambda_a \otimes \Lambda_a \otimes I_n + \frac{1}{2n}\sum_{a=1}^{n^2-1} \Lambda_a \otimes I_n \otimes \Lambda_a$$
$$- \frac{i}{4}\sum_{a=1}^{n^2-1}\sum_{b=1}^{n^2-1}\sum_{c=1}^{n^2-1} f_{abc}\Lambda_a \otimes \Lambda_b \otimes \Lambda_c + \frac{1}{4}\sum_{a=1}^{n^2-1}\sum_{b=1}^{n^2-1}\sum_{c=1}^{n^2-1} d_{abc}\Lambda_a \otimes \Lambda_b \otimes \Lambda_c \tag{5.2}$$

5.2 $U_{2^{\otimes 3}}(\sigma)$, $\sigma \in S_3$

Now we give a formula giving $U_{2^{\otimes 3}}(\sigma)$, naturally in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Using the relation (Cf. for example [8])

$$\sigma_l \sigma_k = \delta_{lk} I_2 + i \sum_{m=1}^{3} \varepsilon_{lkm}\sigma_m$$

where ε_{ijk} is totally antisymmetric in the three indices, which is equal 1 if $(i\;\;j\;\;k) = (1\;\;2\;\;3)$, we have

$$U_{2^{\otimes 3}}(1\;\;2\;\;3) = \frac{1}{4} I_2 \otimes I_2 \otimes I_2 + \frac{1}{4}\sum_{l=1}^{3} I_2 \otimes \sigma_l \otimes \sigma_l + \frac{1}{4}\sum_{l=1}^{3} \sigma_l \otimes I_2 \otimes \sigma_l$$
$$+ \frac{1}{4}\sum_{l=1}^{3} \sigma_l \otimes \sigma_l \otimes I_2 - \frac{i}{4}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\sigma_i \otimes \sigma_j \otimes \sigma_k$$

Conclusion

Based on the fact that a tensor permutation matrix is a product of tensor transposition matrices and on the Theorem 5, with the help of the expression of a tensor commutation matrix in terms of the generalized Gell-Mann matrices, we can express a tensor permutation matrix as linear combination of the tensor products of the generalized Gell-Mann matrices.

We have no intention for searching a general formula. However, we have shown that any tensor permutation matrix can be expressed in terms of the generalized Gell-Mann matrices and then the expression can be simplified by using the relations between these Matrices.

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