

RENORMALIZING THE SCHRÖDINGER EQUATION FOR NN SCATTERING

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Abstract

The renormalization of the Schrödinger equation with regular One Boson Exchange and singular chiral potentials including One and Two-Pion exchanges is analyzed within the context of NN scattering.

1 Introduction

One traditional view of NN force has been through One Boson Exchange (OBE) Models [1, 2]. Recent developments have shown how chiral symmetry may provide NN forces of practical interest in nuclear physics [3–5]. Remarkably, chiral expansions, based on assuming a large scale suppression on the parameters $4\pi f_\pi \sim M_N \sim 1\text{GeV}$ necessarily involve singular potentials at short distances, i.e. $r^2|V(r)| \rightarrow \infty$ for $r \rightarrow 0$. If we take the limit $r \ll 1/m_\pi$ (or equivalently large momenta) pion mass effects are irrelevant and hence at some fixed order of the expansion one has

$$V(r) \sim \frac{M_N}{(4\pi f_\pi)^{2n} M_N^m} \frac{1}{r^{2n+m}} \quad (1)$$

(the only exception is the singlet channel-OPE case which behaves as $\sim m_\pi^2/f_\pi^2 r$, see below). The dimensional argument is reproduced by loop calculations in the so called Weinberg dimensional power counting [6, 7]. Thus, much of our understanding on the physics deduced from chiral potentials might be related to a proper interpretation of these highly singular potentials. Renormalization is the most natural tool provided 1) we expect the potential is realistic at long distances and 2) we want short distance details not to be essential in the description. This is precisely the situation we

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face most often in nuclear physics. Knowledge on the attractive or repulsive character of the singularity turns out to be crucial to successfully achieve this program and ultimately depends on the particular scheme or power counting used to compute the potential. We illustrate our points for the simpler OBE potential in the 1S_0 channel and then review some results for chiral OPE and TPE potentials for all partial waves and the deuteron bound state.

2 Renormalization of OBE potentials

The singularity of chiral potentials raises suspicions and, quite often, much confusion. However, if properly interpreted and handled they do not differ much from the standard well-behaved regular potentials one usually encounters in nuclear physics. Actually, we digress here that renormalization may provide useful insights even if the potential is not singular at the origin ($r^2V(r) \rightarrow 0$!). For definiteness, let us analyze as an illustrative example the phenomenologically successful 1S_0 OBE potential [1, 2] (we take $m_\rho = m_\omega$)

$$V(r) = -\frac{g_{\pi NN}^2 m_\pi^2}{16\pi M_N^2} \frac{e^{-m_\pi r}}{r} - \frac{g_{\sigma NN}^2}{4\pi} \frac{e^{-m_\sigma r}}{r} + \frac{g_{\omega NN}^2}{4\pi} \frac{e^{-m_\omega r}}{r} + \dots \quad (2)$$

where for simplicity we neglect nucleon mass effects and a tiny η contribution. We take $m_\pi = 138\text{MeV}$, $M_N = 939\text{MeV}$, $m_\omega = 783\text{MeV}$ and $g_{\pi NN} = 13.1$ which seem firmly established. Actually, Eq. (2) looks like a long distance expansion of the potential. NN scattering in the elastic region below pion production threshold involves CM momenta $p < p_{\text{max}} = 400\text{MeV}$. Given the fact that $1/m_\omega = 0.25\text{fm} \ll 1/p_{\text{max}} = 0.5\text{fm}$ we expect heavier mesons to be irrelevant, and ω itself to be marginally important. In the traditional approach, however, this is not so [1, 2]. Actually, the problem is essentially handled by solving the Schrödinger equation (S-wave)

$$-u_p''(r) + MV(r)u_p(r) = p^2 u_p(r) \quad (3)$$

with the *regular* solution at the origin, $u_p(0) = 0$. This boundary condition implicitly assumes taking also the potential all the way down to the origin. The asymptotic condition for $r \gg 1/m_\pi$ is taken to be

$$u_p(r) \rightarrow \frac{\sin(pr + \delta_0(p))}{\sin \delta_0(p)} \quad (4)$$

where $\delta_0(p)$ is the phase-shift. For the potential in Eq. (2) the phase shift is an analytic function of p with the closest branch cut located at $p = \pm im_\pi/2$,

so that one can undertake an effective range expansion,

$$p \cot \delta_0(p) = -\frac{1}{\alpha_0} + \frac{1}{2}r_0p^2 + v_2p^4 + \dots \quad (5)$$

within a radius of convergence $|p| \leq m_\pi/2$. A similar expansion for the wave function $u_p(r) = u_0(r) + p^2u_2(r) \dots$ means solving the set of equations

$$-u_0''(r) + MV(r)u_0(r) = 0, \quad (6)$$

$$u_0(r) \rightarrow 1 - r/\alpha_0,$$

$$-u_2''(r) + U(r)u_2(r) = u_0(r), \quad (7)$$

$$u_2(r) \rightarrow (r^3 - 3\alpha_0r^2 + 3\alpha_0r_0r) / (6\alpha_0),$$

where, again, the *regular* solutions, $u_0(0) = u_2(0) = 0$ are taken. With this normalization the effective range r_0 is computed from the standard formula

$$r_0 = 2 \int_0^\infty dr [(1 - r/\alpha_0)^2 - u_0(r)^2]. \quad (8)$$

In the usual approach [1,2] *everything* is obtained from the potential assumed to be valid for $0 \leq r < \infty$. In practice, strong form factors are included mimicking the finite nucleon size and reducing the short distance repulsion of the potential, but the regular boundary condition is always kept.² As it is well known the 1S_0 scattering length is unnaturally large $\alpha_0 = -23.74(2)\text{fm}$, while $r_0 = 2.77(4)\text{fm}$. Let us assume we have fitted the potential, Eq. (2), to reproduce α_0 . Under these circumstances a tiny change in the potential $V \rightarrow V + \Delta V$ has a dramatic effect on α_0 , since one obtains

$$\Delta\alpha_0 = \alpha_0^2 M_N \int_0^\infty \Delta V(r) u_0(r)^2 dr. \quad (9)$$

As a result, potential parameters must be fine tuned. In particular, the resulting ω -repulsive contribution is well determined with an unnaturally large coupling, $g_{\omega NN} \sim 16$. [1,2]. In our case, with no form factors nor relativistic corrections, a fit to Ref. [8] yields $g_{\omega NN} = 12.876(2)$, $g_{\sigma NN} = 12.965(2)$ and $m_\sigma = 554.0(4)\text{MeV}$ with $\chi^2/\text{DOF} = 0.26$. Note the small uncertainties, as expected from our discussion and Eq. (9). As mentioned above $1/m_\omega = 0.25\text{fm} \ll 1/p_{\text{max}} = 0.5\text{fm}$ so ω should not be crucial at least for CM momenta $p \ll p_{\text{max}}$. Thus, despite the undeniable success in fitting the data this sensitivity to short distances looks counterintuitive.

²Calculations solving the equivalent Lippmann-Schwinger equation in momentum space for regular potentials correspond always to choose the regular solution for the Schrödinger equation in coordinate space.

The renormalization viewpoint *refuses* to access physically the very short distance region, but encodes it through low energy parameters described by the effective range expansion, Eq. (5), as renormalization conditions (RC's). In the case of only one RC where α_0 is fixed one proceeds as follows [9, 10]:

- For a given α_0 integrate in the zero energy wave function $u_0(r)$, Eq. (6), down to the cut-off radius r_c . This is the RC.
- Implement self-adjointness through the boundary condition

$$u'_p(r_c)u_0(r_c) - u'_0(r_c)u_p(r_c) = 0, \quad (10)$$

- Integrate out the finite energy wave function $u_p(r)$, from Eq. (3) and determine the phase shift $\delta_0(p)$ from Eq.(4).
- Remove the cut-off $r_c \rightarrow 0$ to strive for model independence.

This allows to compute δ_0 (and hence r_0, v_2) from $V(r)$ and α_0 as *independent* information. Note that this is equivalent to consider, in addition to the regular solution, the *irregular* one³. A beautiful result is the universal low energy theorem which highlights this de-correlation between the potential and the scattering length [10]

$$r_0 = 2 \int_0^\infty dr(1 - u_{0,0}^2) - \frac{4}{\alpha_0} \int_0^\infty dr(r - u_{0,0}u_{0,1}) + \frac{2}{\alpha_0^2} \int_0^\infty dr(r^2 - u_{0,1}^2), \quad (11)$$

based on the superposition principle of boundary conditions, i.e. writing $u_0(r) = u_{0,0}(r) - u_{0,1}(r)/\alpha_0$ with $u_{0,n}(r) \rightarrow r^n$ and using Eq.(8). A fit of the potential (2) with $g_{\omega NN} = 0$ to the effective range yields (Fig. 1) a strong correlation between m_σ and $g_{\sigma NN}$. Over-imposing this correlation to $r_0 = 2.670(4)\text{fm}$, a fit to Ref. [8] yields $m_\sigma = 493(12)\text{MeV}$, $g_{\sigma NN} = 8.8(2)$, $g_{\omega NN} = 0(5)$ with $\chi^2/\text{DOF} = 0.24$ (Fig. 1). Note the larger uncertainties, although correlations allow $g_{\omega NN} \sim 9$ and $m_\sigma \sim 520\text{MeV}$ within $\Delta\chi^2 = 1$. Contrary to common wisdom, but according to our naive expectations, no strong short range repulsion is essential. The moral is that building α_0 *from* the potential is equivalent to absolute knowledge at short distances and in the 1S_0 channel a strong fine tuning is at work. Of course, a more systematic analysis should be pursued in all partial waves and relativistic corrections might be included as well, but this example illustrates our point that the renormalization viewpoint may tell us to what extent short distance physics may be less well determined than the traditional approach assumes. This opens up a new perspective to the phenomenology of OBE potentials in cases where the strong ω -repulsion has proven to be crucial at low energies [12].

³In momentum space this can be shown to be equivalent to introduce one counterterm in the cut-off Lippmann-Schwinger equation, see Ref. [11] for a detailed discussion.

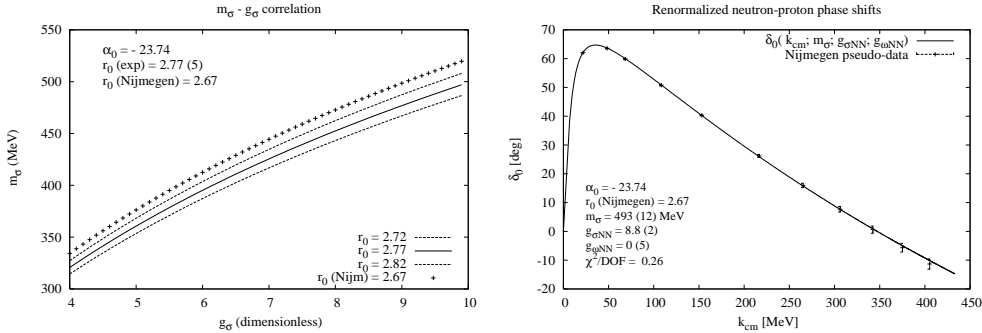


Figure 1: Results in the 1S_0 channel for the renormalized OBE potential. Left: Effective range correlation between $g_{\sigma NN}$ and m_σ for $g_{\omega NN} = 0$. Right: Renormalized phase shift (in degrees) as a function of the CM momentum (in MeV) in the OBE ($\pi + \sigma + \omega$) model. The data are an average of [8].

3 Renormalization of chiral potentials

The generalization of the above method to the singular chiral potentials [6, 7] has been implemented in [10] with promising results for One- and Two Pion Exchange (OPE and TPE). We illustrate again the case of pn scattering in the 1S_0 -channel. For the simplest situation with one RC, corresponding to fix the scattering length as an independent parameter, the method outlined above may be directly applied to singular potentials *provided* they are attractive, i.e. $V(r) \rightarrow -C_n/r^n$ with $n \geq 2$ ⁴. The result for zero energy wave functions as well as the effective range can be seen at Fig. 2. NNLO corresponds to the TPE potential of Ref. [6]. As we see the Nijmegen result $r_0 = 2.67\text{fm}$ is *almost* saturated by the TPE potential yielding $r_0 = 2.87\text{fm}$ already at $r_c \sim 0.5\text{fm}$. Calculations with TPE to N3LO with one RC show convergence but no improvement [11] without or with Δ explicit degrees of freedom. Thus, some physics is missing, perhaps 3π effects. If, in addition to α_0 , we want to fix $r_0 = 2.67\text{fm}$ [8] as a RC we must solve Eqs. (6) and (7). The matching condition at the boundary $r = r_c$ becomes energy dependent [13]

$$\frac{u'_p(r_c)}{u_p(r_c)} = \frac{u'_0(r_c) + p^2 u'_2(r_c) + \dots}{u_0(r_c) + p^2 u_2(r_c) + \dots}. \quad (12)$$

The generalization to arbitrary order is straightforward. For N RC's we have $u_p(r) = \sum_{n=0}^N p^{2n} u_{2n}(r)$ and using the natural extension of the matching relation in Eq. (12) as well as the superposition principle of boundary conditions

⁴If the potential was singular and repulsive one cannot fix any low energy parameters; doing so yields non-converging phase shifts.

one can show the following formula

$$p \cot \delta_0(p) = \frac{\sum_{n=0}^N a_n \mathcal{A}_n(p, r_c)}{\sum_{n=0}^N a_n \mathcal{B}_n(p, r_c)}, \quad (13)$$

where the coefficients a_n can be related to the effective range parameters $a_0 = 1$, $a_1 = -1/\alpha_0$, $a_2 = r_0$, $a_3 = v_2$ etc. and $\mathcal{A}_n(p, r_c)$ and $\mathcal{B}_n(p, r_c)$ are functions which are finite in the limit $r_c \rightarrow 0$ and depend *solely* on the potential. In Eq. (13) the dependence on the low energy parameters used as input is displayed explicitly and can be completely separated from the long range potential [13]. The coupled channel case can be analyzed in terms of eigenpotentials although the result is cumbersome. In Fig. 3 we show the phase shift for the 1S_0 channel when the potential is considered at LO, NLO and NNLO and either one RC (fixing α_0) or two RC's (fixing α_0 and r_0) are considered. LO+1C, NLO+2C and NNLO+2C fix the same number of RC's as LO, NLO and NNLO of the Weinberg counting respectively. As we see, our NNLO+2C does not improve over NLO+2C.

It is worth mentioning that the innocent-looking energy dependent matching condition, Eq. (12), is quite unique since this is the only representation guaranteeing finiteness of results for singular potentials [13]. Polynomial expansions in p^2 such as suggested e.g. in Ref. [7] do not work for $r_c \rightarrow 0$. A virtue of the coordinate over momentum space is that these results can be deduced analytically. For instance, the equivalent representation of Eq.(13) in momentum space may likely exist, but is so far unknown. Actually, the usual polynomial representation of short distance interactions in momentum space $V_S(k', k) = C_0 + C_2(k^2 + k'^2) + \dots$ of standard NLO and NNLO Weinberg counting is renormalizable only when $C_2 \rightarrow 0$ for $\Lambda \rightarrow \infty$ [11].

4 Renormalization of the Deuteron

In the $^3S_1 - ^3D_1$ channel, the relative proton-neutron state for negative energy is described by the coupled equations

$$\begin{pmatrix} -\frac{d^2}{dr^2} + M_N V_s(r) & M_N V_{sd}(r) \\ M_N V_{sd}(r) & -\frac{d^2}{dr^2} + \frac{6}{r^2} + M_N V_d(r) \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = -\gamma^2 \begin{pmatrix} u \\ w \end{pmatrix}. \quad (14)$$

Here $\gamma = \sqrt{M_M B}$, with $B = 2.24\text{MeV}$ is the deuteron binding energy and $u(r)$ and $w(r)$ are S- and D-wave reduced wave functions respectively. At long distances they satisfy,

$$\begin{pmatrix} u \\ w \end{pmatrix} \rightarrow A_S e^{-\gamma r} \begin{pmatrix} 1 \\ \eta \left[1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right] \end{pmatrix}. \quad (15)$$

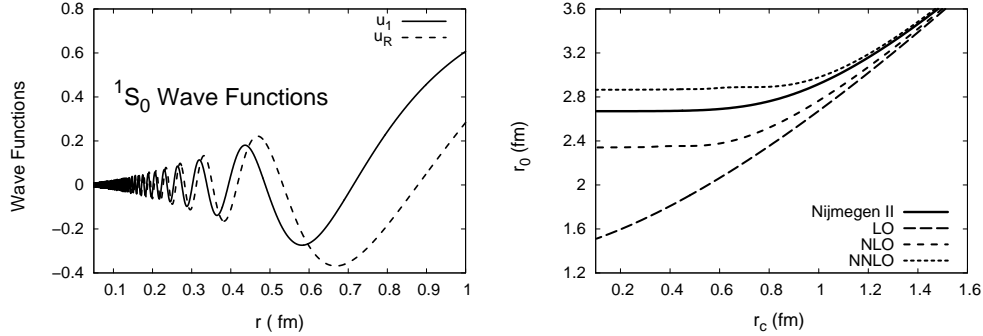


Figure 2: (Left panel) Zero-energy, 1S_0 linearly independent wave functions at NNLO; $u_1 \rightarrow 1$ and $u_r \rightarrow r$ for $r \rightarrow \infty$. (Right Panel) Effective range r_0 as a function to the cut-off for the same channel and different orders; using $r_0(r_c) = 2 \left(\int_0^\infty (1 - r/\alpha_0)^2 dr - \int_{r_c}^\infty u_0^2 dr \right)$, with $\alpha_0 = -23.74\text{fm}$ [8].

where η is the asymptotic D/S ratio parameter and A_S is the asymptotic normalization factor, which is such that the deuteron wave functions are normalized to unity. The OPE $^3S_1 - ^3D_1$ potential is given by $M_N V_s = U_C$, $M_N V_{sd} = 2\sqrt{2}U_T, M_N V_d = U_C - 2U_T$ where for $r \geq r_c > 0$ we have

$$U_C = -\frac{m_\pi^2 M_N g_A^2}{16\pi f_\pi^2} \frac{e^{-m_\pi r}}{r}, \quad U_T = U_C \left(1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2} \right). \quad (16)$$

The tensor force generates a $1/r^3$ singularity at the origin in coupled channel space. This behavior of the potential is strong enough to overcome the centrifugal barrier at short distances, thus modifying the usual threshold behavior of the wave functions. The interesting aspect of this potential is that after diagonalization it has one positive (repulsive) and one (negative) attractive eigenvalue. The proper normalization of the wave functions in the limit $r_c \rightarrow 0$ implies that one can only fix one free parameter, e.g. the deuteron binding energy [10]. Other properties may be predicted, for instance one gets $\eta_{\text{OPE}} = 0.0263$ (exp. 0.0256(4)). The TPE chiral potentials of Ref. [6] have also been renormalized [10], yielding a rather satisfactory picture of the deuteron. The results described here have been reproduced in momentum space [14]. The required cut-off in momentum space is larger than a naive estimate $r_c \sim 1/\Lambda$ because the regularization influences both the counterterms as well as the potential. Deuteron form factors, probing some off-shellness of the potential, have been computed describing surprisingly well the data up to momenta $q \sim 800\text{MeV}$ when LO currents are considered ⁵.

⁵See talk of D. R. Phillips in this conference

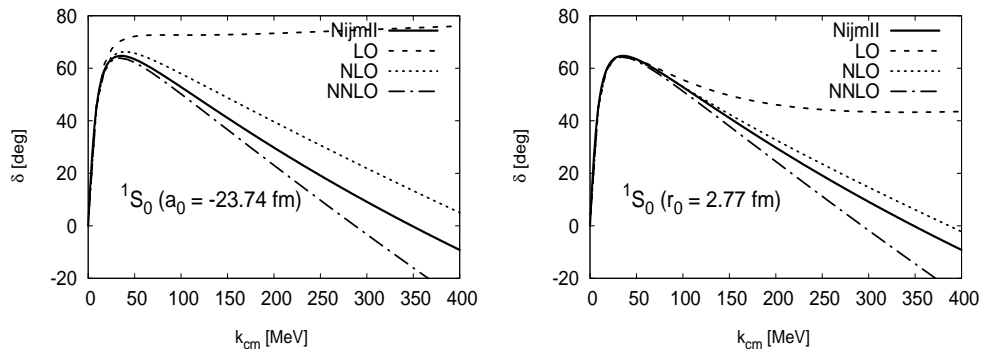


Figure 3: Renormalized 1S_0 phase shifts (in degrees) for chiral LO, NLO and NNLO potentials fixing $\alpha_0 = -23.74\text{fm}$ (Left panel) or $\alpha_0 = -23.74\text{fm}$ and $r_0 = 2.77\text{fm}$ (Right panel) as input parameters. The data are from [8].

5 Power counting and renormalization

The question on how a sensible hierarchy for NN interactions should be organized remains so far open, because it is not obvious if one should renormalize or not and how [10, 15–17]. However, for a given long distance potential, we know *whether* and, in positive case, *how* this can be made compatible with the desired short distance insensitivity [10]. Not all chiral interactions fit into this scheme, and thus it is sometimes preferred to keep finite cut-offs despite results being often strongly dependent on the choice at scales $r_c \sim 0.5 - 1\text{fm}$ similar to the ones we want to probe in NN scattering [10, 15]. Renormalizability of chiral potentials developing a singularity such as Eq. (1) requires that one must choose the regular solution in which case the wave function behaves as $u_p(r) \sim (r4\pi f_\pi)^{\frac{2n+m}{4}}$ and thus increasing insensitivity is guaranteed as the power of the singularity increases. Converging renormalized TPE calculations show insensitivity for reasonable scales of $r_c \sim 0.5\text{fm}$ [10].

The Weinberg counting based in a heavy baryon approach at LO [5] for 1S_0 and $^3S_1 - ^3D_1$ states turns out to be renormalizable. There is at present no necessity argument why this ought to be so, for the simple reason that power counting does not anticipate the sign of the interaction at short distances. When one goes to NLO the short distance $1/r^5$ singular repulsive character of the potential makes the deuteron unbound [10]. Finally, NNLO potentials diverge as $-1/r^6$ and are, again, compatible with Weinberg counting in the deuteron [10]. More failures have been reported in Refs. [11, 15]. Relativistic potentials subjected to different power counting have been renormalized in Ref. [18] yielding much less counterterms due to their different short distance

$1/r^7$ singularities and slightly better overall description, although the 1S_0 phase is not improved as compared to the heavy baryon formulation. These complications in the more fundamental chiral potentials contrast with the simplicity of the $\sigma + \pi$ OBE renormalized results (see Figs. 1 and 3).

In the present state of affairs a clue might come from a remarkable analogy between the NN interaction in the chiral quark model and the Van der Waals molecular interactions in the Born-Oppenheimer approximation [10]. For non-relativistic constituent quarks the direct NN interaction is provided by the convoluted OPE quark-quark potential. Second order perturbation theory in OPE among quarks generates TPE between nucleons yielding

$$V_{NN} = \langle NN | V_{\text{OPE}} | NN \rangle + \sum_{HH' \neq NN} \frac{|\langle NN | V_{\text{OPE}} | HH' \rangle|^2}{E_{NN} - E_{HH'}} + \dots \quad (17)$$

When $HH' = N\Delta$ and $HH' = \Delta\Delta$ this resembles Ref. [19] which for $2\text{fm} < r < 3\text{fm}$ behaves as σ exchange with $m_\sigma = 550\text{MeV}$ and $g_{\sigma NN} = 9.4$. Moreover, the second order perturbative character suggests that the potential becomes singular $\sim 1/r^6$ and attractive, necessarily being renormalizable with an arbitrary number of counterterms through energy dependent boundary conditions [13]. Clearly, the renormalization of such a scheme where the $N\Delta$ splitting is treated as a small scale deserves further investigation [20].

6 Conclusion

Renormalization is the mathematical implementation of the appealing physical requirement of short distance insensitivity and hence a convenient tool to search for model independent results. In a non-perturbative setup such as the NN problem, renormalization imposes rather tight constraints on the interplay between the unknown short distance physics and the perturbatively computable long distance interactions. This viewpoint provides useful insights and it is within such a framework that we envisage a systematic and model independent description of the NN force based on chiral interactions.

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