The Hadron in Black & White

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Abstract

I review the statistical picture for the gluon distribution at high energy, as predicted by the evolution equations with Pomeron loops in perturbative QCD at large N_c and fixed coupling. I emphasize the role of conformal symmetry and present an effective field theory of the Liouville type for the gluon distribution in the two–dimensional impact parameter space.

1 Introduction

Over the recent years, our theoretical understanding of the QCD dynamics at high energy has met with considerable progress, and our physical representation of an energetic hadron has correspondingly evolved. The major, new, observation is that the evolution of a hadronic wavefunction with increasing energy is a *stochastic* process, whose structure is relatively simple, but whose physical consequences can nevertheless be dramatic [1, 2]. The existence of stochastic aspects in the high–energy QCD evolution was first revealed, more than ten years ago, by numerical studies of the BFKL evolution in the framework of the dipole picture [3]. But at that time, the non–linear effects responsible for gluon saturation [4, 5] were not included in the evolution. So, the explosive growth of the correlations induced by statistical fluctuations, as observed in the numerical results, was interpreted as an artifact of the linear, BFKL, evolution and expected to be cut off by saturation in a more realistic, non–linear, evolution. It therefore came as a surprise when, more recently, one has realized that the consequences of the fluctuations remain important even in such a more complete evolution, including saturation (within the leading–order formalism in perturbative QCD, at least) [1, 6]. The fluctuations enhance the effects of saturation (because they generate correlations in the dilute regime which are subsequently amplified by the linear evolution) and thus reduce the phase–space for BFKL evolution. Because of that, the rise of the (average) gluon density with increasing energy is considerably slower [1, 2] than predicted by the mean field approximation. Moreover, the gluon distribution shows important fluctuations from one event to another [1, 6], and also between different points in impact parameter space, for a given event [7].

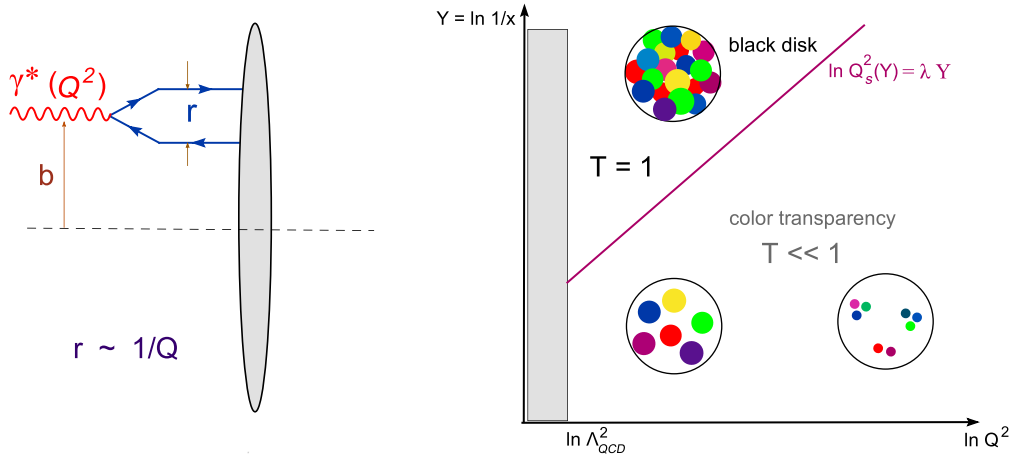

Figure 1: *A cartoon of DIS in the dipole factorization (left) and a 'phase–diagram' showing the different physical regimes in the kinematical plane for DIS (right).*

In what follows, I would like to motivate in simple terms the structure of the stochastic equations describing the evolution with saturation and fluctuations (known as 'Pomeron loop equations' [6, 8]), and then explain the physical picture of an energetic hadron emerging from this evolution. Finally, I shall devote some discussion to a recently proposed effective field theory which is meant to describe the gluon distribution in the two–dimensional impact parameter space, and which turns out to be a simple deformation of the conformal Liouville field theory [7].

2 The Pomeron loop equations

The general equations are most conveniently written in terms of scattering amplitudes for simple projectiles (color dipoles) which scatter off the hadronic target whose partonic structure we would like to investigate. A 'color dipole' is the quark–antiquark fluctuation through which the virtual photon couples to the target in deep inelastic scattering. The dipole scattering amplitude $T(\boldsymbol{x},\boldsymbol{y}) \equiv T(\boldsymbol{r},\boldsymbol{b})$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the transverse coordinates for the quark and the antiquark, $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$ is the dipole size, and $\boldsymbol{b} = (\boldsymbol{x}+\boldsymbol{y})/2$ is its impact parameter (see Fig. 1), is a direct measure of the gluon occupation number $n(\boldsymbol{k},\boldsymbol{b})$ (the number of gluons with transverse momentum $\boldsymbol{k}$ per unit cell in phase–space) in the target wavefunction: $T(r,b) \sim \overline{\alpha}_s n(k,b)$ with $k \sim 1/r$ and $\overline{\alpha}_s = \alpha_s N_c/\pi$. Weak scattering, $T \ll 1$, corresponds to low occupation numbers $n \ll 1/\overline{\alpha}_s$, whereas the 'black disk limit' $T = 1$ (the unitarity bound for dipole scattering) corresponds to *gluon saturation* : $n \sim 1/\overline{\alpha}_s$ (the maximal gluon occupancy allowed by the gluon mutual repulsion). The boundary $Q^2 = Q_s^2(x,b)$

between weak scattering and strong scattering in the kinematical plane (Q^2, x) (at a given impact parameter b) is known as the 'saturation line' (see Fig. 1). In logarithmic variables, $\rho \equiv \ln Q^2$ and $Y \equiv \ln(1/x)$, this is approximately a straight line, $\rho_s(Y) \simeq \lambda Y$, corresponding to the fact that the saturation momentum $Q_s^2(x)$ grows as a power of $1/x$. The saturation line is also the boundary between the dilute and the saturated (or 'color glass condensate') regimes for the gluon distribution in the target. The gluon occupancy $n(k)$ is constant, and large, $n \sim 1/\overline{\alpha}_s$, for $k \leq Q_s(x)$, but it decreases very fast, roughly as $n(k) \sim 1/k^2$, for $k \gg Q_s(x)$. Accordingly $Q_s(x, b)$ is also the *typical gluon momentum* for given x and b.

The dipole amplitudes can be determined by solving appropriate evolution equations, which to lowest order in α_s and for large N_c are known as the 'Pomeon loop equations'. As mentioned in the Introduction, these are *stochastic* equations, which can be equivalently reformulated as an infinite hierarchy of equations for the average κ–dipole amplitudes $\langle T^{(\kappa)}\rangle_Y = \langle T(1)T(2)\ldots T(\kappa)\rangle_Y$, with $\kappa \geq 1$. The first equation in this hierarchy reads (see Fig. 2 for a diagrammatic interpretation):

$$\frac{\partial}{\partial Y}\big\langle T(\boldsymbol{x},\boldsymbol{y})\big\rangle_Y \;=\; \frac{\overline{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \Big\langle -T(\boldsymbol{x},\boldsymbol{y}) + T(\boldsymbol{x},\boldsymbol{z}) + T(\boldsymbol{z},\boldsymbol{y}) - T(\boldsymbol{x},\boldsymbol{z})T(\boldsymbol{z},\boldsymbol{y})\Big\rangle_Y . \quad (1)$$

The linear terms in this equation describe the BFKL evolution (cf. Fig. 2.a), while the last, non–linear, term, represents the unitarity corrections, corresponding to gluon saturation in the target (cf. Fig. 2.b).

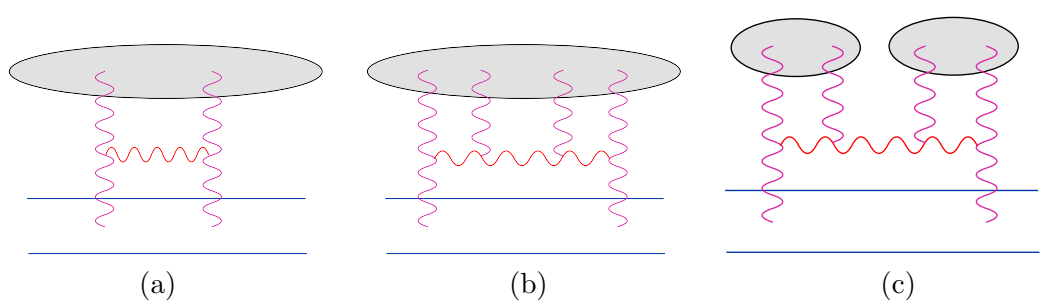

Figure 2: *Diagrams corresponding to Eq. (1) from the perspective of target evolution: (a) the BFKL evolution of the gluon distribution (a $2 \to 2$ gluon vertex); (b) saturation effects in the target (here, a $4 \to 2$ gluon vertex): (c) the mean–field version of the diagram in Fig. 2b, which would lead to BK equation.*

As anticipated, Eq. (1) is not a closed equation (the evolution of the 1–dipole amplitude involves the 2–dipole one), but only the first equation in an infinite hierarchy. A closed equation, known as the BK equation [4, 5], would be obtained under the assumption that $\langle TT\rangle_Y \approx \langle T\rangle_Y\langle T\rangle_Y$; this is a mean field approximation (MFA) which

neglects correlations among the gluons in the target (cf. Fig. 2.c). Here, however, we are precisely interested in the differences between the complete dynamics, which is stochastic, and this MFA. To that aim, we need to inspect the higher equations in the 'Pomeron loop' hierarchy. We shall use here schematic notations which ignore the non–locality of the BFKL kernel and of the various 'Pomeron–number changing' vertices (see [8] for the general equations). Then, the first two equations read

$$\partial_Y \langle T \rangle_Y = \overline{\alpha}_s \langle T - TT \rangle_Y \,, \tag{2}$$

$$\partial_Y \langle TT \rangle_Y = 2\overline{\alpha}_s \langle TT - TTT \rangle_Y + \overline{\alpha}_s \alpha_s^2 \langle T \rangle_Y \,. \tag{3}$$

The first two terms in the r.h.s. of Eq. (3), of $\mathcal{O}(\overline{\alpha}_s)$, are similar to the corresponding terms in Eq. (2); they describe the BFKL evolution and, respectively, saturation effects in the target (see Figs. 3a, b). The last term, proportional to the 1–dipole amplitude $\langle T \rangle_Y$, acts as a source for the 2–dipole amplitude, and hence for *correlations*. As shown by its diagrammatic interpretation in Fig. 3c, this term describes a *gluon–number fluctuation* in the target, which proceeds via the $2 \to 4$ gluon vertex.

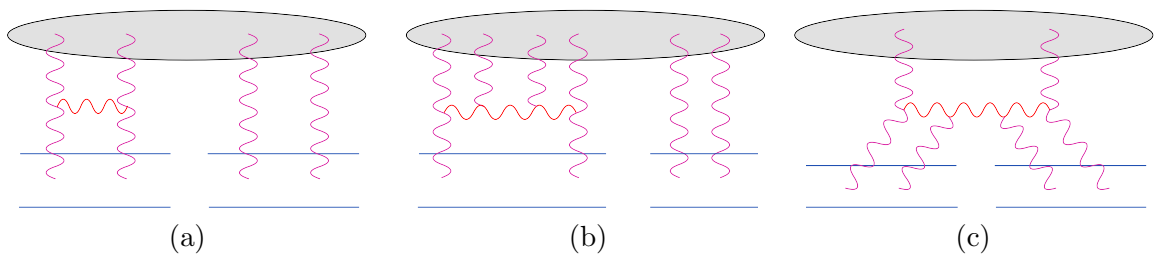

Figure 3: *Diagrams corresponding to the evolution (3) of the 2–dipole amplitude (from the perspective of* target *evolution): (a) the BFKL evolution; (b) saturation effects; (c) a gluon number fluctuation, involving the* $2 \to 4$ *gluon vertex.*

Although suppressed by an additional factor α_s^2, this term is in fact a leading–order effect in the weak–scattering regime where $T \lesssim \alpha_s^2$ (i.e., for a *very dilute* target). Thus, the correlations are first generated through fluctuations in the dilute regime, then they are rapidly amplified by the BFKL evolution, and eventually act as unitarity corrections which oppose to the BFKL growth. We see that, due to the feedback ensured by the non–linear terms in the equations, the particle–number fluctuations, which are a priori important in the *dilute* regime, have a strong influence also on the *approach towards saturation*, and thus on the overall gluon distribution at Y.

3 The statistical gluon distribution

As a result of fluctuations, the gluon distribution produced via the high–energy evolution develops strong variations from one realization to another ('event–by–event

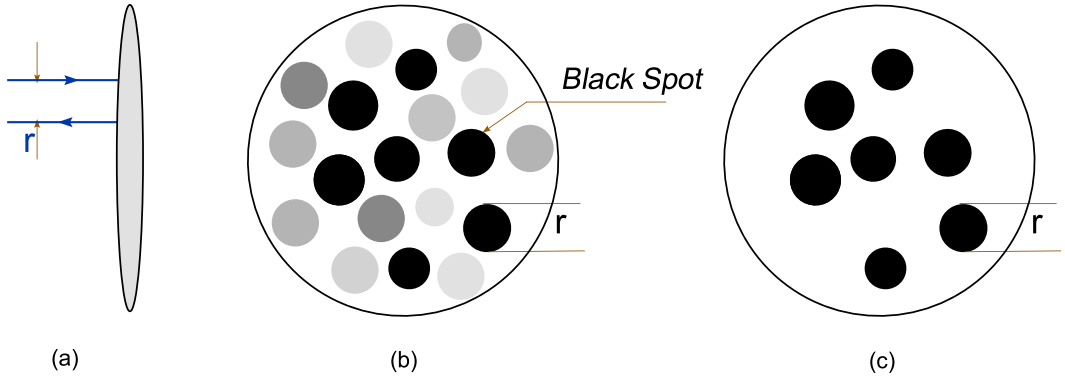

Figure 4: *Dipole–hadron scattering in the fluctuation–dominated regime at very high energy. (a) a view along the collision axis; (b) a transverse view of the hadron, as 'seen' by a small dipole impinging at different impact parameters; (c) the simplified, black&white, picture of the hadron which is relevant for the average dipole amplitude.*

fluctuations'), and also from one impact parameter to another, for a given realization. Accordingly, a small dipole projectile with size r, which explores an area $\sim r^2$ in impact parameter space and couples to gluons with a typical momentum $k \sim 1/r$ (cf. Fig. 4a), can meet with very different gluon density and thus have a very different view of the hadron, depending upon its impact parameter (cf. Fig. 4b): on the *same* resolution scale $Q^2 = 1/r^2$, the hadron can look either 'black' ($T \sim 1$), or 'grey' ($T \ll 1$), or even almost 'white' ($T \lll 1$), because of the strong inhomogeneities in the gluon distribution generated by the high–energy evolution, and this even when starting with a (quasi)homogeneous distribution at low energy.

In principle the whole statistical information about the gluon distribution (or the dipole scattering) at rapidity Y is encoded in the Pomeron loop equations. In practice, however, these equations are too complicated to be solved exactly, and some approximations are required to make progress. In a first approximation, one can assume that the evolution proceeds quasi–locally in impact parameter, which amounts to neglecting the correlations in b [1, 6]. Then the evolution with Y becomes a one–dimensional stochastic process (with the 'dimension' being the dipole size r, or the gluon transverse momentum $k \sim 1/r$), which, remarkably, turns out to be in the same universality class as the 'reaction–diffusion' process, well studied in the context of statistical physics. By translating the corresponding results to QCD, one can deduce important information about the dynamics in QCD at high energy [1, 2].

Namely, the different realizations of the stochastic evolution differ from each other by the value of the saturation momentum, which is now a random variable, with a rel-

atively simple distribution — a Gaussian in the logarithmic variable $\rho_s \equiv \ln(Q_s^2/Q_0^2)$:

$$P_Y(\rho_s) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\rho_s - \langle\rho_s\rangle)^2}{2\sigma^2}\right], \tag{4}$$

where both the central value $\langle\rho_s\rangle$ and the dispersion $\sigma^2 = \langle\rho_s^2\rangle - \langle\rho_s\rangle^2$ rise linearly with Y : $\langle\rho_s\rangle = \lambda\overline{\alpha}_s Y$ and $\sigma^2 \simeq D\overline{\alpha}_s Y$. The interesting observables, like the dipole amplitude $T(r) \equiv T(\rho)$ with $\rho \equiv \ln(1/r^2Q_0^2)$, depend only upon the difference $\rho - \rho_s = \ln(1/r^2Q_s^2)$ (that is, they show 'geometric scaling' *event–by–event* [2, 5]), and hence their statistical distribution is fully determined by the above distribution for ρ_s.

When the dipole is relatively small, $\rho \gg \langle\rho_s\rangle_Y$, the average amplitude is weak, $\langle T(\rho)\rangle_Y \ll 1$, as expected. What is perhaps less expected is that, for energies so large that $\sigma^2(Y) \gg 1$, this (small) average amplitude is controlled by the rare events in which the scattering is *strong*, $T(\rho) \simeq 1$, i.e., by the (unusually dense) target configurations which are saturated on the resolution scale of the dipole, $\rho_s \gtrsim \rho$, and therefore look 'black' to the latter (*black spots*; see Fig. 4c) [6]. The dominance of the 'black spots' has interesting consequences, like a new scaling law ('diffusive scaling') replacing geometric scaling at sufficiently high energy, and which may be relevant for the phenomenology of DIS, or of particle production at LHC [2, 9].

4 Effective Liouville field theory for saturation

As previously explained, the distribution (4) has been obtained under the assumption that different regions in impact parameter space evolve independently from each other. How reasonable is this assumption ? This is approximately correct for regions which are well separated in b, by distances Δb much larger than the (average) saturation length $1/\overline{Q}_s(Y)$, since the correlations produced via gluon splitting (cf. Fig. 3c) get 'frozen' at some intermediate rapidity y at which $\Delta b \sim 1/\overline{Q}_s(y)$. (Indeed, gluon saturation prohibits the radiation of long–wavelength gluons with $\lambda \gg 1/Q_s$ [10].) On the other hand, this cannot be correct for relatively short separations $\Delta b < 1/\overline{Q}_s(Y)$, as it would contradict the *uncertainty principle* [7]. Indeed, the saturation momentum $Q_s(Y, b)$ is the typical transverse momentum of the local gluon distribution at b, so the size of the 'gluon spot' at b cannot be smaller than $1/Q_s(Y, b)$. This size cannot be much larger either, because of the difficulty to radiate gluons with $\lambda \gg 1/Q_s$ out of a saturated distribution. Therefore, the typical size of a 'spot' is fixed by the local saturation momentum, and all gluons inside such a spot must evolve coherently with each other. When further increasing Y, these gluons can emit even smaller gluons, which then become the seeds for new spots, with smaller size and higher density (i.e., large values for their saturation momenta), distributed over the area of the original, larger, spot. Thus, the evolution is naturally biased, by saturation, towards smaller dipole sizes, and it will eventually produce a kind of fractal geometry, with very small

spots distributed on top of larger ones, which are in turn distributed over even larger ones, and so on. In a three–dimensional representation with the gluon density (or the saturation momentum) taken along the vertical axis, the 'landscape' produced by the evolution should exhibit narrow, well–pronounced, peaks, randomly distributed over plateaux of lower elevation, in turn surrounded by valleys of low density.

The short–range ($\Delta b < 1/\overline{Q}_s(Y)$) correlations within this landscape are quasi–local in Y and can be encoded in an effective field theory in 2 dimensions, obtained as a generalization of the local distribution (4) [7]. The general structure of this theory is constrained by the uncertainty principle together with the *conformal symmetry* of the BFKL evolution [11]. The latter is, of course, broken by saturation on large distance scales $\gtrsim 1/Q_s$, but it should nevertheless survive in the correlations on very short distances, which do not know about saturation. This symmetry constraint turns out to be very strong, since there is essentially only one conformally–invariant field theory in two dimensions: the *Liouville field theory* (LFT).

Namely, in order to promote (4) into a field theory, we shall treat the random variable ρ_s as a *field* in impact parameter space: $\rho_s(\boldsymbol{b}) \equiv \ln[Q_s^2(\boldsymbol{b})/Q_0^2]$, and replace the local distribution (4) by the action for this field: $P_Y(\rho_s) \to \exp\{-S_Y[\phi]\}$, with

$$S_Y[\phi] = \frac{1}{2\sigma^2}\int \mathrm{d}^2\boldsymbol{b}\left\{(\nabla^i\phi)^2 + V(\phi)\right\} \quad \text{with} \quad \phi(\boldsymbol{b}) \equiv \rho_s(\boldsymbol{b}) - \langle\rho_s\rangle\,. \tag{5}$$

The Y–dependence of the effective action is implicit in the parameters $\langle\rho_s\rangle$ (the average saturation momentum, assumed to be homogeneous for simplicity) and σ^2. What should be the potential $V(\phi)$? The simplest choice $V(\phi) = \overline{Q}_s^2\,\phi^2$ is not acceptable, since with this choice, the inhomogeneities would be cut off (by the kinetic term) at the *average* scale $\overline{Q}_s^2$, rather than at the *actual* saturation scale $Q_s^2(\boldsymbol{b})$ in a given event. Clearly, the uncertainty principle must be satisfied *event–by–event* (and not only on the average), and this requires $V(\phi)$ to be proportional to $Q_s^2(\boldsymbol{b}) = \overline{Q}_s^2\,\mathrm{e}^{\phi(\boldsymbol{b})}$. This argument implies $V(\phi) = \overline{Q}_s^2\,\mathrm{e}^{\phi}f(\phi)$, where $f(\phi)$ is an arbitrary function which is slowly varying. At this point, conformal symmetry enters the argument and enforces the choice $f = \text{const.}$, since with this choice one obtains the Liouville field theory. After a conveniant rescaling of the field, $\phi \to \sigma\phi$, we obtain the standard form of the Liouville action

$$S_L[\phi] = \int \mathrm{d}^2\boldsymbol{b}\,\left(\frac{1}{2}(\nabla^i\phi)^2 + \frac{\overline{Q}_s^2}{\sigma^2}\,\mathrm{e}^{\sigma\phi}\right)\,. \tag{6}$$

As anticipated, this is invariant under the scale transformations

$$\boldsymbol{b} \to \boldsymbol{b}' = \lambda\boldsymbol{b}, \qquad \phi(\boldsymbol{b}) \to \phi'(\boldsymbol{b}') = \phi(\boldsymbol{b}) - \frac{2}{\sigma}\ln\lambda \tag{7}$$

and, more generally, under the general conformal transformations in the complex plane $z = b_1 + ib_2$. This conformal symmetry ensures that LFT is *integrable*. But

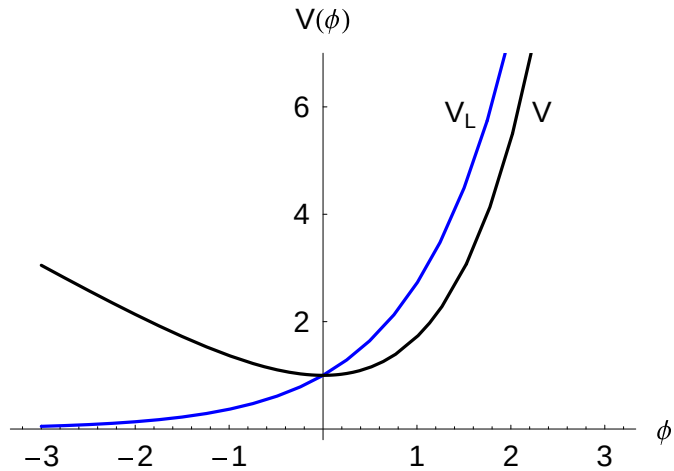

Figure 5: *The potential $V(\phi)$ in the effective action (8) is compared to the Liouville potential $V_L(\phi)$ of (6). Note that $V(\phi)$ has a minimum at $\phi = 0$, corresponding to a stable ground state for the effective theory in perturbation theory.*

this exact symmetry is also a source of annoyance for the QCD problem at hand, as it implies that the average saturation momentum is *zero* : $\langle Q_s^2(\boldsymbol{b})\rangle \equiv \overline{Q}_s^2 \langle \mathrm{e}^{\sigma\phi(\boldsymbol{b})}\rangle = 0$ at any $\boldsymbol{b}$. That is, the mass parameter $\overline{Q}_s^2$ introduced by hand in the action (6) has no intrinsic meaning (as expected for a conformal theory !), and indeed it can be easily scaled away, via a scale transformation in the path–integral for ϕ.

This apparent difficulty reflects the fact that, by itself, Liouville theory is a theory for the *fluctuations* in the gluon distribution, but not also for the *average* value of this distribution: a theory for matter fluctuations, but without the matter ! To obtain a realistic effective theory for the problem at hand, the Liouville actions needs be supplemented with a *source term*, which enforces a non–zero expectation value $\langle Q_s^2\rangle$ for the saturation momentum and thus necessarily breaks down conformal symmetry, *but only on large distances*. To that aim, the source term must modify the shape of the potential at *negative* values of ϕ, in such a way to provide a stable vacuum for the theory (unlike for the Liouville potential; see Fig. 5). As argued in Ref. [7], the simplest choice for this source term, namely a term linear in ϕ :

$$S_Y[\phi] = \int \mathrm{d}^2\boldsymbol{b} \left(\frac{1}{2}(\nabla^i\phi)^2 + \frac{\overline{Q}_s^2}{\sigma^2}\left(\mathrm{e}^{\sigma\phi} - \sigma\phi\right)\right) , \tag{8}$$

is physically acceptable, as it preserves the Liouville potential for $\sigma\phi \gtrsim 1$ and at the same implies the exact result $\langle \mathrm{e}^{\sigma\phi}\rangle = 1$, which ensures the desired property $\langle Q_s^2(\boldsymbol{b})\rangle = \overline{Q}_s^2$ at any $\boldsymbol{b}$. This choice is also mathematically appealing, since the

quantum field theory with action (8) turns out to be *ultraviolet finite* (at least, up to two–loop order in perturbation theory), and thus it requires no UV renormalization.

But independently of the detailed structure of the source term, the effective theory will preserve its conformal properties on short distances, $\Delta b \lesssim 1/\overline{Q}_s(Y)$, and thus generate power–like correlations, with the same powers as in LFT :

$$\langle Q_s^2(\boldsymbol{x})\, Q_s^2(\boldsymbol{y})\rangle \,=\, \frac{C(\sigma)}{|\boldsymbol{x}-\boldsymbol{y}|^4} \quad \text{when} \quad |\boldsymbol{x}-\boldsymbol{y}| \ll 1/\overline{Q}_s(Y)\,. \tag{9}$$

It would be interesting to understand whether this effective theory, and the underlying physical picture, can be extended to accommodate the effects of the *running of the coupling*, which break down conformal symmetry on all scales and may have important consequences for the evolution with Pomeron loops, as shown by the recent analysis in Ref. [12], which was however restricted to a one–dimensional stochastic process, with no impact–parameter dependence.

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