Multiloop Gluon Amplitudes and AdS/CFT

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The meat of this work is a very efficient new algorithm for extracting certain quantities (in particular the cusp anomalous dimension) from gluon amplitudes in $\mathcal{N} = 4$ super-Yang Mills theory. The work does have wider applicability, but our interest in these particular calculations stems from their important impact on studies of integrability in $\mathcal{N} = 4$ super-Yang Mills. Concrete calculations of anomalous dimensions are needed to shed light on certain structures and in particular to test an all-loop conjecture for the cusp anomalous dimension.

The current paradigm is that in the large N (planar) limit, anomalous dimensions of all operators should be determinable by integrability, a technology familiar from the study of spin chains. It is instructive to think of a single-trace operator as a spin chain configuration [10, 11] where the different fields (and their derivatives) appearing inside the trace are the different 'directions' in which each 'spin vector' can point. Under radial evolution (generated by the dilatation operator), the spin vectors on different sites can 'interact' with each other.

An important property of integrable theories is that they can be solved (i.e., the eigenvalues of the Hamiltonian can be found) once the 2-particle S-matrix is known. In order to set up this scattering problem we choose a 'ferromagnetic' ground state $\cdots ZZZZZZZZZ \cdots = Z^J$ which corresponds to a BPS operator whose dimension $\Delta = J$ is protected from quantum corrections. Furthermore we take the infinite volume limit $J \to \infty$ so that we can set up initial and final states of the elementary excitations ('magnons') of the spin chain [7, 13, 20].

The matrix structure of the 2-particle S-matrix is completely fixed by the symmetry [16] up to an overall phase factor which depends on the momenta of the two magnons but not on their species. A conjecture for this phase factor, which satisfies many desirable properties, has been proposed [21, 23]. The motivation for our work was two-fold: to unlock previously hidden mathematical richness lurking deep inside multi-loop gluon amplitudes in $\mathcal{N} = 4$ SYM, and to exploit that structure to help dramatically simplify the difficult calculations necessary to check a consequence of this conjectured S-matrix.

Our target is the cusp anomalous dimension

$$f(\lambda) = \sum_{L=1}^{\infty} \lambda^L f^{(L)}(\lambda) = 4\lambda - 4\zeta(2)\lambda^2 + (4\zeta(2)^2 + 12\zeta(4))\lambda^3 + \mathcal{O}(\lambda^4) \qquad (1)$$

which governs the dimension of twist-two operators in the limit of very large spin:

$$\Delta\left(\mathrm{Tr}[ZD^S Z]\right) = S + f(\lambda)\log S + \mathcal{O}(S^0), \qquad S \gg 1 \tag{2}$$

and has long played an important role in quantitative checks of AdS/CFT [8].

In order to reach this target I will now switch directions and discuss the more familiar S-matrix of Yang-Mills theory—namely the spacetime S-matrix describing the scattering of gluons in four dimensions. Dimensional regularization to $4-2\epsilon$ dimensions is used to tame the infrared divergences.

The first step in calculating an L-loop amplitude is to express it in terms of a relatively small number of scalar integrals. Although it is in general very difficult to determine which integrals contribute to any given amplitude, a systematic procedure for exploiting unitarity to do so leads to what is known as the unitarity-based method or generalized unitarity [2, 3, 12, 15]. This step can be called 'finding the integrand'. For example, generalized unitarity expresses the two-loop amplitude as

$$\frac{\mathcal{A}_4^{(2)}}{\mathcal{A}_4^{(\mathrm{tree})}} = \int \frac{d^D p\, d^D q}{p^2 q^2 (p-k_1)^2 (p-k_1-k_2)^2 (q+k_4)^2 (q+k_3+k_4)^2 (p-q)^2} + (s \leftrightarrow t). \tag{3}$$

Unitarity cuts involving exchange of only two particles in an intermediate channel are particularly easy to analyze, and the relevant algebra extrapolates nicely to all loops. However beginning at four loops [22] the amplitude contains additional integrals that do not participate in any two-particle cuts:

It is an open problem to determine whether there exists an algorithm that would allow a general L-loop amplitude to be easily determined.

Now let us turn to the difficult problem of evaluating multi-loop integrals such as (3), where unitarity is of no help. Consider the L-loop four-gluon amplitude in $D = 4-2\epsilon$. Supersymmetry determines the helicity structure of the amplitude to be the same as that of the tree-level amplitude. The ratio $M^{(L)} = A^{(L)}/A^{(\mathrm{tree})}$ is therefore a function only of ϵ and the Mandelstam variables s, t. By crossing symmetry, the amplitude is symmetric under $s \leftrightarrow t$. Furthermore the amplitude has dimensions of $M^{(L)} \sim [\mathrm{length}]^{2\epsilon L}$. Together, these facts imply that it can be written as

$$M^{(L)}(\epsilon, s, t) = \frac{1}{(st)^{\epsilon L/2}} M^{(L)}(\epsilon, x), \tag{4}$$

where $x = t/s$, and $M^{(L)}(\epsilon, x) = M^{(L)}(\epsilon, 1/x)$.

Amplitudes are almost always studied in an expansion around $\epsilon = 0$. The leading singularity at L loops is ϵ^{-2L}, and higher order terms in the ϵ expansion become

increasingly more complicated. For example the one-loop amplitude is

$$\begin{aligned} M^{(1)} &= -\frac{2}{\epsilon^2} + \Big[\frac{1}{4}L^2 + 4\zeta(2)\Big] + \epsilon\Big[- H_{001}(-x) + LH_{01}(-x) - \frac{1}{2}L^2 H_1(-x) \\ &\qquad -3\zeta(2)H_1(-x) - \frac{3}{2}\zeta(2)L - \frac{L^3}{12} + \frac{17\zeta(3)}{3}\Big] + \epsilon^2\Big[H_{0001}(-x) + \cdots \end{aligned}$$

It is apparently a property of the expansion of any L-loop amplitude that all of the terms which appear at any given power in ϵ have the same degree of transcendentality. Specifically, the coefficient of ϵ^{-2L+k} is a linear combination, with rational coefficients, of terms with degree of transcendentality k.

Transcendentality allows for a tremendous compression of the amount of 'data' required to specify any amplitude. Any amplitude can be expressed, order by order in ϵ, not in terms of completely arbitrary functions of x, but rather in terms of a finite collection of rational numbers. It is unfortunate that current technologies for evaluating multi-loop amplitudes obscure this structure. Reducing a desired amplitude to its rational 'coefficients' is like picking needles out of a haystack.

Moreover, based on resummation work by Magnea, Sterman and Tejeda-Yeomans [1, 9], and infrared singularity work by Catani [4], it is believed [14] that the planar n-particle L-loop MHV amplitude satisfies an iterative relation of the form

$$M_n^{(L)}(\epsilon) = P^{(L)}(M_n^{(1)}(\epsilon), \ldots, M_n^{(L-1)}(\epsilon)) + f^{(L)}(\epsilon) M_n^{(1)}(L\epsilon) + C^{(L)} + \mathcal{O}(\epsilon) \qquad (5)$$

where $P^{(L)}$ are some known polynomials. The only 'unknown' quantities in the iterative relations are $f^{(L)}(\epsilon)$ and $C^{(L)}$, which have been computed analytically through three loops [14].

The iterative relations imply that we havr to sift all the way through to ϵ^{-2} in order to find any 'new' information—the vast majority of the rational coefficients which specify the L-loop amplitude are completely determined in terms of lower loop amplitudes. The one unfixed number at order ϵ^{-2} is $f^{(L)}(0)$. This quantity of particular interest to us; in fact it is the L-loop contribution to the cusp anomalous dimension.

It would be nice to develop some kind of technology which would act like a sieve to help us seek out single numbers from these complicated amplitudes. For example it would be great if there were a procedure to isolate those integrals which contribute to any particular coefficient of interest, say the coefficient of $H_{0001}(-x)$, and that would enable us to calculate the rational number multiplying any given term without calculating everything else. This is probably too much to hope for, but there is an efficient algorithm for reading off the coefficient of any term of the form $\epsilon^i \log^k x$, and it is sufficient to know these particular terms in order to read off the cusp anomalous dimension from the iteration (5).

First observe that any dimensionally regulated L-loop four-particle Feynman integral can be written in the form (Mellin-Barnes representation)

$$\int_{-i\infty}^{+i\infty} dy\ x^y F(y,\epsilon), \qquad F(y,\epsilon) = F(-y,\epsilon) \tag{6}$$

for some function $F(y,\epsilon)$, which is relatively easy to determine [5, 6, 17]. As an example (not representative, because of its simplicity), the one-loop amplitude (5) is completely encapsulated in the expression

$$F(y,\epsilon) = \Gamma(1+\tfrac{1}{2}\epsilon+y)\Gamma^2(y-\tfrac{1}{2}\epsilon)\Gamma^2(-y-\tfrac{1}{2}\epsilon)\Gamma(1-\tfrac{1}{2}\epsilon-y). \tag{7}$$

The final integral over y in (6) is the really nasty one.

Typically $F(y,\epsilon)$, as in the example above, has singularities at $y=0$ in the $\epsilon \to 0$ limit. These singularities can be isolated by using the formula

$$\lim_{\epsilon\to 0} \frac{1}{y \pm \epsilon} = \mathcal{P}\frac{1}{y} \pm \pi\delta(y) \tag{8}$$

(and its derivatives). In this way one obtains a unique decomposition of any amplitude into two pieces,

$$\int_{-i\infty}^{+i\infty} dy\ x^y \left[\mathcal{P}F(y) + G(y)\right], \tag{9}$$

where the first term is nonsingular at $y=0$ and all of the singularities are collected into $G(y)$, which we call the obstruction. If we note that

$$(\log^2 x)^k = \int_{-i\infty}^{+i\infty} dy\ x^y \frac{d^{2k}}{dy^{2k}}\delta(y) \tag{10}$$

then we see that in x space, obstructions are always polynomial in $\log^2 x$.

Hence we find a canonical way of writing any amplitude as a sum of two pieces. The obstructions correspond to the singular part of the amplitude's Mellin transform in y space, and the $\log^k x$ terms in x-space. The remaining smooth part of the Mellin transform corresponds to the harmonic polylogs in x space. A crucial property of these obstructions is that they satisfy a product algebra structure. Namely, the product of two obstructions is again an obstruction simply because the convolution of two delta functions (or derivatives of delta functions) is again a delta function. This means that we can calculate the obstructions in each loop amplitude $M^{(L)}$ individually and then we know the obstructions in any power, or product of powers, of these amplitudes, which feed into the relation (5).

For example, the leading obstructions (the coefficient of $(\log x)^0$) through three loops are

$$M^{(1)}(\epsilon) \ \sim\ -\frac{2}{\epsilon^2} + \frac{2\pi^2}{3} + \epsilon\frac{17\zeta(3)}{3} + \epsilon^2\frac{41\pi^4}{720} + \mathcal{O}(\epsilon^3), \tag{11}$$

$$M^{(2)}(\epsilon) \sim \frac{2}{\epsilon^4} - \frac{1}{\epsilon^2}\frac{5\pi^2}{4} - \frac{1}{\epsilon}\frac{65\zeta(3)}{6} - \frac{\pi^4}{90} + \mathcal{O}(\epsilon), \tag{12}$$

$$M^{(3)}(\epsilon) \sim -\frac{4}{3\epsilon^6} + \frac{1}{\epsilon^4}\frac{7\pi^2}{6} + \frac{1}{\epsilon^3}\frac{31\zeta(3)}{3} - \frac{1}{\epsilon^2}\frac{161\pi^4}{3240} + \mathcal{O}(\epsilon^{-1}). \tag{13}$$

These expressions by themselves obey the expected iterative relation

$$M^{(3)}(\epsilon) = -\frac{1}{3}\left(M^{(1)}(\epsilon)\right)^3 + M^{(1)}(\epsilon)M^{(2)}(\epsilon) + \frac{11\pi^4}{180}M^{(1)}(3\epsilon) + \mathcal{O}(\epsilon^{-1}). \tag{14}$$

To summarize, in order to read off the L-loop cusp anomalous dimension from an L-loop four-particle amplitude, we don't need to calculate the entire amplitude. It is sufficient to start off with the (relatively far simpler) expressions for the Mellin transform and then just read off the coefficient of $\delta(y)/\epsilon^2$ since only this particular coefficient contributes to the cusp anomalous dimension. An algorithm for performing this calculation is easily implemented in Mathematica, building on some code by Czakon [17]. An an application of this method we have obtained a numerical result $f^{(4)} = -117.1789 \pm 0.0002$, in very good agreement with the conjecture [23] $f^{(4)} = -117.1788285\ldots$ and an improvement in precision over the previous numerical result [22] by about a factor of one thousand. The improvement is possible because we only need to compute the obstructions, not the full amplitude, so we can throw most of the terms away.

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References

[1] L. Magnea and G. Sterman, Phys. Rev. D **42**, 4222 (1990).

[2] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425**, 217 (1994) [arXiv:hep-ph/9403226].

[3] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **435**, 59 (1995) [arXiv:hep-ph/9409265].

[4] S. Catani, Phys. Lett. B **427**, 161 (1998) [arXiv:hep-ph/9802439].

[5] V. A. Smirnov, Phys. Lett. B **460**, 397 (1999) [arXiv:hep-ph/9905323].

[6] J. B. Tausk, Phys. Lett. B **469**, 225 (1999) [arXiv:hep-ph/9909506].

[7] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, JHEP **0204**, 013 (2002) [arXiv:hep-th/0202021].

[8] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Nucl. Phys. B **636**, 99 (2002) [arXiv:hep-th/0204051].

[9] G. Sterman and M. E. Tejeda-Yeomans, Phys. Lett. B **552**, 48 (2003) [arXiv:hep-ph/0210130].

[10] J. A. Minahan and K. Zarembo, JHEP **0303**, 013 (2003) [arXiv:hep-th/0212208].

[11] N. Beisert, Phys. Rept. **405**, 1 (2005) [arXiv:hep-th/0407277].

[12] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B **725**, 275 (2005) [arXiv:hep-th/0412103].

[13] M. Staudacher, JHEP **0505**, 054 (2005) [arXiv:hep-th/0412188].

[14] Z. Bern, L. J. Dixon and V. A. Smirnov, Phys. Rev. D **72**, 085001 (2005) [arXiv:hep-th/0505205].

[15] E. I. Buchbinder and F. Cachazo, JHEP **0511**, 036 (2005) [arXiv:hep-th/0506126].

[16] N. Beisert, arXiv:hep-th/0511082.

[17] M. Czakon, Comput. Phys. Commun. **175**, 559 (2006) [arXiv:hep-ph/0511200].

[18] F. Cachazo, M. Spradlin and A. Volovich, JHEP **0607**, 007 (2006) [arXiv:hep-th/0601031].

[19] F. Cachazo, M. Spradlin and A. Volovich, Phys. Rev. D **74**, 045020 (2006) [arXiv:hep-th/0602228].

[20] D. M. Hofman and J. M. Maldacena, J. Phys. A **39**, 13095 (2006) [arXiv:hep-th/0604135].

[21] N. Beisert, R. Hernandez and E. Lopez, JHEP **0611**, 070 (2006) [arXiv:hep-th/0609044].

[22] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower and V. A. Smirnov, Phys. Rev. D **75**, 085010 (2007) [arXiv:hep-th/0610248].

[23] N. Beisert, B. Eden and M. Staudacher, J. Stat. Mech. **0701**, P021 (2007) [arXiv:hep-th/0610251].

[24] F. Cachazo, M. Spradlin and A. Volovich, Phys. Rev. D **75**, 105011 (2007) [arXiv:hep-th/0612309].

[25] Z. Bern, J. J. M. Carrasco, H. Johansson and D. A. Kosower, arXiv:0705.1864.