Vacuum Geometry and the Search for New Physics

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1 Introduction

Obtaining the phenomenology of our real, four-dimensional world based on a fundamental selection principle from the plethora of string theory vacua remains one of the most pertinent challenges to modern physics. The Minimally Supersymmetric Standard Model (MSSM), that Holy Grail, shines upon us with aloof lustre, from an ever-bridging and yet still untangeable distance.

Ever since the pioneering work of [1] some two decades ago, much progress has been made. However, no consensus has been reached and each approach, be it heterotic compactification, intersecting brane models, conformal field theories or branes probing singularities, has its own virtues and vices. Indeed, so vast are the possibilities and methods, a probabilistic "landscape" of Standard-like vacua has been proposed [2]. Yet, along a parallel skein, a model with exact MSSM spectrum without exotic matter has only recently been found [3] and the peculiarity of a special corner of the landscape [4], been investigated. Perhaps, we must pause to question ourselves, there is a yet unfathomed geometry upon which string theory would uniquely give the properties of the MSSM.

Casting aside the top-down stringy perspective momentarily, even as field theorists, the very nature of the MSSM bemuses us. Already, there are unexplained properties: why are there so many free parametres? why is the μ-term so finely tuned? why is RH-neutrino mass so small? etc. Is there, indeed, any further hidden structure which makes our world *sui generis*, rather than generic?

In any quantum mechanical theory, the most fundamental quantity is the vacuum. In a supersymmetric field theory, the vacuum is often parametrised by a multitude of flat directions, the lifting of which is a much pursued enterprise. These flat directions, in general, parametrise a non-trivial geometry, usually called the **vacuum moduli space** $\mathcal{M}$. The specialness of this geometry is the subject my talk. The proposal [5, 6] is that, whether we believe in string theory or not,

> Any special structure of the vacuum moduli space $\mathcal{M}$ should be regarded as fundamental.

This statement is one of **naturalness**. By such a "special structure" I clearly mean properties beyond the apparent gauge and discrete symmetries, and which may be used a indications of new physics. Indeed, one could study the topological and algebro-geometric quantities of $\mathcal{M}$. Deviations from certain special values as one adds higher mass-dimension operators, for example, could be interpreted as a **selection principle**. Is, therefore, the MSSM special in this sense? Does the algebraic geometry of $\mathcal{M}$ provide us with selection rules, both for determining operators in the field theoretic perspective and for string vacuum determination? Are geometry and phenomenology co-extensive? These are the key questions of our investigation.

2 The Vacuum Moduli Space of $\mathcal{N} = 1$ Theories

Let us begin with a lightning review of $\mathcal{N} = 1$ supersymmetric gauge theories, paying particular attention to $\mathcal{M}$, to set the notation. Given an $\mathcal{N} = 1$ theory with chiral (matter) superfields Φ_i in the representation R_i of the gauge group G, a vector (gauge) superfield V valued in the Lie algebra of G, its assciated field strength $\mathcal{W}_\alpha = \overline{D}^2 D_\alpha V$, as well as a superpotential $W(\Phi)$ which is some holomorphic polynomial in the fields Φ_i, together with a coupling constant g, the action, integrating over superspace, is

$$S = \int d^4x \; [\int d^4\theta \; \Phi_i^\dagger e^V \Phi_i + \left(\frac{1}{4g^2} \int d^2\theta \; \mathrm{Tr} \mathcal{W}_\alpha \mathcal{W}^\alpha + \int d^2\theta \; W(\Phi) + \mathrm{h.c.} \right)]. \qquad (1)$$

The scalar potential is

$$\mathcal{V}(\phi_i, \overline{\phi_i}) = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 + \frac{g^2}{4} (\sum_i q_i |\phi_i|^2)^2 \ , \qquad (2)$$

where ϕ_i are the scalar components of Φ_i and q_i, their charges. Hence, the vacuum is defined by (and we label the so-called D- and F-terms):

$$\mathcal{M} = \left\{ \phi_i, \overline{\phi_i} : \mathcal{V}(\phi_i, \overline{\phi_i}) = 0 \right\} \Rightarrow \left\{ \begin{array}{ll} \frac{\partial W}{\partial \phi_i} = 0 & \text{F-TERMS} \\ \sum\limits_i q_i |\phi_i|^2 = 0 & \text{D-TERMS} \end{array} \right. \qquad (3)$$

The vacuum moduli space $\mathcal{M}$ is thus the space of solutions to D- and F-flatness.

It has been the *lex non scripta*, long circulated in the literature, that the most conducive method of solving (3) is to recognise the D-terms as no more than gauge-fixing conditions and that (cf. e.g. [7] which phrases the problem in a language similar to ours) $\mathcal{M}$ can be found by: (1) Find the generating set of all gauge invariant operators $D = \{GIO\}$ and (2) Solve F-flat $\frac{\partial W}{\partial \phi_i} = 0$ and back-substitute into D. As an illustrative example, consider an $SU(N) \times SU(N)$ theory with fields (x_1, x_2) charged as $(\Box, \overline{\Box})$ and (y_1, y_2) as $(\overline{\Box}, \Box)$, together with a superpotential

$$W = \mathrm{Tr}(x_1 y_1 x_2 y_2 - x_1 y_2 x_2 y_1) \ . \qquad (4)$$

Take, for simplicity, $N = 1$; whence $W = 0$ and F-terms are trivial. The generating set D is given by $GIO = \{D_1, \ldots, D_4\} = \{x_i y_j\}_{i,j=1,2}$. Defining $u = x_1 y_1, v = x_2 y_2, z = x_1 y_2, w = x_2 y_1$, we have that

$$\mathcal{M} = \{uw - vz = 0\} \subset \mathbb{C}[u, v, w, z] \tag{5}$$

since there is a single non-trivial relation amongst the GIO's. We have, therefore, realised $\mathcal{M}$ explicitly in coördinates. We instantly recognise (5) as the equation for the conifold, the most well-known local Calabi-Yau threefold. This is no surprise to us; indeed, the original theory is that of the famous Klebanov-Witten gauge theory of D3-branes probing the conifold [8]. This correspondence illustrates an important fact, which was one of the original inspirations to our geometric perspective: *for a gauge theory arising from AdS/CFT scenarios of D-branes probing a transverse space S, its vacuum $\mathcal{M}$ is the space S.* Had the MSSM, alas, been a quiver theory, then finding the precise geometry for the brane-world would be to find $\mathcal{M}$ for the MSSM, and vast degeneracies of string vacua would be eliminated.

The above example of the conifold is a simple one; perhaps we have not found *the* geometry of the real world because we have been looking at such naïve examples. To proceed further, we need significant improvement of computational power; this is the subject of [5, 6].

2.1 Vacuum Moduli Space: Algebro-Geometric Perspective

The fact that we are dealing with essentially solving (complicated) systems of polynomial equations suggests that we should rephrase the problem of determining $\mathcal{M}$ in the language of ideals in polynomial rings. Thus, we map the above method to one of **algorithmic algebraic geometry** and computational commutative algebra:

1. Let there be n-fields: start with polynomial ring $\mathbb{C}[\phi_1, \ldots, \phi_n]$;

2. D = set of k GIO's: prescribes a **ring map** $\mathbb{C}[\phi_1, \ldots, \phi_n] \xrightarrow{D} \mathbb{C}[D_1, \ldots, D_k]$;

3. F-flatness $\langle f_{i=1,\ldots,n} = \frac{\partial W(\phi_i)}{\partial \phi_i} = 0 \rangle$: think of as **ideal** $F := \langle f_i \rangle \subset \mathbb{C}[\phi_1, \ldots, \phi_k]$;

4. $\mathcal{M}$ = image of ring map: $\frac{\mathbb{C}[\phi_1,\ldots,\phi_n]}{\{F=\langle f_1,\ldots,f_n \rangle\}} \xrightarrow{D=GIO} \mathbb{C}[D_1, \ldots, D_k]$, $\quad \mathcal{M} \simeq \mathrm{Im}(D)$.

With this algorithm, $\mathcal{M}$ is explicitly realised as an **affine algebraic variety** in $\mathbb{C}^k$. Fortunately, detailed algorithms realising the above method has been well-developed; the Gröbner bases and syzygies thereof are conveniently implemented by computer software such as *Macaulay2* [9].

Let us apply, for example, the abovementioned procedure to the $U(1)^3$ quiver gauge theory arising from D3-branes probing the Calabi-Yau orbifold singularity $S =$

$\mathbb{C}^3/\mathbb{Z}_3$. As in the conifold example, we expect $\mathcal{M}$ to be the equation for S. Indeed, running the algorithm through Macaulay2, we find that $\mathrm{Im}(D)$ is the (incomplete) intersection of 27 quadrics in $\mathbb{C}^{10}$, which is indeed the equation for the orbifold. The computation, which is rather tedious *per manus*, takes less than a second on a standard laptop.

3 The Moduli Space of the MSSM

Thus emboldened, we can proceed to the *ne plus ultra* of our concern: the MSSM. This, of course, is a much more complicated theory than the examples considered above. It has 7 fields, which, expanded to components, gives $n = 49$:

INDICES	
$i,j,k,l = 1,2,3$	Flavour indices
$a,b,c,d = 1,2,3$	$SU(3)_C$ color indices
$\alpha,\beta,\gamma,\delta = 1,2$	$SU(2)_L$ indices

FIELDS		FIELDS	
u_a^i	$SU(2)_L$ singlet up-quarks	L_α^i	$SU(2)_L$ doublet leptons
d_a^i	$SU(2)_L$ singlet down-quarks	e^i	$SU(2)_L$ singlet leptons
$Q_{a,\alpha}^i$	$SU(2)_L$ doublet quarks	H^α	up Higgs
		$\overline{H}^\alpha$	down Higgs

The superpotential, including all renormalisable terms compatible with R-parity is

$$\begin{aligned} W_{renorm} &= C^0 \sum_{\alpha,\beta} H^\alpha \overline{H}^\beta \epsilon_{\alpha\beta} + \sum_{i,j} C^1_{ij} \sum_{\alpha,\beta,a} Q^i_{a,\alpha} u^j_a H_\beta \epsilon_{\alpha\beta} + \\ &\quad + \sum_{i,j} C^2_{ij} \sum_{\alpha,\beta,a} Q^i_{a,\alpha} d^j_a \overline{H}_\beta \epsilon_{\alpha\beta} + \sum_{i,j} C^3_{ij} \sum_{\alpha,\beta,a} L^i_\alpha \overline{H}_\beta e_j \epsilon_{\alpha\beta} \ , \end{aligned} \tag{6}$$

where the C's are couplings. The generating set D of GIO's is also significantly more involved. Luckily, this has been lucidly presented in [10]. In component form, $k = |D| = 991$ and for brevity we will not present them here. A frontal attack of substituting the above data into the algorithm is overly optimistic; currently we do not have the computer resources to address ring maps of this magnitude. Let us, therefore, first resort to some warm-up exercises.

3.1 Case Study: One Generation

Let us consider the toy example of the one-generation MSSM. Dropping family i, j, k indices, n is reduced to 9 and k also, to 9:

$$D = \{LH, H\overline{H}, QdL, QuH, Qd\overline{H}, L\overline{H}e, QuQd, QuLe, Qu\overline{H}e\}. \tag{7}$$

The renormalisable superpotential becomes:

$$W_{renorm} = \sum_{\alpha,\beta} H^\alpha \overline{H}^\beta \epsilon_{\alpha\beta} + C^1 \sum_{\alpha,\beta,a} Q_{a,\alpha} u^a H^\beta \epsilon_{\alpha\beta} + C^2 \sum_{\alpha,\beta,a} Q_{a,\alpha} d_a \overline{H}_\beta \epsilon_{\alpha\beta} + C^3 \sum_{\alpha,\beta,a} L_\alpha \overline{H}_\beta e \epsilon_{\alpha\beta} \ . \tag{8}$$

Now, we are easily within our calculational grasp. In fact, to illustrate an aforementioned point of selecting operators of certain mass dimension, we can deform W_{renom}

with various operators. We tabulate below our results:

$W_{renorm}+?$	$\dim\mathcal{M}$	$\mathcal{M}$	$W_{renorm}+?$	$\dim\mathcal{M}$	$\mathcal{M}$
0	1	$\mathbb{C}$	$QuQd$	1	$\mathbb{C}$
LH	0	point	$QuLe$	1	$\mathbb{C}$
QdL	0	point	$Qu\overline{H}e$	1	$\mathbb{C}$

We see that $\mathcal{M}$ is not too interesing in all of the above cases, being either the complex line or a point. Perhaps Nature needs more than one generation to be interesting!

3.2 Case Study: Electro-weak MSSM

Our next example is much more tantalising. Let us consider the electro-weak sector of the MSSM (and for the moment forget that it is anomalous). Ignoring all quarks, we now have $n = 13$ component fields with $k = 22$ basic GIO's:

Type	Explicit Sum	Index	Number
LH	$L_i^\alpha H^\beta \epsilon_{\alpha\beta}$	$i = 1, 2, 3$	3
$H\overline{H}$	$H_\alpha \overline{H}_\beta \epsilon_{\alpha\beta}$		1
LLe	$L^i_\alpha L^j_\beta e^k \epsilon_{\alpha\beta}$	$i, j = 1, 2, 3; k = 1, \ldots, j-1$	9
$L\overline{H}e$	$L^i_\alpha \overline{H}_\beta \epsilon_{\alpha\beta} e^j$	$i, j = 1, 2, 3$	9

with renormalisable superpotential:

$$W_{renorm} = C^0 \sum_{\alpha,\beta} H^\alpha \overline{H}^\beta \epsilon_{\alpha\beta} + \sum_{i,j} C^3_{ij} \sum_{\alpha,\beta,a} L^i_\alpha \overline{H}_\beta e_j \epsilon_{\alpha\beta} \ . \tag{9}$$

Now we determine that $\dim(\mathcal{M}) = 5$. If we deformed W_{renorm} by adding all up to dimension four operators, habitually performed in lifting Higgs vevs, we would have

$$W = W_{renorm} + \lambda(\sum_{\alpha,\beta} H^\alpha \overline{H}^\beta \epsilon_{\alpha\beta})^2 + \sum_{\alpha,\beta,i,j} \lambda^{ij} (L_i H_\alpha)(L_j H_\beta) \epsilon_{\alpha\beta} \ . \tag{10}$$

Now, running through our algorithm, we obtain that $\dim(\mathcal{M}) = 3$. Specifically, we can find that $\mathcal{M}$ is affine cone over a compact 2-dimensional base B and B is the (non-complete) degree 4 intersection of 6 quadrics in $\mathbb{P}^5$ as a projective variety with Hodge diamond $h^{p,q}(B) = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$. In fact, we can pin-point B to a classical variety: the **Veronese surface**, a progeny of the *Scuola Italiana* of algebraic geometry at the turn of twentieth century. Amazingly, adding *any* R-parity violating operator ruins this *special structure*. For example, adding LH to (9) reduces $\mathcal{M}$ to $\mathbb{C}$ and adding LLe collapses $\mathcal{M}$ to a point. Moreover, intriguingly, if we added the right-handed neutrino, ν, then we need not go higher in mass dimension and already at the renormalisable level, i.e.,

$$W_{renorm} = C^0 \sum_{\alpha,\beta} H^\alpha \overline{H}^\beta \epsilon_{\alpha\beta} + \sum_{i,j} C^3_{ij} \sum_{\alpha,\beta,a} L^i_\alpha \overline{H}_\beta e_j \epsilon_{\alpha\beta} + C^4_{ij} \nu^i \nu^j + \sum_i C^5_{ij} \nu^i \sum_{\alpha,\beta} L^\alpha_j H^\beta \epsilon_{\alpha\beta} \ , \tag{11}$$

we find that $\mathcal{M}$ is our cone over the Veronese surface. Nature seems to desire the righ-handed neutrino for the sake of special geometry! *There is no explanation of this simple structure from field theory.*

3.3 The Special Structure of the Veronese

Is it a mere coincidence that we have chanced on a named geometry? How special is the Veronese surface? Let us play a statistical game. Consider all operators, including the right-handed neutrino, up to renormalisable level:

Mass Level	Interactions
1	ν
2	LH, $H\overline{H}$, ν^2
3	$L\overline{H}e$, LLe, $LH\nu$, $H\overline{H}\nu$, ν^3

There are 9 operators, thus assuming *flavour democracy* - i.e., if one family should appear then all three families are present - there would be $2^9 = 512$ possible super-potentials and subsequent theories. We can compute $\mathcal{M}$ for each member of this data set. We find 58 cases which have $\dim(\mathcal{M}) = 3$, of these, 4 are cones over the Veronese and 4 are cones over the discrete union of the Veronese and $\mathbb{C}$. Of these 8, most are either unphysical (e.g., absence of leptons) or ineffectual (e.g., replacing Higgs μ-parametres with the right-handed neutrino vev). The only non-trivial, physical theory with the Veronese moduli space is the one in (11). Our special structure is indeed a *rara avis*!

4 Conclusions and Prospectus

We have developed an efficient method for explicitly finding the vacuum of supersymmetric gauge theories. Applying the algorithm to the MSSM, we indeed find unexpected algebro-geometrical structure: in the electro-weak sector, $\mathcal{M}$ is a complex affine cone over the Veronese surface. Various deformations show that our result is consistent with right-handed neutrino, R-parity, and flavour phenomenology. Most importantly, our Guiding Principle of the *coextensivity of geometry and phenomenology* exhibits herself with seductive encouragement.

The most pressing challenge which confronts us is, of course, to find $\mathcal{M}$ for the full MSSM and to analyse it fully. This is doubtlessly one of the most important quantities in modern field theory to compute. This challenge is manifold and inter-disciplinary. To the field theorist, can we use $\mathcal{M}_{MSSM}$ as a new guide to phenomenology? To the string theorist, can this be an aid to find *the correct* compactification? To the mathematician, will it give novel geometries to study? To the computer scientist, what new, efficient algorithms can help us with finding $\mathcal{M}_{MSSM}$?

Our preliminary results are intriguing and appealing, prompting us with the vigour to further the programme. Let me conclude with a bold addendum to that ancient adage of "*Natura abhorret a vacuo*", and say that "*cum Natura abhorrat a vacuo, tamen spatium modulorum vacui amat.*"

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