Lecture 1 Outline

1. Levels of approximation
2. Modern color and helicity organization of amplitudes
3. Soft and collinear behavior
Levels of Approximation

• Monte Carlos (PYTHIA, HERWIG, …)
• LO, fixed-order matrix elements (ALPGEN, …)
• LO MEs matched to parton showers
• NLO MEs (parton level)
• NLO MEs matched to showers (MC@NLO)
• NNLO MEs
• MC@NNLO?
Monte Carlos

- Based on properties of **soft and collinear radiation** in QCD
- Partons surrounded by “cloud” of soft and collinear partons
- Leading double logs of $Q_{\text{hard}}/Q_{\text{soft}}$ exponentiate, can be generated **probabilistically**
- Shower starts with **basic** $2 \rightarrow 2$ parton scattering
  -- or **basic production process** for $W$, $Z$, $tt$, etc.
- Further radiation **approximate**, requires infrared cutoff
- Shower can be evolved down to very low $Q_{\text{soft}}$, where models for **hadronization** and **spectator interactions** can be applied
- Complete hadron-level event description attained
- Normalization of event rates **unreliable**
- Event “shapes” **sometimes unreliable**
Monte Carlos in pictures

Splitting probability: \[ P_g(q^2) = \int_0^1 dz \frac{\alpha_s(q^2)}{2\pi} \tilde{P}_{gg}(z) \Theta(q^2 - q_0^2) \]

\[ \Delta_g(Q^2, q^-) = \exp \left[ - \int_{q^2} q^- \frac{q^-}{q^2} P_g(q^2) \right] \]

make hadrons

Sudakov

L. Dixon, 7/20/06

Higher Order QCD: Lect. 1
Leading order matrix elements

- Based on sum of all tree-level Feynman diagrams in QCD
- Generates correct hard radiation pattern (at tree level)
- Event “shapes” often fairly reliable
- Event rates (normalization) still fairly unreliable, especially if:
  - more jets \( \rightarrow \) more powers of \( \alpha_s(\mu_{R,F}) \)
  - gluons in the initial state (lots of extra soft radiation)
  - cases where new subprocesses appear at NLO \( (q\bar{q} \rightarrow \gamma\gamma) \)
- Description is only at parton level
- Sophisticated programs can now rapidly produce tree-level cross sections for very high multiplicity
- Some use Feynman diagrams \( \text{MadGraph; GRACE; CompHEP, etc.} \)
- Other use recursive or iterative organization \( \text{Berends, Giele, VECBOS, NJETS; HELAC; ALPHA \rightarrow ALPGEN} \)
- Recent techniques spun off from “twistor string theory”:
  - MHV vertices; on-shell recursive; other scalar-type graphs
  \( \text{Cachazo, Svrcek, Witten; Britto, Cachazo, Feng (2004); Schwinn, Weinzierl (2005)} \)
Leading order state of art

Number of Feynman diagrams grows very rapidly with number of legs!

<table>
<thead>
<tr>
<th>Process</th>
<th>$n = 7$</th>
<th>$n = 8$</th>
<th>$n = 9$</th>
<th>$n = 10$</th>
</tr>
</thead>
</table>
| $g
g\rightarrow n\ g$ | 559,405 | 10,525,900 | 224,449,225 | 5,348,843,500 |
| ME per minute | 28000   | 9170    | 2870    | 870     |
| $q\bar{q}\rightarrow n\ g$ | 231,280 | 4,016,775 | 79,603,720 | 1,773,172,275 |

Table 1:

Number of Feynman diagrams corresponding to amplitudes with different numbers of quarks and gluons. CPU performance on a pentium III 850MH

(notice: $n = 10$ expected in some R-parity breaking scenarios)

Caravaglios, Moretti (1995); Caravaglios, Mangano, Moretti, Pittau (1999); Mangano, Moretti, Piccinini, Pittau, Polosa (2002)
Not just quarks and gluons

Shopping list

- $W^* Q\bar{Q} + n$-jets,
- $W^* + n$-jets
- $Z^*/\gamma^* Q\bar{Q} + n$-jets
- $Z^*/\gamma^* + n$-jets
- $Q\bar{Q} + n$-jets
- $Q\bar{Q}Q\bar{Q} + n$-jets
- $Q\bar{Q} + H + n$-jets
- $n$-$W + m$-$Z + l$-$H + n$-jets
- $n$-jets
- $m$-$\gamma + n$-jets
- $t(+W, +b, +Wh) + n$-jets
- $H + n$-jets
- $W^*(Z^*/\gamma^*) + m$-$\gamma + n$-jets
- $Q\bar{Q} + m$-$\gamma + n$-jets

ALPGEN

$W^* \equiv l\nu_l$ and $Q = b, t, (c)$.

jets $= \text{“light” quarks, gluons}$

$ggH$ effective coupling ($m_t \rightarrow \infty$)

in progress

in progress
Matching MEs to showers

- Would like to have both:
  - accurate hard radiation pattern of MEs
  - hadron-level event description of parton-shower MCs
- Why not just use $2 \rightarrow 3, 4, \ldots$ parton processes as starting point for the shower?
- Problem of **double-counting**: When does radiation “belong” to the shower, and when to the hard matrix element?
ME/shower matching

- **CKKW matching:**
  - separate ME and shower domains using a common jet cluster algorithm variable ($k_T$ algorithm with $y = y_{ini}$)

- an example in pictures:  

  Catani, Kuhn, Krauss, Webber, hep-ph/0109231

Nagy, Soper, hep-ph/0607046

|ME|$^2_{n=6}$  

|ME|$^2_{n=4}$  

ratio regulated by jet algorithm; results depend on $y_{ini}$

L. Dixon, 7/20/06

Higher Order QCD: Lect. 1
Several other general matching schemes available or in the works, e.g.:

- MLM scheme (ALPGEN)
- Lonnblad, hep-ph/0112284 (Ariadne)
- CKKW (Sherpa)
- Mrenna, Richardson, hep-ph/0312274
- Nagy, Soper, hep-ph/0601021
- Skands, Giele, Kosower

ALPGEN, Ariadne, Sherpa compared in Hoche et al., hep-ph/0602031

$\bar{p}p \rightarrow W + 4 \text{ jets at Tevatron}$

reasonable agreement between different schemes
NLO ME calculations

- Based on sum of all one-loop QCD Feynman diagrams for a given $n$-parton process (plus any “electroweak” particles)
- Also need to square tree amplitudes for $(n+1)$-parton process
  - these contribute at same order in $\alpha_s$
  - infrared singularities cancel between virtual and real terms
- Event “shapes” usually quite reliable
  - except near kinematic boundaries (e.g. $p_T(W) \to 0$)
- Normalization of event rates usually pretty reliable (10% level)
- Description is only at parton level
- One-loop amplitudes are still generally hand-crafted
  - often with agonizing care taken over the finished product!
- NLO programs scattered about
- Feynman diagrams very often used
- Techniques spun off from “twistor string theory” – MHV vertices, on-shell recursive bootstrap – now almost ready for phenomenology
Infrared cancellations at NLO

LO

NLO

real

virtual

soft singularities: \( k_s \to 0 \)

collinear singularities: \( k_{ab}^2 \to 0 \) (\( k_a \parallel k_b \))

virtual soft/collinear singularities:

\[ \sigma_{\text{vir}} \sim \frac{1}{\epsilon^2} \sum_i C_i - \frac{1}{\epsilon} \sum_{i,j} D_{ij} \ln (\frac{\mu^2}{s_{ij}}) \] \( \sigma^{\text{LO}} \)

Use dimensional regularization,

\( D = 4 - 2\epsilon \)

\( d^4 k \to d^{4-2\epsilon} k \)
in all phase-space and loop integrals

- Virtual corrections cancel real singularities, but only for quantities insensitive to soft/collinear radiation → infrared-safe observables \( O \)
Infrared safety

**infrared-safe observables** $O$:
- Behave smoothly in soft limit as any parton momentum $\to 0$
- Behave smoothly in collinear limit as any pair of partons $\to$ parallel (||)

$$O_n(\ldots, k_s, \ldots) \to O_{n-1}(\ldots, x_s, \ldots) \quad k_s \to 0$$
$$O_n(\ldots, k_\alpha, k_\beta, \ldots) \to O_{n-1}(\ldots, k_P, \ldots) \quad k_\alpha \parallel k_\beta$$

- Cannot predict perturbatively any infrared-unsafe quantity, such as:
  - the number of partons (hadrons) in an event
  - observables requiring no radiation in some region (rapidity gaps or overly strong isolation cuts)
  - $p_T(W)$ precisely at $p_T = 0$
Infrared safety (cont.)

Examples of IR safe quantities:
- *jets*, defined by cluster or (suitable) cone algorithm
- most kinematic distributions of “electroweak” objects, *W*, *Z*, *Higgs*
  (photons tricky because they can come from fragmentation)

**$k_T$ jet cluster algorithm:**
- Construct a list of objects, starting with particles $i$
  (or maybe calorimeter towers), plus “the beam” $b$
- Define a “distance” between objects, which vanishes in soft/collinear limits:
  $$d_{ij} = 2 \min\{k_T^{(i)}, k_T^{(j)}\} \{\cosh(\eta^{(i)} - \eta^{(j)}) - \cos(\phi^{(i)} - \phi^{(j)})\}$$
  $$d_{ib} = k_T^b$$
- Cluster together the 2 objects with smallest distance; combine their 4-momenta into one.
- Repeat until all $d_{ij} > d_{ij}^{\text{cut}}$
- The remaining objects are jets
MC@NLO

- As with LO matching of MEs to MCs, goal is to combine best features of two approaches: more accurate normalization of event rates (NLO) and hadron-level event descriptions (MC).
- More intricate than LO matching – must perform an exact NLO subtraction, then correct it to remove the parton-shower double-count.

Working example: **MC@NLO**  
Frixione, Webber, hep-ph/0204244

Based on HERWIG MC
LHC processes available to date:
- single vector and Higgs bosons
- vector boson pairs
- heavy quark pairs
- single top
- lepton pairs
- Higgs bosons in association with a W or Z.
NNLO

- Based on sum of 2-loop $n$-parton process, plus 1-loop $(n+1)$-parton process, tree-level $(n+2)$-parton processes
- Required for high precision at LHC, because NLO results often have 10% or more residual uncertainties
- Where is high precision warranted?
  - parton distributions
    - evolution (NNLO DGLAP kernels)
    - fits to DIS, Drell-Yan, and jet data
  - LHC production of single $W$s and $Z$s
    - “partonic” luminosity monitor
    - precision $m_W$
  - Higgs production via gluon fusion and extraction of Higgs couplings
  - Inclusive jets? Vector boson pairs? Not yet available
- Use NNLO studies to reweight MC[@NLO]

Davatz et al. hep-ph/0604077
How to organize pQCD amplitudes

- Avoid tangled algebra of color and Lorentz indices generated by Feynman rules

\[ i g f^{abc} [\eta_{\nu \rho}(p - q)_{\mu} + \eta_{\rho \mu}(q - k)_{\nu} + \eta_{\mu \nu}(k - p)_{\rho}] \]

structure constants

- Take advantage of physical properties of amplitudes

  - dual (trace-based) color decompositions
  - spinor helicity formalism
Standard color factor for a QCD graph has lots of structure constants contracted in various orders; for example:

We can write every $n$-gluon tree graph color factor as a sum of traces of matrices $T^a$ in the fundamental (defining) representation of $SU(N_c)$:

$$\text{Tr}(T^{a_1}T^{a_2}\ldots T^{a_n}) + \text{all non-cyclic permutations}$$

Use definition:

$$[T^a, T^b] = i f^{abc} T^c$$

+ normalization:

$$\text{Tr}(T^a T^b) = \delta^{ab} \quad \Rightarrow \quad f^{abc} = -i \text{Tr}([T^a, T^b] T^c)$$
Color in pictures

Insert

\[ f^{abc} = \text{Tr}([T^a, T^b] T^c) = \]

where

\[ (T^a)^j_i = \]

is color factor for \(qqg\) vertex

into typical string of \(f^{abc}\) structure constants for a Feynman diagram:

- Always single traces (at tree level)
- \(\text{Tr}(T^a_1 T^a_2 \ldots T^a_n)\) comes only from those planar diagrams with cyclic ordering of external legs fixed to 1, 2, \ldots, n
Trace-based (dual) color decomposition

Similarly

\[ q\bar{q}gg \cdots g \text{ amplitudes} \Rightarrow (T^{a_1}T^{a_2} \cdots T^{a_n})^j_i \]

+ permutations

In summary, for the \( n \)-gluon trees, the color decomposition is

\[ A^\text{tree}_n(\{k_i, a_i, h_i\}) = g^{n-2} \text{Tr}(T^{a_1}T^{a_2} \cdots T^{a_n}) A^\text{tree}_n(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) \]

+ non-cyclic perm’s

- color-ordered subamplitude only depends on momenta.
- Compute separately for each cyclicly inequivalent helicity configuration \((h_1, h_2, \ldots, h_n)\)

- Because \( A^\text{tree}_n(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) \) comes from planar diagrams with cyclic ordering of external legs fixed to 1,2,\ldots,n, it only has singularities in cyclicly-adjacent channels \( s_{i,i+1} \), \ldots
Aside: Strings and Color

- The “trace” color basis for QCD is also called the “dual” basis because it first arose, as Chan-Paton factors in the description of $SU(N)$ symmetric dual models (string theories) – initially it was describing flavor!
- A modern string theorist would say that a string end moves from one of $N$ D-branes to another by emitting a green-antiblue gluon
- Also related to ‘t Hooft double-line formalism
Color-ordered Feynman rules

In Feynman gauge, use these “color-stripped” rules

\[
\begin{align*}
\frac{q}{\nu} & \quad \frac{q}{\nu} \quad \frac{p}{\nu} \\
\mu & = \frac{i}{\sqrt{2}} \left( \eta_{\nu \rho} (p-q)_{\mu} + \eta_{\rho \nu} (q-k)_{\nu} + \eta_{\mu \nu} (k-p)_{\rho} \right) \\
\lambda & = i \eta_{\mu \rho} \eta_{\nu \lambda} - \frac{i}{2} (\eta_{\mu \nu} \eta_{\rho \lambda} + \eta_{\mu \lambda} \eta_{\nu \rho}) \\
\mu & = \frac{i}{\sqrt{2} \gamma_{\mu}} \\
\nu & = \frac{1}{p^2} \eta_{\mu \nu} \\
\mu & = -\frac{i}{\sqrt{2} \gamma_{\mu}} \\
\nu & = \frac{i}{p^2}
\end{align*}
\]

in the (1,2,…,n)-ordered planar diagrams, to compute

\[ A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) \quad A_n^{\text{tree}}(1^{-q}, 2^{+q}, 3^{h_3}, \ldots, n^{h_n}) \]

\[ gg \cdots g \quad q\bar{q} gg \cdots g \]

etc.
Color sums

In the end, we want to sum/average over final/initial colors (as well as helicities):

\[ d\sigma_{\text{tree}} \propto \sum_{a_i} \sum_{h_i} |A_n^{\text{tree}}(\{k_i, a_i, h_i\})|^2 \]

Inserting:

\[ A_n^{\text{tree}}(\{k_i, a_i, h_i\}) = g^{n-2} \text{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n}) A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) \]

\[ + \text{ non-cyclic perm's} \]

and doing the color sums diagrammatically:

we get:

\[ d\sigma_{\text{tree}} \propto N_c^n \sum_{\sigma \in S_n/Z_n} \sum_{h_i} |A_n^{\text{tree}}(\sigma(1^{h_1}), \sigma(2^{h_2}), \ldots, \sigma(n^{h_n}))|^2 + \mathcal{O}(N_c^{-2}) \]

\[ \rightarrow \text{Up to } 1/N_c^2 \text{ suppressed effects, squared subamplitudes have definite color flow -- important for handoff to parton shower programs} \]
Spinor helicity formalism

Scattering amplitudes for massless plane waves of definite momentum:
Lorentz 4-vectors $k_i^\mu$ $k_i^2 = 0$

Natural to use Lorentz-invariant products (invariant masses): $s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$

But for elementary particles with spin (e.g. all observed ones!)
there is a better way:

Take “square root” of 4-vectors $k_i^\mu$ (spin 1)
use Dirac (Weyl) spinors $u_\alpha(k_i)$ (spin $\frac{1}{2}$)

right-handed: $(\lambda_i)_\alpha = u_+(k_i)$  
left-handed: $(\bar{\lambda}_i)^{\dot{\alpha}} = u_-(k_i)$

$q, g, \gamma$, all have 2 helicity states, $h = \pm$
Spinor products

Instead of Lorentz products:

\[ s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2 \]

Use spinor products:

\[ \bar{u}_-(k_i)u_+(k_j) = \varepsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_j)_\beta = \langle i \, j \rangle \]
\[ \bar{u}_+(k_i)u_-(k_j) = \varepsilon^{\dot{\alpha}\dot{\beta}}(\bar{\lambda}_i)_{\dot{\alpha}}(\bar{\lambda}_j)_{\dot{\beta}} = [i \, j] \]

Identity

\[ k^\mu_i (\sigma_{\mu})_{\alpha\dot{\alpha}} = (k_i)_\alpha^{\dot{\alpha}} = u_+(k_i)\bar{u}_+(k_i) = (\lambda_i)_\alpha(\bar{\lambda}_i)_{\dot{\alpha}} \]

These are complex square roots of Lorentz products:

\[ \langle i \, j \rangle [j \, i] = \frac{1}{2} \text{Tr} \left[ k_i \; k_j \right] = 2k_i \cdot k_j = s_{ij} \]

\[ \langle i \, j \rangle = \sqrt{s_{ij}} e^{i\phi_{ij}} \quad [j \, i] = \sqrt{s_{ij}} e^{-i\phi_{ij}} \]
Most famous (simplest) Feynman diagram

\[ A_4 = 2i \epsilon^2 Q_\ell Q_q \delta_{i3}^4 A_4 \]

\[ A_4 = \frac{1}{2 s_{12}} \bar{v}_-(k_2) \gamma^\mu u_-(k_1) \bar{u}_+(k_3) \gamma_\mu v_+(k_4) \]

\[ = \frac{1}{2 s_{12}} (\sigma^\mu)_{\alpha\dot{\alpha}}(\lambda_2)^\alpha(\tilde{\lambda}_1)^{\dot{\alpha}} (\sigma_\mu)^{\dot{\beta}\beta}(\tilde{\lambda}_3)^{\dot{\beta}}(\lambda_4)_\beta \]

\[ = \frac{1}{s_{12}} (\lambda_2)^\alpha(\tilde{\lambda}_1)^{\dot{\alpha}}(\lambda_4)_\alpha(\tilde{\lambda}_3)^{\dot{\alpha}} \]

\[ A_4 = \frac{\langle 2 4 \rangle [1 3]}{s_{12}} = e^{i\phi} \frac{s_{13}}{s_{12}} = -\frac{e^{i\phi}}{2} (1 - \cos \theta) \]

Fierz identity

\[ (\sigma^\mu)_{\alpha\dot{\alpha}} (\sigma_\mu)^{\dot{\beta}\beta} = 2 \delta_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}} \]

L. Dixon, 7/20/06

Higher Order QCD: Lect. 1
Sometimes useful to rewrite answer

Crossing symmetry more manifest if we switch to all-outgoing helicity labels (flip signs of incoming helicities)

\[ A_4 = \frac{\langle 2\ 4 \rangle [1\ 3]}{s_{12}} = \frac{\langle 2\ 4 \rangle [1\ 3] \langle 1\ 3 \rangle}{\langle 1\ 2 \rangle [2\ 1] \langle 1\ 3 \rangle} = -\frac{\langle 2\ 4 \rangle [2\ 4] \langle 2\ 4 \rangle}{\langle 1\ 2 \rangle [2\ 4] (4\ 3)} \]

useful identities

\[ \langle i\ j \rangle = -\langle j\ i \rangle \]
\[ [i\ j] = -[j\ i] \]
\[ \langle i\ i \rangle = [i\ i] = 0 \]
\[ [i\ j] [j\ i] = s_{ij} \]

\[ \sum_{j=1}^{4} \langle i\ j \rangle [j\ k] = 0 \]

\[ s_{12} = s_{34} \]
\[ s_{13} = s_{24} \]

L. Dixon, 7/20/06

Higher Order QCD: Lect. 1

28
Symmetries for all other helicity config’s

\[ A_4 = \frac{\langle 24 \rangle^2}{\langle 12 \rangle\langle 34 \rangle} \]

\[ A_4 = -\frac{\langle 14 \rangle^2}{\langle 12 \rangle\langle 34 \rangle} \]

\[ A_4 = \frac{[24]^2}{[12][34]} \]

\[ A_4 = -\frac{[14]^2}{[12][34]} \]
Unpolarized, helicity-summed cross sections

(the norm in \text{QCD})

\[
\frac{d\sigma(e^+e^- \rightarrow q\bar{q})}{d\cos\theta} \propto \sum_{\text{hel.}} |A_4|^2 = 2 \left\{ \left| \frac{\langle 2\,4 \rangle^2}{\langle 1\,2 \rangle \langle 3\,4 \rangle} \right|^2 + \left| \frac{\langle 1\,4 \rangle^2}{\langle 1\,2 \rangle \langle 3\,4 \rangle} \right|^2 \right\} \\
= 2 \frac{s_{24}^2 + s_{14}^2}{s_{12}^2} \\
= \frac{1}{2} \left[ (1 - \cos\theta)^2 + (1 + \cos\theta)^2 \right] \\
= 1 + \cos^2\theta
\]
Reweight helicity amplitudes \( \rightarrow \) electroweak/QCD processes

For example, \( Z \) exchange

\[
Q_e Q_q \quad \Rightarrow \quad Q_e Q_q + \frac{v_{L,R}^e v_{L,R}^q}{s - M_Z^2 + i\Gamma_Z M_Z} \frac{s}{s - M_Z^2 + i\Gamma_Z M_Z}
\]

\[
v_L^f = \frac{2I_{3}^f - 2Q_f \sin^2 \theta_W}{\sin 2\theta_W} \quad \quad v_R^f = -\frac{2Q_f \sin^2 \theta_W}{\sin 2\theta_W}
\]
Next most famous pair of Feynman diagrams

(to a higher-order QCD person)

\[ A_5 = 2i e^2 g Q_e Q_q (T^a_{4})_{\bar{1}}^5 A_5 \]

\[ A_5 = \frac{\langle 2 \bar{5} \rangle \langle 1^+ | (k_3 + k_4) \bar{\gamma}^+ | 3^- \rangle}{s_{12} \sqrt{2} s_{34}} + \frac{[1 \ 3] \langle 2^- | (k_4 + k_5) \bar{\gamma}^+ | 5^+ \rangle}{s_{12} \sqrt{2} s_{45}} \]
Helicity formalism for massless vectors

\[
(\varepsilon_i^+)_{\mu} = \varepsilon^+_\mu (k_i, q) = \frac{\langle i^+ | \gamma_{\mu} | q^+ \rangle}{\sqrt{2} \langle i q \rangle},
\]

\[
(\varepsilon_i^+)_{\alpha \dot{\alpha}} = \varepsilon^{\alpha \dot{\alpha}} (k_i, q) = \frac{\sqrt{2} \lambda^\alpha_i \lambda_{\dot{\alpha}}}{\langle i q \rangle}.
\]

\(\varepsilon_i^+ \cdot k_i = 0\) (required transversality)

\(\varepsilon_i^+ \cdot q = 0\) (bonus)

under azimuthal rotation about \(k_i\) axis, helicity +1/2

\[\lambda_i^{\dot{\alpha}} \rightarrow e^{i\phi/2} \lambda_i^{\dot{\alpha}}\]

\[\lambda_i^{\alpha} \rightarrow e^{-i\phi/2} \lambda_i^{\alpha}\]

so

\[\varepsilon_i^+ \propto \frac{\lambda_i^{\dot{\alpha}}}{\lambda_i^\alpha} \rightarrow e^{i\phi} \varepsilon_i^+\]

as required for helicity +1

Berends, Kleiss, De Causmaecker, Gastmans, Wu (1981); De Causmaecker, Gastmans, Troost, Wu (1982); Xu, Zhang, Chang (1984); Kleiss, Stirling (1985); Gunion, Kunszt (1985)
\[ A_5 = \frac{\langle 2 \ 5 \rangle \langle 1^+ | (k_3 + k_4) \varphi^+_4 | 3 \rangle}{s_{12} \sqrt{2} s_{34}} + \frac{[1 \ 3] \langle 2^- | (k_4 + k_5) \varphi^+_4 | 5^+ \rangle}{s_{12} \sqrt{2} s_{45}} \]

Choose \( q = k_5 \)
to remove 2\textsuperscript{nd} graph
Properties of $\mathcal{A}_5(e^+e^- \to qg\overline{q})$

1. Soft gluon behavior

$k_4 \to 0$

$$A_5 = \frac{\langle 2 \ 5 \rangle^2}{\langle 1 \ 2 \rangle \langle 3 \ 4 \rangle \langle 4 \ 5 \rangle} = \frac{\langle 3 \ 5 \rangle}{\langle 3 \ 4 \rangle \langle 4 \ 5 \rangle} \times \frac{\langle 2 \ 5 \rangle^2}{\langle 1 \ 2 \rangle \langle 3 \ 5 \rangle}$$

$$\to S(3, 4^+, 5) \times A_4(1^+, 2^-, 3^+, 5^-)$$

Universal “eikonal” factors

for emission of soft gluon $s$

between two hard partons $a$ and $b$

$$S(a, s^+, b) = \frac{\langle a \ b \rangle}{\langle a \ s \rangle \langle s \ b \rangle}$$

$$S(a, s^-, b) = -\frac{[a \ b]}{[a \ s][s \ b]}$$

Soft emission is from the classical chromoelectric current:

independent of parton type ($q$ vs. $g$) and helicity

– only depends on momenta of $a, b$, and color charge
Properties of $A_5(e^+e^- \rightarrow qg\bar{q})$ (cont.)

2. Collinear behavior

\[ k_3 \parallel k_4 : \quad k_3 = z k_P, \quad k_4 = (1 - z) k_P \]

\[ k_P \equiv k_3 + k_4, \quad k_P^2 \rightarrow 0 \]

\[ \lambda_3 \approx \sqrt{z} \lambda_P, \quad \lambda_4 \approx \sqrt{1 - z} \lambda_P, \quad \text{etc.} \]

\[
A_5 = \frac{\langle 2 \ 5 \rangle^2}{\langle 1 \ 2 \rangle \langle 3 \ 4 \rangle \langle 4 \ 5 \rangle} \approx \frac{1}{\sqrt{1 - z} \langle 3 \ 4 \rangle} \times \frac{\langle 2 \ 5 \rangle^2}{\langle 1 \ 2 \rangle \langle P \ 5 \rangle} \\
\rightarrow \quad \text{Split} \_ (3^+_q, 4^+_g) \times A_4(1^+, 2^-, P^+, 5^-)
\]

Universal collinear factors, or splitting amplitudes

\[ \text{Split} \_ h_P(a^{h_a}, b^{h_b}) \] depend on parton type and helicity $h$

Time-like kinematics (fragmentation). Space-like (parton evolution) related by crossing
Collinear limits (cont.)

We found, from $k_3 \parallel k_4$:

\[ \text{Split}_- (a_q^+, b_g^+) = \frac{1}{\sqrt{1-z} \langle a \ b \rangle} \]

Similarly, from $k_4 \parallel k_5$:

\[ \text{Split}_+ (a_g^+, b_q^-) = \frac{1-z}{\sqrt{z} \langle a \ b \rangle} \]
\[ \downarrow \]
\[ \text{Split}_- (a_q^+, b_g^-) = -\frac{z}{\sqrt{1-z} \langle a \ b \rangle} \]
Simplest pure-gluonic amplitudes

**Note:** helicity label assumes particle is outgoing; reverse if it's incoming

Strikingly, many vanish:

\[ A_n^{\text{tree}}(1^\pm, 2^+, \ldots, n^+) = \begin{array}{c}
\text{+} \\
\text{+} \\
\text{+} \\
\text{+} \\
\text{+} \\
\text{+}
\end{array} = \begin{array}{c}
\text{+} \\
\text{+} \\
\text{+} \\
\text{+} \\
\text{+} \\
\text{+}
\end{array} = 0\]

Maximally helicity-violating (MHV) amplitudes:

\[ A_n^{ij, \text{MHV}} = A_n^{\text{tree}}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \begin{array}{c}
\text{+} \\
\text{+} \\
\text{+} \\
\text{+} \\
\text{+} \\
\text{+}
\end{array} = \left\langle i \ j \right\rangle^4 \frac{1}{\left\langle 1 \ 2 \right\rangle \left\langle 2 \ 3 \right\rangle \cdots \left\langle n \ 1 \right\rangle} \]

Parke-Taylor formula (1986)

Remarkable simplicity – has inspired many formal developments
MHV amplitudes with massless quarks

Helicity conservation on fermion line \[ A_n^{\text{tree}}(1^\pm_q, 2^\pm_q, 3^{h_3}, \ldots, n^{h_n}) \equiv 0 \]

more vanishing ones:

the MHV amplitudes:

\[ A_n^{\text{tree}}(1_q^-, 2_q^+, 3^+, \ldots, n^+) = \begin{pmatrix} n^- & 1^- & \cdots & 2^+ \end{pmatrix} = \frac{\langle 1 i \rangle^3 \langle 2 i \rangle}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \]

Related to pure-gluon MHV amplitudes by a secret supersymmetry: after stripping off color factors, massless quarks \( \sim \) gluinos

Grisaru, Pendleton, van Nieuwenhuizen (1977); Parke, Taylor (1985); Kunszt (1986)
Properties of MHV amplitudes

1. Verify soft limit

\[ k_s \to 0 \]

\[
\frac{\langle i, j \rangle^4}{\langle 12 \rangle \cdots \langle a \ s \rangle \langle s \ b \rangle \cdots \langle n \ 1 \rangle} = \frac{\langle a \ b \rangle}{\langle a \ s \rangle \langle s \ b \rangle} \frac{\langle i, j \rangle^4}{\langle 12 \rangle \cdots \langle a \ b \rangle \cdots \langle n \ 1 \rangle} \\
\Rightarrow \text{Soft}(a, s^+, b) \times A_{n-1}^{ij, \text{MHV}}
\]

2. Extract gluonic collinear limits:

\[ k_a || k_b \quad (b = a + 1) \]

\[
\frac{\langle i, j \rangle^4}{\langle 12 \rangle \cdots \langle a - 1, a \rangle \langle a \ b \rangle \langle b, b + 1 \rangle \cdots \langle n \ 1 \rangle} = \frac{1}{\sqrt{z(1-z)} \langle a \ b \rangle} \frac{\langle i, j \rangle^4}{\langle 12 \rangle \cdots \langle a - 1 \rangle \langle P \rangle \langle P, b + 1 \rangle \cdots \langle n \ 1 \rangle} \\
\Rightarrow \text{Split}_{-}(a^+, b^+) \times A_{n-1}^{ij, \text{MHV}}
\]

So

\[
\text{Split}_{-}(a^+, b^+) = \frac{1}{\sqrt{z(1-z)} \langle a \ b \rangle}
\]

and

\[
\text{Split}_{+}(a^-, b^+) = \frac{z^2}{\sqrt{z(1-z)} \langle a \ b \rangle}
\]

\[
\text{Split}_{+}(a^+, b^-) = \frac{(1-z)^2}{\sqrt{z(1-z)} \langle a \ b \rangle}
\]

plus parity conjugates
Spinor Magic

Spinor products precisely capture **square-root + phase** behavior in **collinear limit**. Excellent variables for **helicity amplitudes**

**Scalars**

\[ \sim \frac{1}{p^2} \sim \frac{1}{s_{ij}} \]

**Gauge Theory**

**Angular Momentum Mismatch**

\[ \sim \frac{e^{\pm i\phi}}{\sqrt{p^2}} \sim \frac{1}{\langle i, j \rangle} \text{ or } \frac{1}{[i, j]} \]
From splitting amplitudes to probabilities

\[ d\sigma_n \sim d\sigma_{n-1} \times \frac{1}{s_{ab}} \times P(z) \]

\[ P(z) \propto \sum_{h_P, h_a, h_b} |\text{Split}_{h_P}(a^{h_a}, b^{h_b})|^2 s_{ab} \]

\[ q \rightarrow qg: \quad P_{qq}(z) \propto C_F \left\{ \left| \frac{1}{\sqrt{1-z}} \right|^2 + \left| \frac{z}{\sqrt{1-z}} \right|^2 \right\} \]

\[ = C_F \frac{1 + z^2}{1-z} \quad z < 1 \]

Note soft-gluon singularity as \( z_g = 1 - z \rightarrow 0 \)
Similarly for gluons

\[ g \rightarrow gg: \]

\[ P_{gg}(z) \propto C_A \left\{ \left| \frac{1}{\sqrt{z(1-z)}} \right|^2 + \left| \frac{z^2}{\sqrt{z(1-z)}} \right|^2 + \left| \frac{(1-z)^2}{\sqrt{z(1-z)}} \right|^2 \right\} \]

\[ = C_A \left( 1 + z^4 + (1-z)^4 \right) \frac{1}{z(1-z)} \]

\[ = 2C_A \left[ \frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right] \quad C_A = N_c \]

Again a soft-gluon singularity. Gluon number not conserved. But momentum is. Momentum conservation mixes \( g \rightarrow gg \) with

\[ g \rightarrow q\bar{q}: \]

\[ P_{qg}(z) = T_R \left[ z^2 + (1-z)^2 \right] \quad T_R = \frac{1}{2} \]

(can deduce, up to color factors, by taking \( e^+ \parallel e^- \) in \( \mathcal{A}_5(e^+e^- \rightarrow qg\bar{q}) \) )
Gluon splitting (cont.)

\( g \rightarrow gg: \)

Applying momentum conservation,

\[
\int_0^1 dz \, z \left[ P_{gg}(z) + 2n_f P_{qg}(z) \right] = 0
\]

gives

\[
P_{gg}(z) = 2C_A \left[ \frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + b_0 \delta(1-z)
\]

\[
b_0 = \frac{11C_A - 4n_f T_R}{6}
\]

Amusing that first \( \beta \)-function coefficient enters, since no loops were done, except implicitly via unitarity:
Recursive Tree Techniques

Illustrate with Berends-Giele (1987) [off-shell] recursion relations

Other [off-shell] recursive approaches underly HELAC; ALPHA

• Follow an off-shell gluon (or quark) line into “forest”
  of color-ordered tree graphs
• All other legs on shell
• Trail forks into either 2 or 3 more lines, via 3- or 4-gluon vertex
• Each new path enters a forest with fewer on-shell legs
• Put last leg on shell to get $A_{n}^{\text{tree}}$
Extra Slides
QCD factorization & parton model

- Asymptotic freedom guarantees that at short distances (large transverse momenta), partons in the proton are almost free.
- They are sampled “one at a time” in hard collisions.
- Leads to QCD-improved parton model:

\[
\sigma^{pp \rightarrow X}(s; \alpha_s, \mu_R, \mu_F) = \sum_{a,b} \int_0^1 dx_1 \int_0^1 dx_2 f_a(x_1, \alpha_s, \mu_F) f_b(x_2, \alpha_s, \mu_F) \times \hat{\sigma}^{ab \rightarrow X}(sx_1x_2; \alpha_s, \mu_R, \mu_F)
\]

Parton distribution function: prob. of finding parton \(a\) in proton 1, carrying fraction \(x_1\) of its momentum

“suitable” final state

Partonic cross section, computable in perturbative QCD

factorization scale (“arbitrary”)

renormalization scale (“arbitrary”)

L. Dixon, 7/20/06
Higher Order QCD: Lect. 1
Parton evolution

- Partons in the proton are not quite free.
- Distributions $f_a(x, \mu_F)$ evolve as scale $\mu_F$ at which they are resolved varies.

\[
\sigma^{pp \to X}(s; \alpha_s, \mu_R, \mu_F) = \sum_{a,b} \int_0^1 dx_1 \int_0^1 dx_2 \ f_a(x_1, \alpha_s, \mu_F) f_b(x_2, \alpha_s, \mu_F) \times \sigma^{ab \to X}(s x_1 x_2; \alpha_s, \mu_R, \mu_F)
\]

Example:

- $p_T(g) \sim 10$ GeV, say
- $\mu_F$, large, $\sim m_Z$
- $\mu_F$, small, $\sim$ few GeV

L. Dixon, 7/20/06
Parton evolution (cont.)

- parton distributions are nonperturbative
- must be measured experimentally
- experimental data at much lower $\mu_F^2$ than $(100-1000 \text{ GeV})^2$
- fortunately, evolution at $\mu_F > 1-2 \text{ GeV}$ is perturbative
- DGLAP equation (return to later)

\[ \mu^2 \frac{\partial}{\partial \mu^2} f_a(x, \mu) = \frac{\alpha_s(\mu)}{2\pi} \sum_b \int_x^1 \frac{d\xi}{\xi} P_{ab}(x/\xi, \alpha_s(\mu)) f_b(\xi, \mu) \]

\[ x = \frac{x}{\xi} \times \xi \]

\[ P_{ab}(x, \alpha_s) = P_{ab}^{(0)}(x) + \frac{\alpha_s}{2\pi} P_{ab}^{(1)}(x) + \left( \frac{\alpha_s}{2\pi} \right)^2 P_{ab}^{(2)}(x) + \cdots \]

Also expand partonic cross section:

$$\bar{\sigma}(\alpha_s, \mu_F, \mu_R) = \left[\alpha_s(\mu_R)\right]^{n_\alpha} \left[ \bar{\sigma}^{(0)} + \frac{\alpha_s}{2\pi} \bar{\sigma}^{(1)}(\mu_F, \mu_R) + \left(\frac{\alpha_s}{2\pi}\right)^2 \bar{\sigma}^{(2)}(\mu_F, \mu_R) + \cdots \right]$$

LO  NLO  NNLO

**Problem:** Leading-order, tree-level predictions often only qualitative due to **poor convergence** of expansion in $\alpha_s(\mu)$

(setting $\mu_R = \mu_F = \mu$)

**Example:** $Z$ production at LHC. Predict distribution in rapidity

$$Y = \frac{1}{2} \ln \left( \frac{E + p_z}{E - p_z} \right)$$

$$\frac{d\sigma}{dY}$$ has $n_\alpha = 0$

still 30% NLO corrections
Space-like splitting

- The case relevant for parton evolution
- Related by crossing to time-like case
- Have to watch out for flux factor, however

\[ q \rightarrow qg: \quad k_P = x k_5, \quad k_4 = (1 - x) k_5 \]

\[ \Lambda_5 = \frac{\langle 2 \ 5 \rangle^2}{\langle 1 \ 2 \rangle \langle 3 \ 4 \rangle \langle 4 \ 5 \rangle} \approx \frac{1}{x} \frac{1}{\sqrt{x} \sqrt{1 - x}} \frac{\langle 4 \ 5 \rangle}{\langle 1 \ 2 \rangle \langle 3 \ 4 \rangle} \times \frac{\langle 2 \ P \rangle^2}{\langle 1 \ 2 \rangle \langle 3 \ P \rangle} \]

\[ = \frac{1}{\sqrt{x} \sqrt{1 - x}} \frac{1}{\langle 4 \ 5 \rangle} \times \frac{\langle 2 \ P \rangle^2}{\langle 1 \ 2 \rangle \langle 3 \ P \rangle} \]

absorb into flux factor:

\[ d\sigma_5 \propto \frac{1}{s_{15}} \]

\[ d\sigma_4 \propto \frac{1}{s_{1P}} = \frac{1}{x \ s_{15}} \]

When dust settles, get exactly the same splitting kernels (at LO)