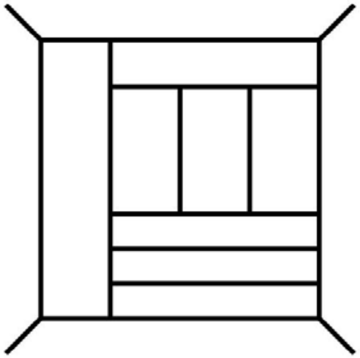
Multi-Loop Miracles in N=4 Super-Yang-Mills Theory



- Z. Bern, L.D., V. Smirnov, hep-th/0505205
- F. Cachazo, M. Spradlin, A. Volovich, hep-th/0602228
- Z. Bern, M. Czakon, D. Kosower, R. Roiban, V. Smirnov, hep-th/0604074

LoopFest V, SLAC

June 19, 2006

Outline

- N=4 SYM and AdS/CFT
- Use of unitarity to construct N=4 SYM amplitude integrands
- Exponentiated infrared structure of amplitudes and Sudakov form factors
- Exponentiation of full amplitudes (including IR finite terms) in planar N=4 SYM at 2 and 3 loops
- “Leading transcendentality” connection between terms in QCD and N=4 SYM
- Conclusions

N=4 SYM, AdS/CFT and perturbative scattering

- N=4 SYM is most supersymmetric theory possible without gravity
- Uniquely specified by gauge group, say $SU(N_c)$
- Exactly scale-invariant (conformal) field theory: $\beta(g) = 0$
- AdS/CFT duality suggests that weak-coupling perturbation series for planar (large N_c) N=4 SYM should have special properties:
  - strong-coupling limit equivalent to weakly-coupled gravity theory
- Some quantities are protected, unrenormalized, so the series is trivial (e.g. energies of BPS states)
- Scattering amplitudes (near $D=4$) are not protected – how does series organize itself into a simple result, from gravity point of view?

N=4 SYM particle content

massless spin 1 gluon

4 massless spin 1/2 gluinos

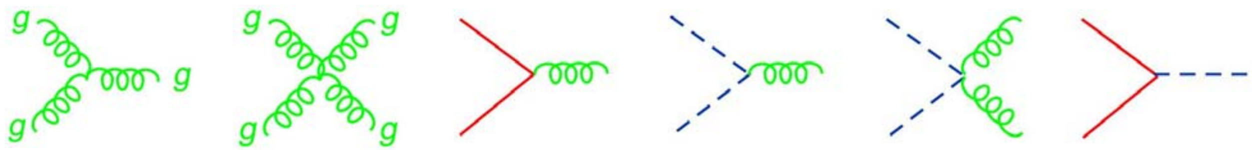
6 massless spin 0 scalars

SUSY
Q_a, a=1,2,3,4
shifts helicity
by 1/2 $\longleftrightarrow$

$\mathcal{N}=4$	1	$\longleftrightarrow$	4	$\longleftrightarrow$	6	$\longleftrightarrow$	4	$\longleftrightarrow$	1
	g^-		$\lambda_{\bar{i}}^-$		$\bar{\phi}_{\bar{i}\bar{j}}, \phi_{ij}$		λ_i^+		g^+
helicity	-1		$-\frac{1}{2}$		0		$\frac{1}{2}$		1

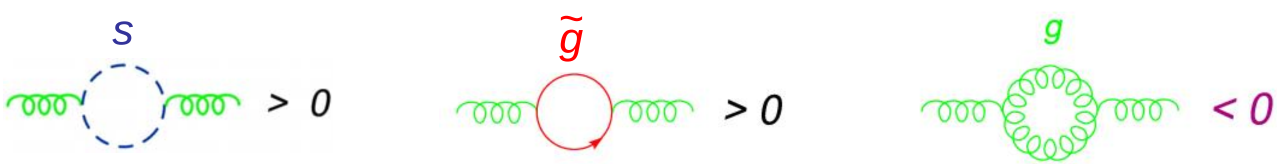
all in adjoint representation

N=4 SYM interactions



All proportional to same *dimensionless* coupling constant, *g*

- SUSY cancellations: scale invariance *preserved* quantum mechanically



$$\frac{d\alpha}{d \ln \mu^2} = \beta(\alpha) = \left[6 \times \frac{1}{6} + 4 \times \frac{2}{3} - \frac{11}{3} \right] \frac{N_c \, \alpha^2}{4\pi} = 0 \qquad \text{(true to all orders in } \alpha\text{)}$$

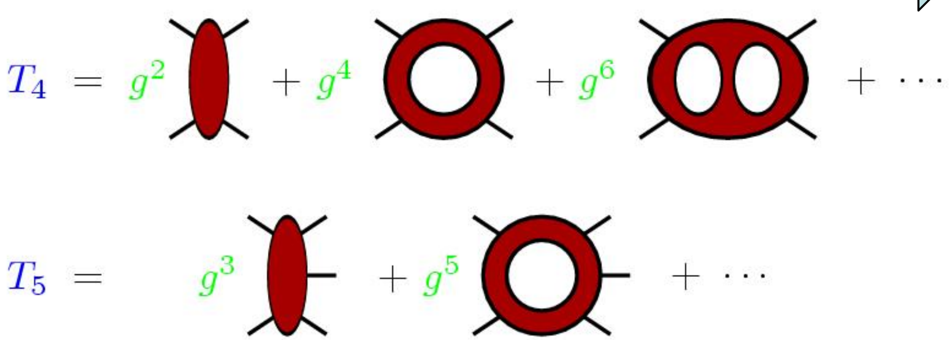
Perturbative Unitarity

- *S*-matrix is a unitary operator between in and out states

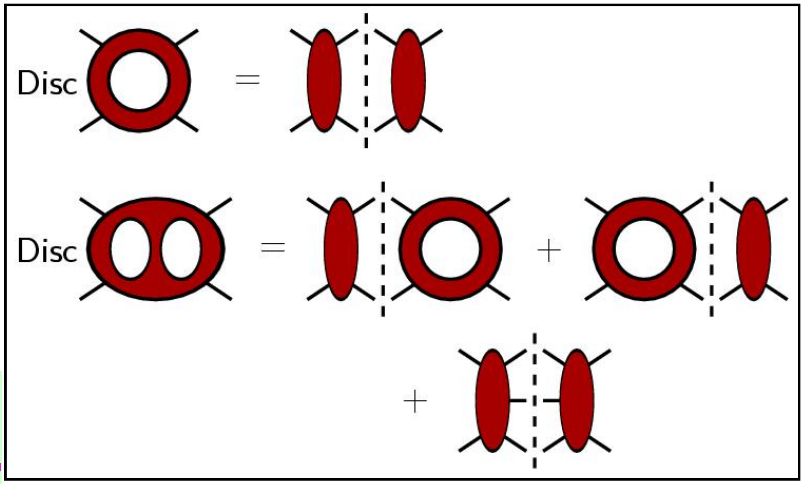
$$1 = S^\dagger S = (1 - iT^\dagger)(1 + iT)$$

$$2\,\mathrm{Im}T = T^\dagger T$$

- Expand *T*-matrix in *g*



Unitarity relations (cutting rules) for amplitudes



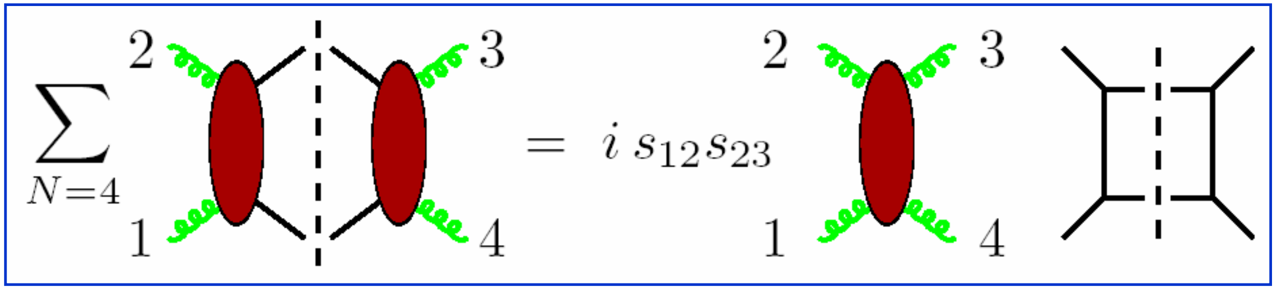
- Very efficient due to simple structure of tree helicity amplitudes

Bern, LD, Dunbar, Kosower (1994)

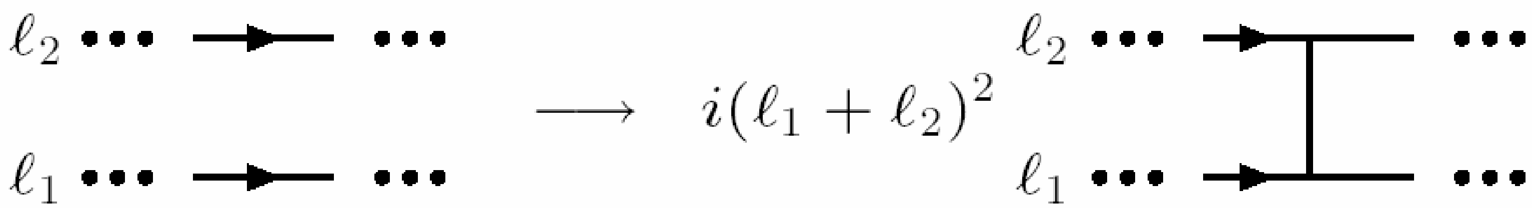
Unitarity and N=4 SYM

- Many higher-loop contributions to *gg -> gg* scattering can be deduced from a simple property of the 2-particle cuts at one loop

Bern, Rozowsky, Yan (1997)

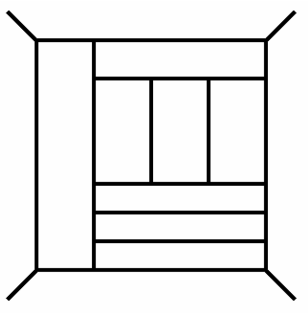


- Leads to "rung rule" for easily computing all contributions which can be built by iterating 2-particle cuts



“Iterated 2-particle cut-constructible contributions”

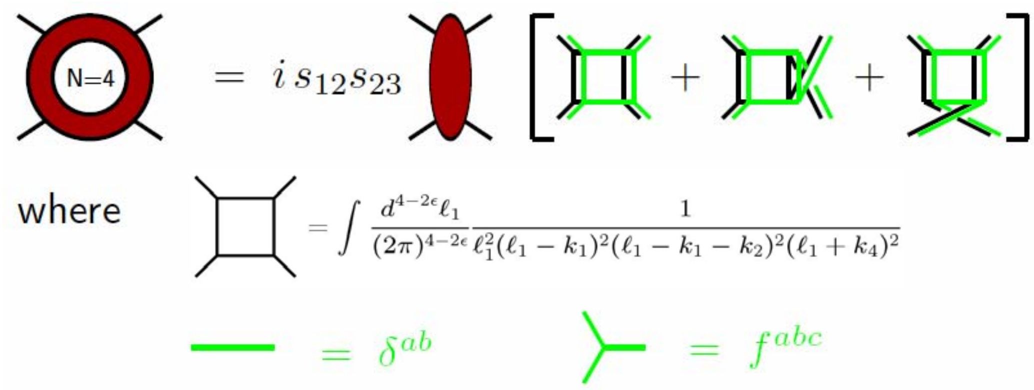
For example, the coefficient of this topology is easily computable from the rung rule



More concise terminology:
(planar) Mondrian diagrams

Simplicity of N=4 SYM 4-point amplitudes

- 1 loop:

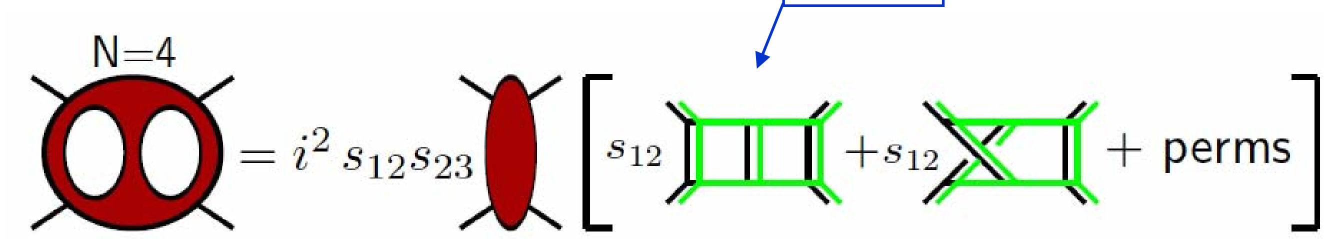


Green, Schwarz, Brink (1982)

“color dresses kinematics”

planar

- 2 loops:



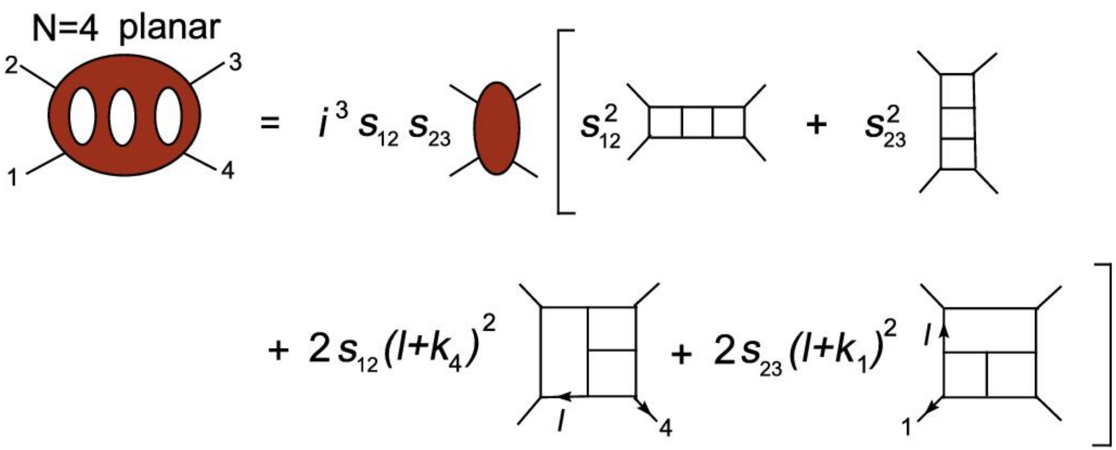
Bern, Rozowsky, Yan (1997); Bern, LD, Dunbar, Perelstein, Rozowsky (1998)

- Analogous computation in QCD not completed until 2001

Glover, Oleari, Tejeda-Yeomans (2001); Bern, De Freitas, LD (2002)

Three loop planar amplitude

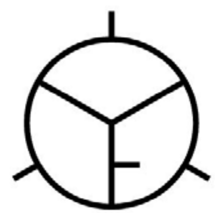
- 3-loop planar diagrams (leading terms for large N_c):



BRY (1997); BDDPR (1998)

(3-particle cuts also checked)

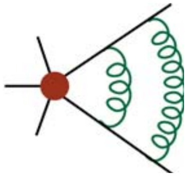
- 3-loop non-planar result includes non-Mondrian diagrams – not completed yet



IR Structure of QCD and N=4 SYM Amplitudes

- Expand multi-loop amplitudes in $d=4-2\varepsilon$ around $d=4$ ($\varepsilon=0$)

- Overlapping soft + collinear divergences at each loop order imply leading poles are $\sim 1/\varepsilon^{2L}$ at L loops

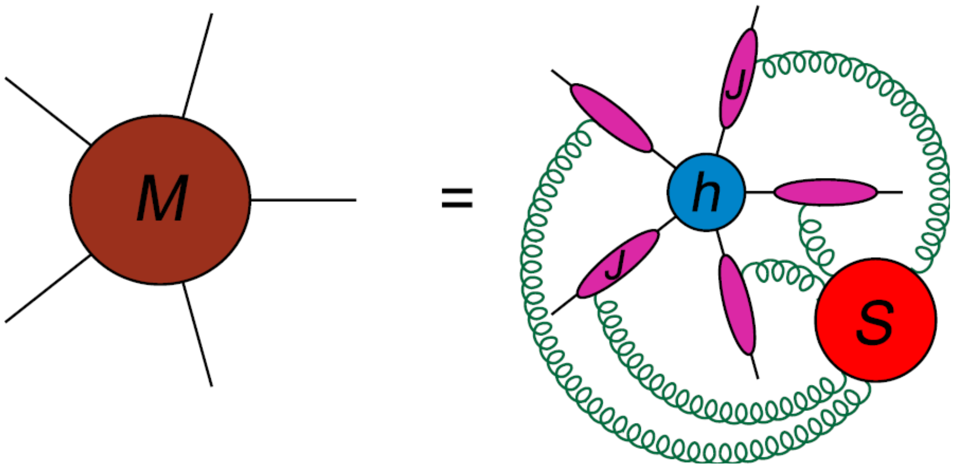


- Pole terms are predictable, up to constants, for QCD & N=4 SYM, due to soft/collinear factorization and exponentiation

Mueller (1979); Collins (1980); Sen (1981); Sterman (1987)
Catani, Trentadue (1989); Korchemsky (1989)
Magnea, Sterman (1990) ; Korchemsky, Marchesini, hep-ph/9210281
Catani, hep-ph/9802439 ; Sterman, Tejeda-Yeomans, hep-ph/0210130

- Surprise is that, for planar N=4 SYM (only), the finite (ε^0) terms also exponentiate!

Soft/Collinear Factorization

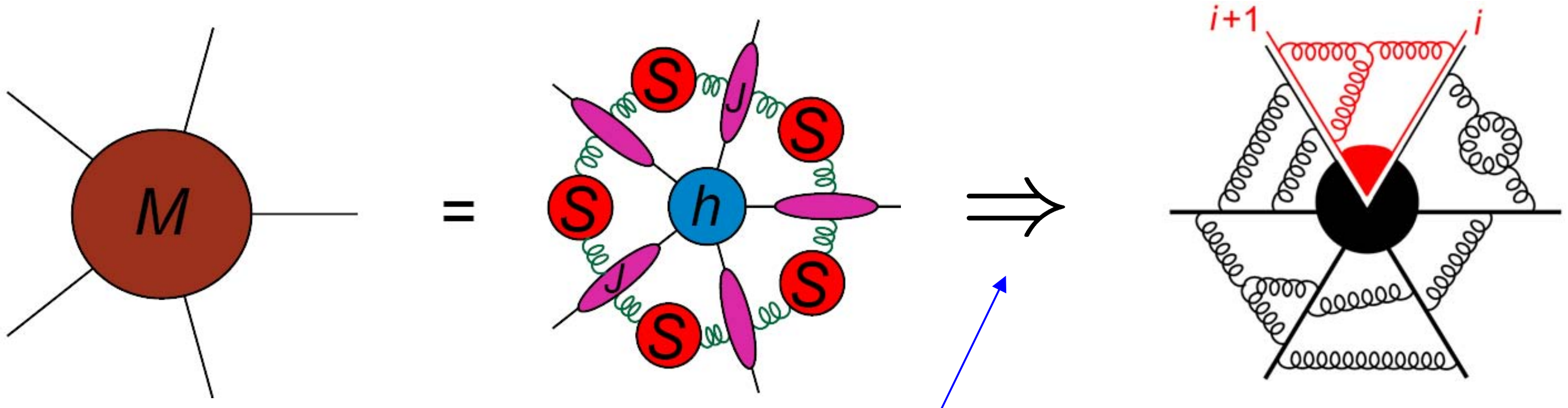


Akhoury (1979);
Sen (1983);
Botts, Sterman (1989);
Magnea, Sterman (1990);
Sterman, Tejeda-Yeomans, hep-ph/0210130

$$\mathcal{M}_n = S(k_i, \mu, \alpha_s(\mu), \epsilon) \times \left[\prod_{i=1}^{n} J_i(\mu, \alpha_s(\mu), \epsilon) \right] \times h_n(k_i, \mu, \alpha_s(\mu), \epsilon)$$

- *S = soft function (only depends on color of i^{th} particle)*
- *J = jet function (color-diagonal; depends on i^{th} spin)*
- *h_n= hard remainder function (finite as $\epsilon \to 0$)*

Simplification at Large N_c (Planar Case)



- *Soft function* only defined up to a multiple of the identity matrix in color space
- Planar limit is color-trivial; can absorb S into J_i
- If all n particles are identical, say gluons, then each “wedge” is the square root of the “gg -> 1” process (Sudakov form factor):

$$\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[gg\to 1]} \left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} \times h_n(k_i, \mu, \alpha_s, \epsilon)$$

Sudakov form factor

- By analyzing structure of soft/collinear terms in axial gauge, find differential equation for form factor $\mathcal{M}^{[gg\to 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon)$:

Mueller (1979); Collins (1980); Sen (1981); Korchemsky, Radyushkin (1987); Korchemsky (1989); Magnea, Sterman (1990)

$$\frac{\partial}{\partial \ln Q^2} \mathcal{M}^{[gg\to 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon)$$
$$= \frac{1}{2}\Big[K(\epsilon, \alpha_s) + G(Q^2/\mu^2, \alpha_s(\mu), \epsilon)\Big] \times \mathcal{M}^{[gg\to 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon)$$

finite as $\varepsilon \to 0$; contains all Q^2 dependence

Pure counterterm (series of $1/\varepsilon$ poles);
like $\beta(\varepsilon,\alpha_s)$, single poles in ε determine K completely

K, G also obey differential equations (ren. group):

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\Big)(K+G) = 0 \qquad \Big(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\Big)K = -\gamma_K(\alpha_s)$$

soft or cusp anomalous dimension

Sudakov form factor (cont.)

- Solution to differential equations for $\mathcal{M}^{[gg\to 1]}$, G:

$$\mathcal{M}^{[gg\to 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon)$$
$$= \exp\Big\{\frac{1}{2}\int_0^{-Q^2}\frac{d\xi^2}{\xi^2}\Big[K(\epsilon,\alpha_s(\mu)) + G(-1,\bar{\alpha}_s(\mu^2/\xi^2,\alpha_s(\mu),\epsilon),\epsilon)$$
$$+\frac{1}{2}\int_{\xi^2}^{\mu^2}\frac{d\tilde{\mu}^2}{\tilde{\mu}^2}\gamma_K(\bar{\alpha}_s(\mu^2/\tilde{\mu}^2,\alpha_s(\mu),\epsilon))\Big]\Big\}$$

- N=4 SYM: $\beta=0$, so $\alpha_s(\mu) = \alpha_s$ = constant,
  Running coupling in $d=4-2\varepsilon$ has only trivial (engineering) dependence on scale μ:

$$\bar{\alpha}_s\Big(\frac{\mu^2}{\tilde{\mu}^2},\epsilon\Big) \;=\; \alpha_s \times \Big(\frac{\mu^2}{\tilde{\mu}^2}\Big)^{\epsilon}(4\pi e^{-\gamma})^{\epsilon}$$

which makes it simple to perform integrals over ξ, $\tilde{\mu}$

Sudakov form factor in planar N=4 SYM

• Expand $K,\ \gamma_K,\ G$ in terms of $\quad a \equiv \frac{N_c \alpha_s}{2\pi}(4\pi e^{-\gamma})^{\epsilon}$

$$K(\alpha_s, \epsilon) = \sum_{l=1}^{\infty} \frac{a^l}{2l\epsilon} \hat{\gamma}_K^{(l)}$$

$$\gamma_K(\bar{\alpha}_s(\mu^2/\tilde{\mu}^2, \alpha_s(\mu), \epsilon)) = \sum_{l=1}^{\infty} a^l \Big(\frac{\mu^2}{\tilde{\mu}^2}\Big)^{l\epsilon} \hat{\gamma}_K^{(l)}$$

$$G(-1, \bar{\alpha}_s(\mu^2/\xi^2, \alpha_s(\mu), \epsilon), \epsilon) = \sum_{l=1}^{\infty} a^l \Big(\frac{\mu^2}{\xi^2}\Big)^{l\epsilon} \hat{\mathcal{G}}_0^{(l)}$$

• Perform integrals over $\xi,\ \tilde{\mu}$

$$\mathcal{M}^{[gg\to 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon) = \exp\Big[-\frac{1}{4}\sum_{l=1}^{\infty} a^l \Big(\frac{\mu^2}{-Q^2}\Big)^{l\epsilon}\Big(\frac{\hat{\gamma}_K^{(l)}}{(l\epsilon)^2} + \frac{2\hat{\mathcal{G}}_0^{(l)}}{l\epsilon}\Big)\Big]$$

General amplitude in planar N=4 SYM

Insert result for form factor into

$$\boxed{\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[gg\to 1]} \left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} \times h_n(k_i, \mu, \alpha_s, \epsilon)}$$

$$\Rightarrow \quad \mathcal{M}_n = 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)} = \exp\Big[\underbrace{-\frac{1}{8} \sum_{l=1}^{\infty} a^l \Big(\frac{\hat{\gamma}_K^{(l)}}{(l\epsilon)^2} + \frac{2\hat{\mathcal{G}}_0^{(l)}}{l\epsilon} \Big) \sum_{i=1}^{n} \Big(\frac{\mu^2}{-s_{i,i+1}} \Big)^{l\epsilon}}_{} \Big] \times h_n$$

this looks like the one-loop amplitude, but with ε shifted to (l ε), up to finite terms

which we can rewrite as

$$\boxed{\mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty} a^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + h_n^{(l)}(\epsilon, s_{i,i+1}) \Big) \Big]}$$

where $f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon\, f_1^{(l)} + \epsilon^2\, f_2^{(l)}$

with $f_0^{(l)} = \frac{1}{4}\hat{\gamma}_K^{(l)}$ $\quad f_1^{(l)} = \frac{l}{2}\hat{\mathcal{G}}_0^{(l)}$ $\quad f_2^{(l)} = (???)$

"mixes" with $h_n^{(l)}$

Exponentiation in planar N=4 SYM

- **Miracle:** In planar N=4 SYM the finite terms also exponentiate. That is, the hard remainder function $h_n^{(l)}$ defined by

$$\mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty} a^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + h_n^{(l)}(\epsilon, s_{i,i+1}) \Big)\Big]$$

is also a series of simple constants, $C^{(l)}$ [for **MHV** amplitudes]:

$$\mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty} a^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \Big)\Big]$$

where $E_n^{(l)}(\epsilon) \to 0$ as $\epsilon \to 0$

Evidence based so far on two loops (*n=4,5*, plus collinear limits) and three loops (for *n=4*)

In contrast, for QCD, and non-planar N=4 SYM, two-loop amplitudes have been computed, and the hard remainders are a polylogarithmic mess!

Exponentiation at two loops

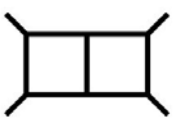
• The general formula,

$$\mathcal{M}_n \;=\; 1+\sum_{L=1}^{\infty} a^L M_n^{(L)} \;=\; \exp\Big[\sum_{l=1}^{\infty} a^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\Big)\Big]$$

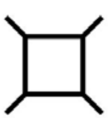
implies at two loops:

$$M_n^{(2)}(\epsilon) \;=\; \frac{1}{2}\Big[M_n^{(1)}(\epsilon)\Big]^2 + f^{(2)}(\epsilon) M_n^{(1)}(2\epsilon) + C^{(2)} + E_n^{(2)}(\epsilon)$$

• To check at *n=4*, need to evaluate just 2 integrals:



$\frac{1}{\epsilon^4},\; \frac{1}{\epsilon^3},\; \frac{1}{\epsilon^2},\; \frac{1}{\epsilon},\; \epsilon^0$ Smirnov, hep-ph/0111160



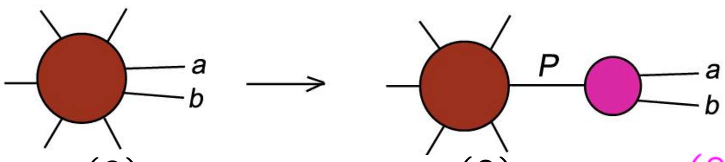
$\frac{1}{\epsilon^2},\; \frac{1}{\epsilon},\; \epsilon^0,\; \epsilon,\; \epsilon^2$ elementary

Two-loop exponentiation & collinear limits

- Evidence for *n>4*: Use limits as 2 momenta become **collinear:**

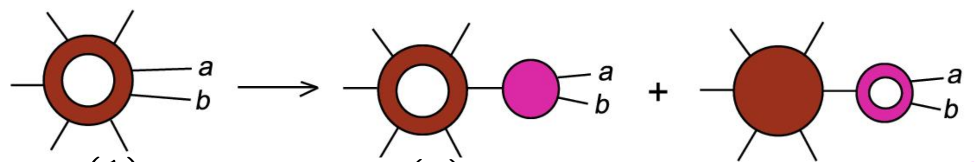
$$k_a \to z k_P$$
$$k_b \to (1-z) k_P$$

- Tree amplitude behavior:



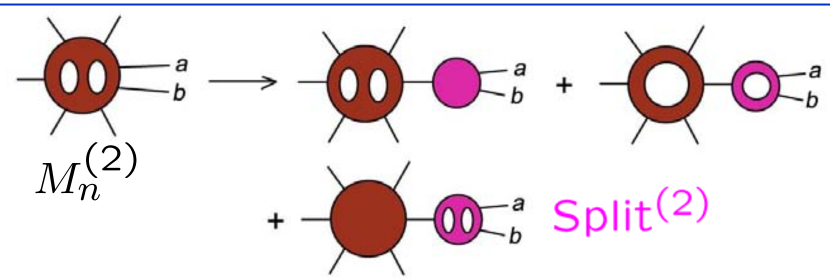
$M_n^{(0)}$ $M_{n-1}^{(0)}$ $\text{Split}^{(0)}(\lambda_P, \lambda_a, \lambda_b)$

- One-loop behavior:



$M_n^{(1)}$ $M_{n-1}^{(1)}$ $\text{Split}^{(1)}$

- Two-loop behavior:



$M_n^{(2)}$ $\text{Split}^{(2)}$

Two-loop splitting amplitude iteration

• In N=4 SYM, all helicity configurations are equivalent, can write

$$\text{Split}^{(l)}(\lambda_P, \lambda_a, \lambda_b) = r_S^{(l)}(z, s_{ab}, \epsilon) \times \text{Split}^{(0)}(\lambda_P, \lambda_a, \lambda_b)$$

• Two-loop splitting amplitude obeys:

Anastasiou, Bern, LD, Kosower, hep-th/0309040

$$\boxed{r_S^{(2)}(\epsilon) \;=\; \frac{1}{2}\left[r_S^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon) r_S^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)}$$

Consistent with the *n*-point amplitude ansatz

$$\mathcal{M}_n^{(2)}(\epsilon) \;=\; \frac{1}{2}\left[M_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon) M_n^{(1)}(2\epsilon) + C^{(2)} + E_n^{(2)}(\epsilon)$$

and fixes $f_0^{(2)} = -\zeta_2$ $f_1^{(2)} = -\zeta_3$ $f_2^{(2)} = -\zeta_4$ $C^{(2)} = -\frac{(\zeta_2)^2}{2}$

n-point information required to separate these two

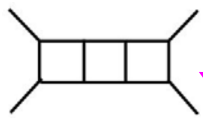
$$\boxed{\text{Note: by definition } f_0^{(1)} = 1, \; f_1^{(1)} = f_2^{(1)} = C^{(1)} = E_n^{(1)}(\epsilon) = 0}$$

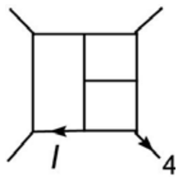
Exponentiation at three loops

• *L*-loop formula implies at three loops

$$M_n^{(3)}(\epsilon) \;=\; -\frac{1}{3}\Big[M_n^{(1)}(\epsilon)\Big]^3 + M_n^{(1)}(\epsilon)M_n^{(2)}(\epsilon) + f^{(3)}(\epsilon)M_n^{(1)}(3\epsilon) + C^{(3)} + E_n^{(3)}(\epsilon)$$

• To check at *n=4*, need to evaluate just 4 integrals:

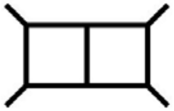




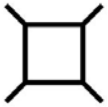
$$\frac{1}{\epsilon^6},\ \frac{1}{\epsilon^5},\ \frac{1}{\epsilon^4},\ \frac{1}{\epsilon^3},\ \frac{1}{\epsilon^2},\ \frac{1}{\epsilon},\ \epsilon^0$$

Smirnov, hep-ph/0305142

Use Mellin-Barnes integration method



$$\frac{1}{\epsilon^4},\ \frac{1}{\epsilon^3},\ \frac{1}{\epsilon^2},\ \frac{1}{\epsilon},\ \epsilon^0,\ \epsilon,\ \epsilon^2$$



$$\frac{1}{\epsilon^2},\ \frac{1}{\epsilon},\ \epsilon^0,\ \epsilon,\ \epsilon^2,\ \epsilon^3,\ \epsilon^4$$

elementary

Harmonic polylogarithms

• Integrals are all transcendental functions of $x = -\frac{s}{t}$

• Expressed in terms of
harmonic polylogarithms (HPLs)
$H_{a_1 a_2 \ldots a_n}(x) \equiv H(a_1, a_2, \ldots, a_n; x)$
with indices $a_i \in \{0, 1\}$,
defined recursively by:

Remiddi, Vermaseren, hep-ph/9905237;
Gehrmann, Remiddi, hep-ph/0107173;
Vollinga, Weinzierl, hep-ph/0410259

number of indices = *weight w*

$$H_{a_1 a_2 \ldots a_n}(x) \;=\; \int_0^x dt\, f_{a_1}(t)\, H_{a_2 \ldots a_n}(t)$$

with $$f_1(t) = \frac{1}{1-t} \qquad f_0(t) = \frac{1}{t}$$

• For *w = 0,1,2,3,4*, these HPLs can all be reduced to ordinary polylogarithms,
$\mathrm{Li}_w(z)$ with $z = x$, $\frac{1}{1-x}$, or $\frac{-x}{1-x}$

• But here we need *w = 5,6* too

Exponentiation at three loops

- Inserting the values of the integrals (including those with $s \leftrightarrow t$) into

$$M_4^{(3)}(\epsilon) = -\frac{1}{3}\left[M_4^{(1)}(\epsilon)\right]^3 + M_4^{(1)}(\epsilon)M_4^{(2)}(\epsilon) + f^{(3)}(\epsilon)M_4^{(1)}(3\epsilon) + C^{(3)} + E_4^{(3)}(\epsilon)$$

and using HPL identities relating $1/x \leftrightarrow x$, etc.,
we verify the relation, and extract

$$f_0^{(3)} = \frac{11}{2}\zeta_4 \qquad f_1^{(3)} = 6\zeta_5 + 5\zeta_2\zeta_3 \qquad f_2^{(3)} = c_1\zeta_6 + c_2\zeta_3^2$$

$$C^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_2$$

n-point information still required to separate

"Leading transcendentality" relation between QCD and N=4 SYM

- KLOV (Kotikov, Lipatov, Onishschenko, Velizhanin, hep-th/0404092) noticed (at 2 loops) a remarkable relation (**miracle**) between kernels for:
  - BFKL evolution (strong rapidity ordering)
  - DGLAP evolution (pdf evolution = strong collinear ordering)

  in QCD and N=4 SYM**:**
- Set fermionic color factor $C_F = C_A$ in the QCD result and keep only the "leading transcendentality" terms. They coincide with the full N=4 SYM result (even though theories differ by scalars)
- Conversely, N=4 SYM results predict pieces of the QCD result

- "transcendentality": 1 for π
  n for $\zeta_n = \mathrm{Li}_n(1)$

similar counting for HPLs and for related harmonic sums used to describe DGLAP kernels

3-loop DGLAP splitting functions *P(x)* in QCD

Related by a Mellin transform to the anomalous dimensions $\gamma(j)$ of leading-twist operators with spin *j* $\bar{q}(\gamma^+\partial_+)^j q$

$$\gamma(j) \equiv -\int_0^1 dx\, x^{j-1} P(x)$$

- Computed by MVV (Moch, Vermaseren, Vogt, hep-ph/0403192, hep-ph/0404111)

	tree	1-loop	2-loop	3-loop
qγ	1	3	25	359
gγ		2	17	345
hγ			2	56
qW	1	3	32	589
qϕ		1	23	696
gϕ	1	8	218	6378
hϕ		1	33	1184
sum	3	18	350	9607

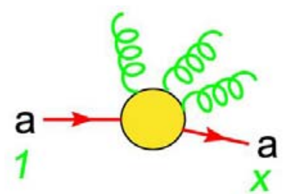
Table 1. The number of diagrams employed in our calculation of the three-loop splitting functions.

- KLOV obtained the N=4 SYM results by keeping only the "leading transcendentality" terms of MVV

3-loop planar N=4 amplitude checks QCD

- From the value of the $1/\varepsilon^2$ pole in the scattering amplitude, we can check the KLOV observation, plus the MVV computation, in the large-spin j limit of the leading-twist anomalous dimensions $\gamma(j)$ ($x \to 1$ limit of the x-space DGLAP kernel), also known as the soft or cusp anomalous dimension:

$$P^{(n)}_{\mathrm{aa},x\to 1}(x) = \frac{A^{\mathrm{a}}_{n+1}}{(1-x)_+} + B^{\mathrm{a}}_{n+1}\,\delta(1-x) + C^{\mathrm{a}}_{n+1}\,\ln(1-x) + \mathcal{O}(1)$$



or

$$\gamma(j) = \frac{1}{2}\gamma_K(\alpha_s)\,(\ln(j) + \gamma_e) - B(\alpha_s) + \mathcal{O}(\ln(j)/j)$$

where $A(\alpha_s) = \frac{1}{2}\gamma_K(\alpha_s)$

Korchemsky (1989); Korchemsky, Marchesini (1993)

3-loop planar N=4 check (cont.)

MVV $x \to 1$ limit:

$f_0^{(1)} = 1$

$f_0^{(2)} = -\zeta_2$

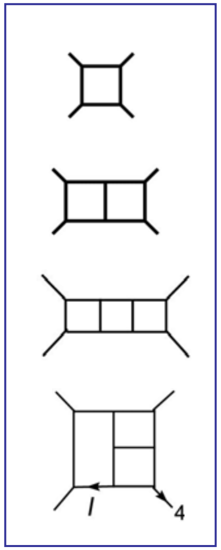
$f_0^{(3)} = \frac{11}{2}\zeta_4 = \frac{11}{5}(\zeta_2)^2$

$$A_1^{\rm q} = 4\,C_F$$

$$A_2^{\rm q} = 8\,C_F \left[\left(\frac{67}{18} - \zeta_2 \right) C_A - \frac{5}{9}\,n_f \right]$$

$$A_3^{\rm q} = 16\,C_F C_A^2 \left[\frac{245}{24} - \frac{67}{9}\,\zeta_2 + \frac{11}{6}\,\zeta_3 + \frac{11}{5}\,\zeta_2^2 \right] + 16\,C_F^2 n_f \left[-\frac{55}{24} + 2\,\zeta_3 \right]$$
$$+ 16\,C_F C_A n_f \left[-\frac{209}{108} + \frac{10}{9}\,\zeta_2 - \frac{7}{3}\,\zeta_3 \right] + 16\,C_F n_f^2 \left[-\frac{1}{27} \right] .$$

only 4 integrals required



3-loop planar N=4 check (cont.)

- *1/ε* pole is sensitive to *G* in Sudakov form factor.
- We can apply KLOV's prescription here as well,

to predict the leading transcendentality terms in the three-loop QCD form factor:

$$f_1^{(3)} = 6\zeta_5 + 5\zeta_2\zeta_3 \;\Rightarrow\; \mathcal{G}_0^{(3)} = C_A^3\left(4\zeta_5 + \tfrac{10}{3}\zeta_2\zeta_3\right)$$

- Prediction confirmed by MVV (hep-ph/0508055)

$$\begin{aligned} G_3^g \;=\; & C_A^3\left(-\frac{373975}{729} - \frac{27320}{81}\zeta_2 + \frac{4096}{27}\zeta_3 + \frac{1276}{15}\zeta_2^2 + \frac{80}{3}\zeta_2\zeta_3 + 32\zeta_5\right) \\ & + C_A^2 n_f\left(\frac{266072}{729} + \frac{7328}{81}\zeta_2 + \frac{56}{9}\zeta_3 - \frac{328}{15}\zeta_2^2\right) + C_A C_F n_f\left(\frac{3833}{27} + 8\zeta_2 \right. \\ & \left. - \frac{752}{9}\zeta_3 + \frac{32}{5}\zeta_2^2\right) - 4C_F^2 n_f + C_A n_f^2\left(-\frac{28114}{729} - \frac{160}{27}\zeta_2 - \frac{256}{27}\zeta_3\right) \\ & + C_F n_f^2\left(-\frac{104}{3} + \frac{64}{3}\zeta_3\right). \end{aligned}$$

Integrability & anomalous dimensions

- The dilatation operator for N=4 SYM, treated as a Hamiltonian, is integrable at one loop.
- E.g. in *SU(2)* subsector, tr($Z^L X^j$), it is a Heisenberg model

Minahan, Zarembo, hep-th/0212208; Beisert, Staudacher, hep-th/0307042

- Much accumulating evidence of multi-loop integrability in various sectors of the theory

Beisert, Dippel, Eden, Jarczak, Kristjansen, Rej, Serban, Staudacher, Zwiebel, Belitsky, Gorsky, Korchemsky, …

including *sl(2)* sector, tr($D^j Z^L$), directly at two loops

Eden, Staudacher, hep-th/0603157

A conjectured all-orders asymptotic (large *j*) Bethe ansatz has been obtained by deforming the "spectral parameter" *u* to *x:*

$$\boxed{\gamma(j) \equiv -\int_0^1 dx\, x^{j-1} P(x)}$$

$$u \pm \frac{i}{2} = x^\pm + \frac{g^2}{2x^\pm}$$

Staudacher, hep-th/0412188; Beisert, Staudacher, hep-th/0504190; Beisert, hep-th/0511013, hep-th/0511082; Eden, Staudacher, hep-th/0603157

All-orders proposal

The all-orders asymptotic Bethe ansatz leads to the following proposal for the soft/cusp anomalous dimension in N=4 SYM:

Eden, Staudacher, hep-ph/0603157

$$f(g) \;=\; 4\,g^2 \;-\; 16\,g^4 \int_0^\infty dt\,\hat\sigma(t)\,\frac{J_1(\sqrt{2}\,g\,t)}{\sqrt{2}\,g\,t}$$

where

$$\hat\sigma(t) \;=\; \frac{t}{e^t-1}\left[\,\frac{J_1(\sqrt{2}\,g\,t)}{\sqrt{2}\,g\,t} - 2\,g^2\int_0^\infty dt'\;\hat K(\sqrt{2}\,g\,t,\sqrt{2}\,g\,t')\;\hat\sigma(t')\right]$$

is the solution to an integral equation with Bessel-function kernel

$$\hat K(t,\,t') \;=\; \frac{J_1(t)\,J_0(t') \;-\; J_0(t)\,J_1(t')}{t\;-\;t'}$$

Perturbative expansion:

$$f(g) = 4\,g^2 - \frac{2}{3}\,\pi^2\,g^4 + \frac{11}{45}\,\pi^4\,g^6 - \left(\frac{73}{630}\,\pi^6 \underset{?}{-} 4\,\zeta(3)^2\right)\,g^8 + \ldots$$

Beyond three loops

Bethe ansatz “wrapping problem” when
interaction range exceeds spin chain length,
implies proposal needs checking via other methods,
e.g. gluon scattering amplitudes – particularly at 4 loops.

Recently two programs have been written to automate the
extraction of $1/\varepsilon$ poles from Mellin-Barnes integrals, and
set up numerical integration over the multiple inversion contours.

Anastasiou, Daleo, hep-ph/0511176; this workshop;
Czakon, hep-ph/0511200

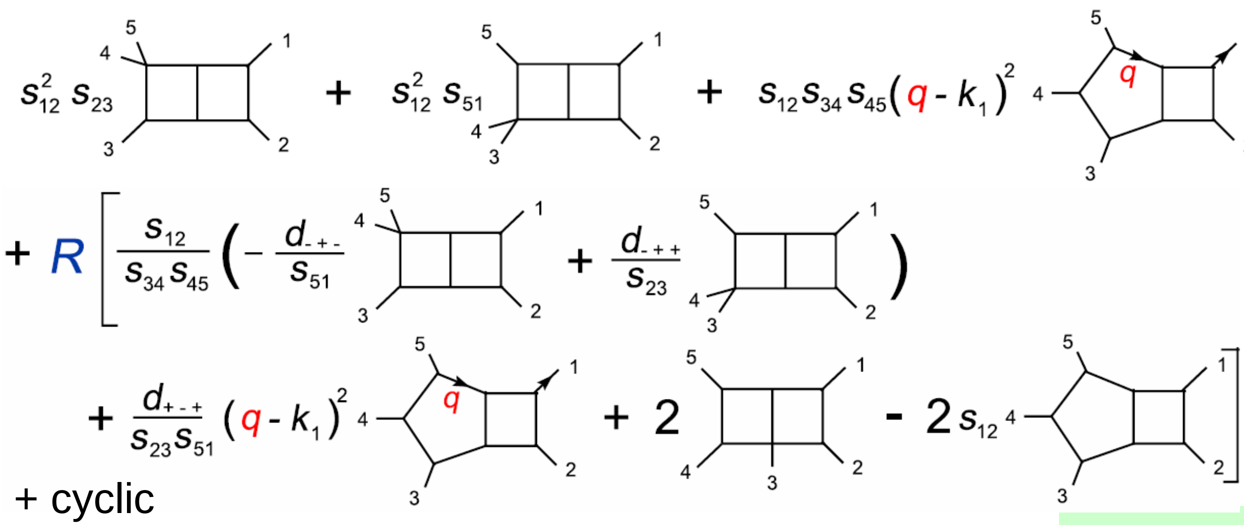
Numerics should be enough to check four-loop ansatze.

Z. Bern et al, in progress

Numerical two-loop check for *n=5*

Collinear limits are highly suggestive, but not quite a proof.

Using unitarity, first in *D=4*, later in *D=4-2ε*, the two-loop *n=5* amplitude was found to be:



Bern, Rozowsky, Yan, hep-ph/9706392

Cachazo, Spradlin, Volovich, hep-th/0602228

$$R = \varepsilon(k_1,k_2,k_3,k_4) \times s_{12}s_{23}s_{34}s_{45}s_{51}/\det(s_{ij})|_{i,j=1,2,3,4}$$

Even and odd terms checked numerically with aid of Czakon, hep-ph/0511200

Bern, Czakon, Kosower, Roiban, Smirnov, hep-th/0604074

Conclusions & Outlook

- N=4 SYM captures most singular infrared behavior of QCD.
- Finite terms exponentiate in a very similar way to the IR divergent ones, in the planar, large N_c limit
- How is this related to the AdS/CFT correspondence?
- "Leading transcendentality" relations for some quantities. Why?
- Is the Eden-Staudacher all-order prediction for the soft anomalous dimension correct at 4 loops?

Extra Slides

Simpler ways to check ansatz?

One can apply suitable differential operators to terms in the ansatz, which reduce their degree of infrared divergence

Cachazo, Spradlin, Volovich, hep-th/0601031

At two loops, $\mathcal{L}^{(2)} = \left[\frac{d^2}{d(\ln x)^2} - \epsilon^2\right]^3$ annihilates

$$(st)^\epsilon M_4^{(2)}(\epsilon) = \frac{2}{\epsilon^4} - \frac{1}{\epsilon^2}\left[\frac{1}{2}\ln^2(-x) + \frac{5}{4}\pi^2\right] + \mathcal{O}(\epsilon^{-1}),$$

$$(st)^\epsilon \frac{1}{2}\left(M_4^{(1)}(\epsilon)\right)^2 = \frac{2}{\epsilon^4} - \frac{1}{\epsilon^2}\left[\frac{1}{2}\ln^2(-x) + \frac{4}{3}\pi^2\right] + \mathcal{O}(\epsilon^{-1}).$$

and greatly simplifies the MB integral evaluation.

Verifies two-loop ansatz up to functions in the kernel of $\mathcal{L}^{(2)}$

$$\frac{1}{(st)^\epsilon}K(x,\epsilon) = C + E\ln^2(-x) + F\ln^4(-x) + \mathcal{O}(\epsilon),$$

Generalization to multi-loops may also be quite useful

Other theories

Khoze, hep-th/0512194

Two classes of (large N_c) conformal gauge theories "inherit" the same large N_c perturbative amplitude properties from N=4 SYM:

1. Theories obtained by orbifold projection
– product groups, matter in particular bi-fundamental rep's

Bershadsky, Johansen, hep-th/9803249

2. The N=1 supersymmetric "beta-deformed" conformal theory
– same field content as N=4 SYM, but superpotential is modified:

$$ig\,\mathrm{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) \to ig\,\mathrm{Tr}(e^{i\pi\beta_R}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta_R}\Phi_1\Phi_3\Phi_2)$$

Leigh, Strassler, hep-th/9503121

Supergravity dual known for this case, deformation of $AdS_5 \times S^5$

Lunin, Maldacena, hep-th/0502086

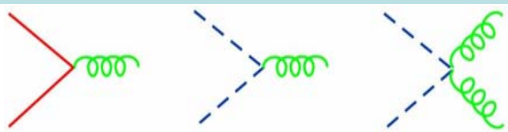
Breakdown of inheritance at five loops (!?) for more general marginal perturbations of N=4 SYM? Khoze, hep-th/0512194

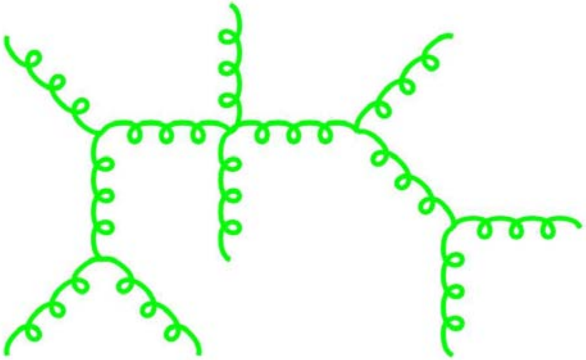
How are QCD and N=4 SYM related?

At tree-level they are essentially identical

Consider a tree amplitude for *n* gluons.

Fermions and scalars cannot appear because they are produced in pairs



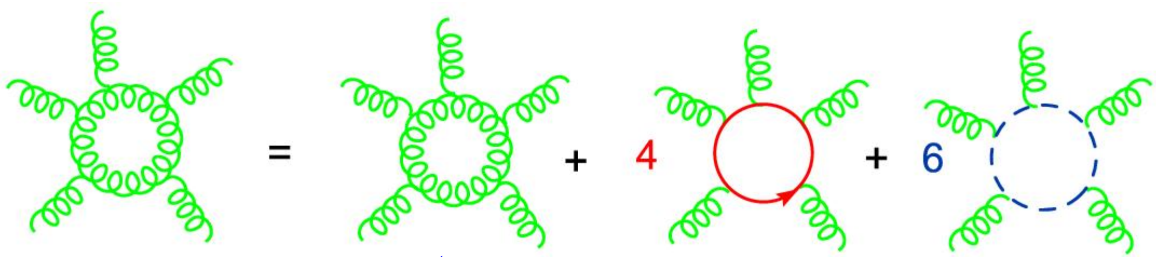


Hence the amplitude is the same in QCD and N=4 SYM. The QCD tree amplitude “secretly” obeys all identities of N=4 supersymmetry:

At loop-level, QCD and N=4 SYM differ

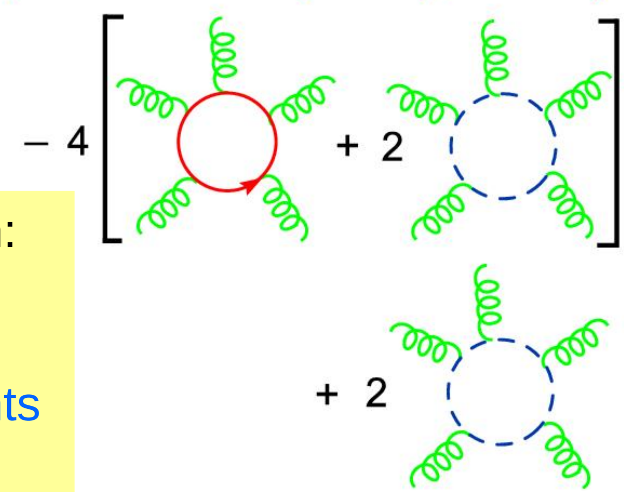
However, it is profitable to rearrange the QCD computation to exploit supersymmetry

gluon loop



N=4 SYM

contains most of the $1/\varepsilon$ poles

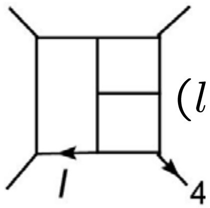


N=1 multiplet

scalar loop --- no SUSY, but also no spin tangles

Perhaps best virtue of decomposition: gives excuse to compute beautiful, simple amplitudes in N=4 SYM -- consider them as one of 3 components of practical QCD amplitudes.

The “tennis court” integral



$$(l+k_4)^2 \quad = \quad I_4^{(3)b}(s,t) = -\frac{1}{(-s)^{1+3\epsilon}\, t^2} \sum_{j=0}^{6} \frac{c_j(x,L)}{\epsilon^j}$$

$$x = -t/s,\ L = \ln(s/t)$$

$$c_6 = \frac{16}{9}\,, \quad c_5 = \frac{13}{6}L\,, \quad c_4 = \frac{1}{2}L^2 - \frac{19}{12}\pi^2\,,$$

$$c_3 = \frac{5}{2}\left[H_{0,0,1}(x) + LH_{0,1}(x)\right] + \frac{5}{4}\left[L^2 + \pi^2\right] H_1(x)$$

$$-\frac{7}{12}L^3 - \frac{157}{72}L\pi^2 - \frac{241}{18}\zeta_3\,,$$

$$c_2 = \frac{1}{2}\left[11H_{0,0,0,1}(x) - 5H_{0,0,1,1}(x) - 5H_{0,1,0,1}(x) - 5H_{1,0,0,1}(x)\right]$$

$$+\frac{1}{2}L\left[14H_{0,0,1}(x) - 5H_{0,1,1}(x) - 5H_{1,0,1}(x)\right] + \frac{1}{4}L^2\left[17H_{0,1}(x) - 5H_{1,1}(x)\right]$$

$$+\frac{4}{3}\pi^2 H_{0,1}(x) - \frac{5}{4}\pi^2 H_{1,1}(x) + \frac{5}{3}L^3 H_1(x) + \frac{25}{12}L\pi^2 H_1(x)$$

$$-\frac{41}{3}L\zeta_3 + \frac{5}{2}H_1(x)\zeta_3 - \frac{1}{3}L^4 - \frac{1}{4}L^2\pi^2 + \frac{2429}{6480}\pi^4\,,$$

“Tennis court” integral (cont.)

$$c_1 = \frac{1}{2}\left[-55H_{0,0,0,0,1}(x) - 59H_{0,0,0,1,1}(x) - 31H_{0,0,1,0,1}(x) + 5H_{0,0,1,1,1}(x)\right.$$
$$-3H_{0,1,0,0,1}(x) + 5H_{0,1,0,1,1}(x) + 5H_{0,1,1,0,1}(x) + 25H_{1,0,0,0,1}(x)$$
$$+\frac{1}{2}L\left[22H_{0,0,0,1}(x) - 46H_{0,0,1,1}(x) - 18H_{0,1,0,1}(x) + 5H_{0,1,1,1}(x)\right.$$
$$\left.+10H_{1,0,0,1}(x) + 5H_{1,0,1,1}(x) + 5H_{1,1,0,1}(x)\right]$$
$$+\frac{1}{4}L^2\left[64H_{0,0,1}(x) - 33H_{0,1,1}(x) - 5H_{1,0,1}(x) + 5H_{1,1,1}(x)\right]$$
$$+\frac{1}{24}\pi^2\left[25H_{0,0,1}(x) - 128H_{0,1,1}(x) + 40H_{1,0,1}(x) + 30H_{1,1,1}(x)\right]$$
$$+\frac{1}{12}L^3\left[71H_{0,1}(x) - 20H_{1,1}(x)\right]$$
$$+\frac{1}{24}L\pi^2\left[153H_{0,1}(x) - 50H_{1,1}(x)\right] + \frac{1}{2}\left[8H_{0,1}(x) - 5H_{1,1}(x)\right]\zeta_3$$
$$+\frac{43}{48}L^4H_1(x) + \frac{71}{48}L^2\pi^2H_1(x) - \frac{5}{144}\pi^4H_1(x) - \frac{5}{2}LH_1(x)\zeta_3 + \frac{7}{48}L^5$$
$$+\frac{227}{144}L^3\pi^2 + \frac{13}{4}L^2\zeta_3 + \frac{10913}{8640}L\pi^4 + \frac{3257}{216}\pi^2\zeta_3 - \frac{889}{10}\zeta_5\,,$$

“Tennis court” integral (cont.)

$$\begin{aligned}
c_0 = \frac{1}{2}\,&[379H_{0,0,0,0,0,1}(x) + 343H_{0,0,0,0,1,1}(x) + 419H_{0,0,0,1,0,1}(x) + 347H_{0,0,0,1,1,1}(x)\\
&+355H_{0,0,1,0,0,1}(x) + 175H_{0,0,1,0,1,1}(x) + 223H_{0,0,1,1,0,1}(x) - 5H_{0,0,1,1,1,1}(x)\\
&+151H_{0,1,0,0,0,1}(x) + 3H_{0,1,0,0,1,1}(x) + 51H_{0,1,0,1,0,1}(x) - 5H_{0,1,0,1,1,1}(x)\\
&+99H_{0,1,1,0,0,1}(x) - 5H_{0,1,1,0,1,1}(x) - 5H_{0,1,1,1,0,1}(x) - 193H_{1,0,0,0,0,1}(x)\\
&-169H_{1,0,0,0,1,1}(x) - 121H_{1,0,0,1,0,1}(x) - 5H_{1,0,0,1,1,1}(x) - 73H_{1,0,1,0,0,1}(x) - 5H_{1,0,1,0,1,1}(x)\\
&-5H_{1,0,1,1,0,1}(x) - 25H_{1,1,0,0,0,1}(x) - 5H_{1,1,0,0,1,1}(x) - 5H_{1,1,0,1,0,1}(x) - 5H_{1,1,1,0,0,1}(x)]\\
&+\frac{1}{2}L\,[98H_{0,0,0,0,1}(x) - 22H_{0,0,0,1,1}(x) + 98H_{0,0,1,0,1}(x) + 238H_{0,0,1,1,1}(x) + 78H_{0,1,0,0,1}(x)\\
&+66H_{0,1,0,1,1}(x) + 114H_{0,1,1,0,1}(x) - 5H_{0,1,1,1,1}(x) - 82H_{1,0,0,0,1}(x) - 106H_{1,0,0,1,1}(x)\\
&-58H_{1,0,1,0,1}(x) - 5H_{1,0,1,1,1}(x) - 10H_{1,1,0,0,1}(x) - 5H_{1,1,0,1,1}(x) - 5H_{1,1,1,0,1}(x)]\\
&+\frac{1}{4}L^2\,[124H_{0,0,0,1}(x) - 208H_{0,0,1,1}(x) - 44H_{0,1,0,1}(x) + 129H_{0,1,1,1}(x)\\
&-20H_{1,0,0,1}(x) - 43H_{1,0,1,1}(x) + 5H_{1,1,0,1}(x) - 5H_{1,1,1,1}(x)]\\
&+\frac{1}{24}\pi^2\,[183H_{0,0,0,1}(x) - 121H_{0,0,1,1}(x) + 375H_{0,1,0,1}(x) + 704H_{0,1,1,1}(x)\\
&+31H_{1,0,0,1}(x) - 328H_{1,0,1,1}(x) - 40H_{1,1,0,1}(x) - 30H_{1,1,1,1}(x)]\\
&+\frac{1}{12}L^3\,[260H_{0,0,1}(x) - 215H_{0,1,1}(x) - 7H_{1,0,1}(x) + 20H_{1,1,1}(x)]\\
&+\frac{1}{24}L\pi^2\,[326H_{0,0,1}(x) - 633H_{0,1,1}(x) + 127H_{1,0,1}(x) + 50H_{1,1,1}(x)]\\
&-\frac{1}{2}\,[-3LH_{0,1}(x) - 5LH_{1,1}(x) + 165H_{0,0,1}(x) + 104H_{0,1,1}(x) - 68H_{1,0,1}(x) - 5H_{1,1,1}(x)]\,\zeta_3\\
&+\frac{1}{48}L^4\,[309H_{0,1}(x) - 43H_{1,1}(x)] + \frac{1}{48}L^2\pi^2\,[725H_{0,1}(x) - 71H_{1,1}(x)]\\
&+\frac{1}{720}\pi^4\,[1848H_{0,1}(x) + 25H_{1,1}(x)]\\
&+\frac{37}{120}L^5H_1(x) + \frac{11}{8}L^3\pi^2H_1(x) + \frac{641}{720}L\pi^4H_1(x) + \frac{38}{3}L^3\zeta_3 + \frac{479}{18}L\pi^2\zeta_3\\
&-2L^2H_1(x)\zeta_3 - \frac{269}{24}\pi^2H_1(x)\zeta_3 + \frac{129}{2}H_1(x)\zeta_5 + \frac{151}{720}L^6 + \frac{373}{288}L^4\pi^2\\
&+\frac{3163}{2880}L^2\pi^4 - \frac{1054}{5}L\zeta_5 + \frac{1391417}{3265920}\pi^6 + \frac{197}{6}\zeta_3^2\,.
\end{aligned}$$