Numerical Computation of a Non-Planar Two-Loop Vertex Diagram

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Overview.

♢ The two-loop crossed vertex diagram gives rise to a six-dimensional integral, where the outer integration is over the simplex $z_1+z_2+z_3 = 1$ and the inner integration over the hyper-rectangle $[-1,+1]^3$. The factor $1/D_3^2$ in the integrand has a non-integrable singularity interior to the integration domain and a singularity on the boundary.

♢ The integral can be evaluated by iterated numerical integration.

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$\diamondsuit$ We also study a sector transformation which rewrites the original problem as a sum of three five-dimensional integrals (two of which are equal through symmetry) and eliminates the boundary singularity.

$\diamondsuit$ The interior singularity is handled by replacing D_3 in the denominator by $D_3 - i\varepsilon$ and treating the integral in the limit as $\varepsilon \longrightarrow 0$. This is accomplished numerically via an extrapolation.

$\diamondsuit$ The integration and extrapolation are performed automatically.

$\diamondsuit$ We verify the results with data published in the literature.

1 Introduction

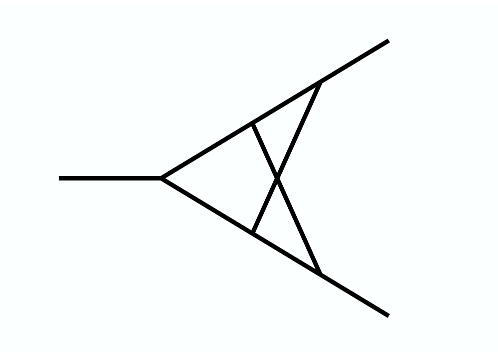
♢ The scalar non-planar two-loop vertex integral, according to Kurihara et al. (2005) [10], is

$$IF = \frac{1}{8}\int_0^\infty \mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\delta(1-\Sigma_{j=1}^3 z_j)\,z_1z_2z_3 \int_{-1}^1 \mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3\,\frac{1}{(D_3-i\varepsilon)^2},$$

where D_3 is a quadratic in $\vec{y}=(y_1,y_2,y_3)^\tau$, and D_3 depends on the masses m_j, $1\le j\le 6$ and on $s_\ell = p_\ell^2$, $\ell=1,2,3$.

♢ The problem is scalar in view of the constant numerator in the integrand. If the numerator is a polynomial, the problem is non-scalar.

♢ The integral is interpreted in the limit as the parameter in the denominator, $\varepsilon \longrightarrow 0$.



Nonplanar vertex two-loop diagram

Specifically,

$$D_3 = \vec{y}^{\tau} A \vec{y} + \vec{b}^{\tau} \vec{y} + c, \tag{1}$$

where

$$A = \frac{1}{4} \begin{pmatrix} -z_1^2(z_2+z_3) & z_1z_2z_3(-s1-s2+s3)/2 & z_1z_2z_3(-s_1+s_2-s_3)/2 \\ z_1z_2z_3(-s_1-s_2+s_3)/2 & -z_2^2(z_3+z_1)s_2 & z_1z_2z_3(s_1-s_2-s_3)/2 \\ z_1z_2z_3(-s_1+s_2-s_3)/2 & z_1z_2z_3(s_1-s_2-s_3)/2 & -z_3^2(z_1+z_2)s_3 \end{pmatrix},$$

$$\vec{b} = \frac{1}{2}\, U \begin{pmatrix} z_1(m_3^2 - m_4^2) \\ z_2(m_5^2 - m_6^2) \\ z_3(m_2^2 - m_1^2) \end{pmatrix},$$

$$c = \frac{1}{4}\, U(z_1s_1+z_2s_2+z_3s_3-2(m_3^2+m_4^2)z_1-2(m_5^2+m_6^2)z_2-2(m_1^2+m_2^2)z_3)$$

and

$$U = z_1z_2 + z_2z_3 + z_3z_1.$$

♢ The outer integral (in z_1, z_2, z_3) of 1 is taken over the unit simplex, $1 - \Sigma_{j=1}^{3} z_j = 0$.

♢ The inner integral is over the three-dimensional hyper-rectangle $-1 \le y_j \le 1, 1 \le j \le 3$.

♢ Note that $D_3 = 0$ at $z_1 = z_2 = z_3 = 0$.

2 Transformation

$\diamondsuit$ We apply a transformation which was used to handle infrared divergent loop integrals by Binoth et al. [3].

$\diamondsuit$ This casts the integral IF in the form

$$IF = I_1F + I_2F + I_3F,$$

where $F = F(\vec{z})$ represents the inner integral, and the integrals in the sum are taken over sectors of the first octant in three-space, i.e.,

$$\begin{aligned} I_1 &= \int_0^\infty dz_1 \int_0^{z_1} dz_2 \int_0^{z_1} dz_3\, F(\vec{z}), \\ I_2 &= \int_0^\infty dz_2 \int_0^{z_2} dz_1 \int_0^{z_2} dz_3\, F(\vec{z}), \\ I_3 &= \int_0^\infty dz_3 \int_0^{z_3} dz_1 \int_0^{z_3} dz_2\, F(\vec{z}). \end{aligned}$$

$\diamondsuit$ I_1 is transformed according to

$$\begin{aligned} &z_1 \\ &z_2 = t_1 z_1 \\ &z_3 = t_2 z_1 \end{aligned}$$

This yields I_1 in the form

$$I_1 = \frac{1}{8}\int_0^\infty \mathrm{d}z_1 \int_0^1 \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2\, t_1 t_2\ \delta(1{-}z_1(1{+}t_1{+}t_2))\, z_1^5 \int_{-1}^1 \mathrm{d}\vec{y}\, \frac{1}{(D_3 - i\varepsilon)^2}$$

where

$$D_3 = z_1^3(A_1 + B_1 + C_1).$$

♢ Furthermore, writing

$$z_1^5 \frac{1}{(D_3 - i\varepsilon)^2} = \mathcal{R} + i\mathcal{I},$$

we have

$$\mathcal{R} = \frac{1}{z_1}\,\frac{(A_1+B_1+C_1)^2 - \varepsilon^2/z_1^6}{((A_1+B_1+C_1)^2 + \varepsilon^2/z_1^6)^2}$$

and

$$\mathcal{I} = \frac{2\varepsilon}{z_1^4}\,\frac{A_1+B_1+C_1}{(A_1+B_1+C_1)^2 + \varepsilon^2/z_1^6)^2}.$$

♢ The dimension reduction is achieved by the transformation

$$z_1 = \frac{u_1}{1 + t_1 + t_2}$$

so that $dz_1/z_1 = du_1/u_1$ and

$$\delta(1 - z_1(1 + t_1 + t_2)) = \delta(1 - u_1).$$

The integration in u_1 thus reduces to setting $u_1 = 1$ in the integrand.

♢ The resulting integral for I_1 is:

$$I_1 = \frac{1}{8}\int_0^1 \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2 \int_{-1}^1 \mathrm{d}\vec{y}\, \frac{(A_1 + B_1 + C_1)^2 - \varepsilon^2(1 + t_1 + t_2)^6}{((A_1 + B_1 + C_1)^2 + \varepsilon^2(1 + t_1 + t_2)^6)^2}$$
$$+ \frac{2i\varepsilon(A_1 + B_1 + C_1)(1 + t_1 + t_2)^3}{((A_1 + B_1 + C_1)^2 + \varepsilon^2(1 + t_1 + t_2)^6)^2}.$$

♢ I_2 and I_3 are derived in a similar manner.

3 Numerical Integration

♢ In [6, 7] we used iterated integration together with extrapolation methods to compute various one-loop (scalar and nonscalar) integrals and a two-loop planar vertex integral.

♢ E.g., the three-dimensional non-scalar box integral in [7] was evaluated by iterated integration as a 1D×1D×1D integral by applying a one-dimensional adaptive method in every coordinate direction.

♢ Iterated integration methods have further been examined theoretically and experimentally in [12, 11].

♢ For the current computation we can apply iterated adaptive numerical integration to the 5D integral as a 2D×1D×1D×1D problem (after the sector transformation which transforms the outer 3D integral to 2D). The inner three dimensions need substantial subdivision in view of the quadratic hypersurface singularity.

♢ An alternative approach is by treating the original problem as a $(1D)^6$ iterated integral.

$\diamondsuit$ A general outline of the adaptive numerical integration algorithm (applied for each group of iterated dimensions) is given below.

Evaluate initial region and initialize results
Put initial region on priority queue
while (evaluation limit not reached
    and estimated error too large)
  Retrieve region from priority queue
  Split region
  Evaluate new subregions and update results
  Insert subregions into priority queue

$\diamondsuit$ The user specifies the function $f(\mathbf{x})$, integration limits (for a domain $\mathcal{D}$), requested absolute and relative accuracies ε_a and ε_r, respectively, and determines a limit on the number of subdivisions.

$\diamondsuit$ The (black box) algorithm calculates an integral approximation $Qf \approx If = \int_{\mathcal{D}} f(\vec{x})\, d\vec{x}$ and an absolute error estimate Ef, with the aim to satisfy a criterion of the form $|Qf - If| \leq Ef \leq \max\{\varepsilon_a, \varepsilon_r |If|\}$ within the allowed number of subdivisions, or indicate an error condition if the subdivision limit has been reached.

♢ The QUADPACK [13] adaptive routine DQAGE was used for the 1D quadrature problems, with a 7 and 15-point Gauss-Kronrod quadrature rule pair on each subinterval.

♢ The multivariate integration was based on DCUHRE [9, 2] and its cubature rule of polynomial degree 7 for integration over the subregions. A parallel implementation of this method is layered over MPI in PARINT [1].

4 Extrapolation

♢ Assuming the integral $I = I(\varepsilon)$ of (1) has an asymptotic expansion in terms of the form $\varepsilon^k \log^\ell \varepsilon$, $k \geq 0, \ \ell \geq 0$ integer, algorithms such as the ε algorithm [14, 16] are valid for accelerating convergence when supplied with a sequence of $I(\varepsilon_j)$ for a geometric progression of ε_j.

♢ Table 1 shows a sample extrapolation table obtained for the crossed vertex two-loop problem with parameters $m_1 = m_2 = m_4 = m_5 = 150$ GeV, $m_3 = m_6 = 91.17$ GeV; $s_1 = s_2 = 150^2$ GeV^2 and $s_3/m_1^2 = 5$.

Table 1: Sample extrapolation table

j				
32	0.1019E-08			
31	0.1096E-08	0.1480E-08		
30	0.1160E-08	0.1411E-08	0.1441E-08	
29	0.1211E-08	0.1478E-08	0.1469E-08	0.1464E-08
28	0.1254E-08	0.1468E-08	0.1451E-08	
27	0.1290E-08	0.1462E-08		
26	0.1319E-08			

◇ The entries in the leftmost column of the table are approximations to $I(\varepsilon_j)$ computed by numerical integration of the 5D integral for requested relative tolerances of 10^{-3}.

◇ It should be noted that it is generally preferable to increase the accuracy requirement toward the inner integrations. A scheme for setting the error toleranced for the iterated integrations is under study [5].

◇ The extrapolation shown here is performed with $\varepsilon = \epsilon^j$ where $\epsilon = 1.2$ and $j = 32\ (-1)\ 26$. The result agrees with the data in [10].

Table 2: Real Part (in units of 10^{-9})

$(s_3/(m**2)$	Tarasov [15] (hep ph/9505277)	Ferroglia [8] (hep ph/0311186)	KEK Minami Tateya
4.0	0.733120(0.02)	0.7331(1)	0.733120(2)
4.5	0.61644824(0.1)	0.6216(78)	0.61650(2)
5.0	0.5184444(0.3)	0.5203(40)	0.51845(1)
8.0	0.14555(0.7)	0.1455(20)	0.1455223(5)
20.0	-0.2047(0.8)	-0.2058(5)	-0.20471(4)
100.0	-0.0382(3)	-0.0385(1)	-0.0382(2)

Table 3: Imaginary Part (in units of 10^{-9})

$(s_3/m**2)$	Tarasov [15] (hep ph/9505277)	Ferroglia [8] (hep ph/0311186)	KEK Minami Tateya
4.5	-0.3349475(1)	-0.3402(71)	-0.3349(1)
5.0	-0.430997(0.3)	-0.4442(93)	-0.43100(5)
8.0	-0.5460(0.5)	-0.5491(40)	-0.54594(1)
20.0	-0.1876(4)	-0.1864(4)	-0.187578(10)

$\diamondsuit$ Table 2 shows results obtained with the $(1D)^6$ approach for parameters $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m = 150$ GeV; $s_1 = s_2 = 0$ (Real Part). Table 3 lists the corresponding Imaginary Part data.

5 Conclusions

◇ The scalar crossed two-loop Feynman diagram gives rise to a six-dimensional integral. The integration in the outer three dimensions is over a simplex, while the inner integration is taken over a three-dimensional hyper-rectangle. The integrand has singularities on the boundary and within the domain.

◇ The integral can be approximated directly by iterated integration over the six dimensions.

◇ Alternatively, we can apply a sector transformation which rewrites the problem as a sum of three (two, through symmetry) five-dimensional integrals. Apart from the removal of the boundary singularity and the mapping to a hyper-rectangular domain, the reduction in dimension is significant for reducing the cost of the subsequent numerical cubature.

$\diamondsuit$ The transformation can be implemented automatically via symbolic manipulation (cf, [3]). For the subsequent automatic cubature, the software is supplied with the integrand, domain, requested accuracies and limits on the number of subdivision; it returns a result and estimated error.

$\diamondsuit$ As such, this paper is part of an effort to increase the automatization in computing Feynman diagrams.

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