On equations for the multi-quark bound states in Nambu – Jona-Lasinio model

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INTRODUCTION

Multi-particle equations are a traditional basis for a quantum-field description of bound states in particle physics. A well-known example of these equations is the two-particle Bethe-Salpeter equation (BSE) for the two-particle amplitude and for the two-particle bound state:

Bethe H. and Salpeter E.E.: Phys.Rev. 82 (1951) 309

The multi-particle (three or more particle) generalization of BSE have been also studied. A straightforward generalization of two-particle BSE has been intensively studied in sixties-seventies of last century.

A best exposition of these studies can be found in the work of Huang and Weldon:

Huang K. And Weldon H.A.: Phys.Rev D11 (1975) 257

An essential imperfection of the original Bethe-Salpeter approach to multi-particle equations was a full disconnection of the approach with the field-theoretical equations for Green functions (which are known as Schwinger-Dyson equations).

This imperfection has been eliminated by Dahmen and Jona-Lasinio, which had included the BSE to the field-theoretical Lagrangian formalism with the consideration of functional Legendre transformation with respect to bilocal source of fields:

Dahmen H.D. and Jona-Lasinio G.: Nuovo Cim. 52A (1967) 807

Then this approach has been generalized for multi-particle equations with consideration of Legendre transformation with respect to multi-local sources:

Rochev V.E.: Theor.Mat.Fiz. 51 (1982) 22

However, these theoretical constructions had not solved the principial dynamical problem of quantitative description of real bound states (nucleons, mesons etc.). A solution of BSE-type equations has been founded as a very complicated mathematical tool even for simple dynamical model. There is a main reason of a comparatively small popularity of the method of multi-particle BSE- type equations among the theorists.

Much more popular approach to the problem of hadronization in QCD is based on the ’t Hooft’s conjecture that QCD can be regarded as an effective theory of mesons and glueballs:

‘t Hooft G.: Nucl. Phys. B74 (1974) 461

Subsequently, it was shown by Witten that the baryons could be viewed as the solitons of the meson theory:

Witten E.: Nucl. Phys. B160 (1979) 57

Further development of these ideas has been successful and has leaded to the prediction of pentaquark states in baryon spectrum:

Diakonov D., Petrov V. and Polyakov M.: Z.Phys, A359 (1997) 305.

(It should be noted that the idea of pentaquarks was put forward for the first time in: **Zeldovich Ya. B. and Sakharov A.D.: Yadernaya Fizika, 4 (1966) 395.)**

Nevertheless, the investigations of multi-quark equations are of significant interest due to the much less model assumptions in this approach in comparison with the chiral-soliton models.

The solutions of multi-quark equations will provide us almost exhaustive information about the structure of hadrons.
There is the basic motivation of present work.

We shall investigate Nambu - Jona-Lasinio (NJL) model with quark content which is one of the most successful effective models of QCD in the non-perturbative region.

(for review see:

Vogl U. and Weise W.: Prog. Part. and Nucl. Phys. 27 (1991) 195

Klevansky S.P.: Rev. Mod. Phys. 64 (1992) 649)

In overwhelming majority of the investigations, the NJL model has been considered in the mean-field approximation or in the leading order of **1/n**-expansion. However, a number of perspective physical applications of NJL model is connected with multi-quark functions (for example: meson decays, pion-pion scattering, baryons, pentaquarks etc.). These multi-quark functions arise in higher orders of the mean-field expansion (MFE) for NJL model.

In present report we review some preliminary results of investigation of higher orders of MFE for NJL model. We have used an iteration scheme of solution of Schwinger-Dyson equation with fermion bilocal source, which has been developed in work:

Rochev V.E.: J. Phys. A: Math. Gen. 33 (2000) 7379

We have considered equations for Green functions of NJL model in MFE up to third order. The leading approximation and the first order of MFE maintain equations for the quark propagator and the two-particle function and also the first-order correction to the quark propagator. A consideration of these equations is the usual field of investigations of NJL model. The second order of MFE maintains the equations for four-particle and three-particle functions, and the third order maintains the equations for six-particle and five-particle functions. Here we discuss first results of investigation of the second-order equations for four-particle and three-particle Green functions.

Mean-field expansion in bilocal-source formalism

We consider NJL model with the Lagrangian

$$L = \overline{\psi} i \hat{\partial} \psi + \frac{g}{2}\left[\left(\overline{\psi}\psi\right)^2 + \left(\overline{\psi} i \gamma_5 \tau^a \psi\right)^2 \right] \quad .$$

A generating functional of Green functions (vacuum expectation values of **T-**products of fields) can be represented as the functional integral with bilocal source:

$$G(\eta) = \int D(\psi, \overline{\psi}) \exp i \left[\int dx L - \int dx dy \overline{\psi}(y) \eta(y,x) \psi(x) \right].$$

Here $\eta(y,x)$ is the bilocal source of the quark field.

The **n**-th functional derivative of **G** over source η is the **n**-particle **(2n**-point**)** Green function:

$$\left.\frac{\delta^n G}{\delta\eta(y_1,x_1)\cdots\delta\eta(y_n,x_n)}\right|_{\eta=0} =$$

$$\langle 0|T\{\psi(x_1)\overline{\psi}(y_1)\cdots\psi(x_n)\overline{\psi}(y_n)\}|0\rangle \equiv$$

$$\equiv S\begin{pmatrix} x_1 y_1 \\ \cdots\ \cdots \\ x_n y_n \end{pmatrix}.$$

Translational invariance of the functional measure gives us the functional-differential Schwinger-Dyson equation for the generating functional **G**:

$$\delta(x-y)G + i\hat{\partial}_x \frac{\delta G}{\delta\eta(y,x)} +$$

$$+ig\left\{\frac{\delta}{\delta\eta(y,x)} tr\left[\frac{\delta G}{\delta\eta(x,x)}\right] - \gamma_5 \frac{\delta}{\delta\eta(y,x)} tr\left[\gamma_5 \frac{\delta G}{\delta\eta(x,x)}\right]\right\} =$$

$$= \int dx_1 \eta(x,x_1) \frac{\delta G}{\delta\eta(y,x_1)}.$$

We shall solve this equation employing the method of work:

Jafarov R.G. and Rochev V.E.: Centr. Eur. J. Phys. 2 (2004) 367

A leading approximation is the functional

$$G^{(0)} = \exp Tr\left(\eta * S^{(0)}\right)$$

The leading approximation generates the linear iteration scheme:

$$G = G^{(0)} + G^{(1)} + \cdots G^{(n)} + \cdots,$$

Functional $G^{(n)}$

$$G^{(n)} = P^{(n)} G^{(0)},$$

where $P^{(n)}$ is a polynomial of **2n**-th order over the bilocal source η:

$$P^{(n)} = \frac{1}{(2n)!} S^{(n)}_{2n} * \eta^{2n} + \frac{1}{(2n-1)!} S^{(n)}_{2n-1} * \eta^{2n-1}$$

$$+ \frac{1}{2} S^{(n)}_{2} * \eta^{2} + S^{(n)} * \eta \,.$$

The unique connected Green function of the leading approximation is the quark propagator $S^{(0)}$. Other connected Green functions appear in the following iteration steps. The quark propagator in the chiral limit is

$$S^{(0)}(p) = (m - \hat{p})^{-1}$$

where m is the dynamical quark mass, which is a solution of gap equation:

$$1 = -\frac{8ign_c}{(2\pi)^4} \int \frac{dp}{m^2 - p^2}$$

The iteration-scheme equations give us the equations for first-order two-particle function and for propagator **(R.G. Jafarov and V.E. Rochev: Centr. Eur. J. of Phys. <u>2</u> (2004) 367.)**

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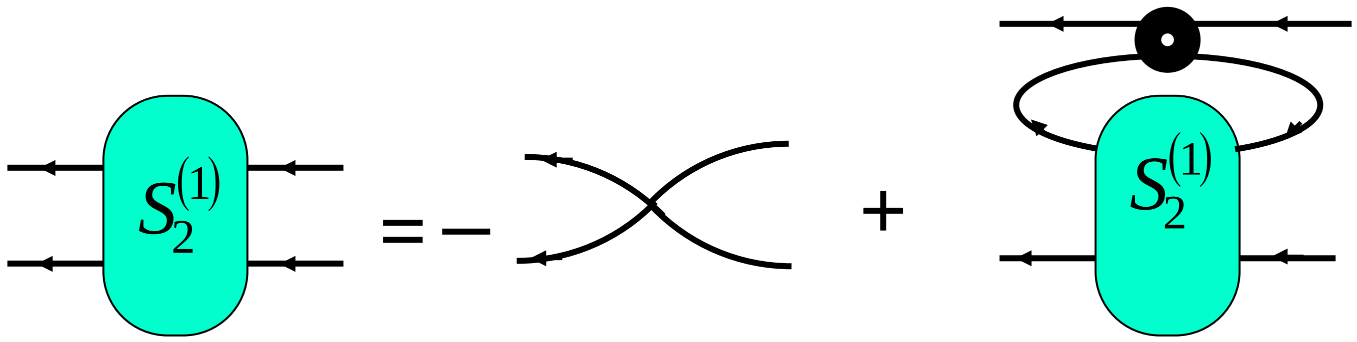
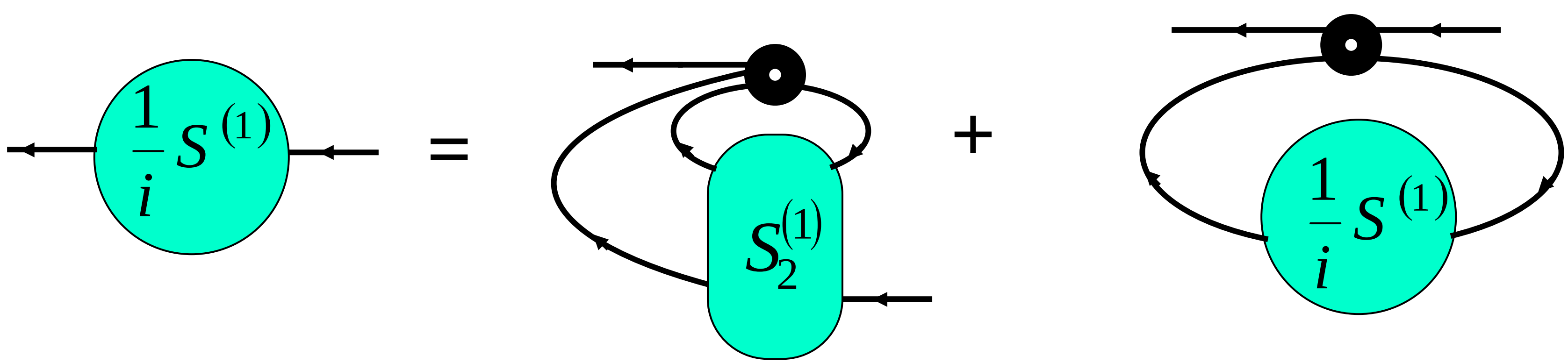


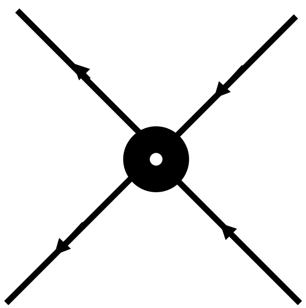
Two-quark function equation



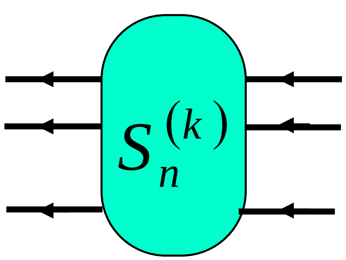
The equation of quark propagator

Here the graphical notation are used:

$$= \frac{1}{i} S^{(0)}$$



$$= ig\left(1 \otimes 1 - \gamma_5 \frac{\tau}{2} \otimes \gamma_5 \frac{\tau}{2}\right)$$



$$= S_n^{(k)}$$

To describe the solution of the first-order equation for two-particle function and for future purposes we introduce the composite meson propagators by following way:

a) Let us define scalar-scalar function

$$S_\sigma(x-x') \equiv \mathrm{tr}\left[S_2^{(1)}\begin{pmatrix} x & x \\ x' & x' \end{pmatrix}\right] \quad \sim \quad \langle \overline{\psi}\psi(x)\overline{\psi}\psi(x')\rangle$$

From the equation for two-particle function we obtain (in momentum space)

$$S_\sigma(p) = \frac{1}{ig}\left(1 - i\Delta_\sigma(p)\right)$$

Here we define the following function, which we call σ –meson propagator:

$$\Delta_\sigma(p) = \frac{Z(p)}{4m^2 - p^2}$$

where $Z(p) = \dfrac{I_0(4m^2)}{I_0(p^2)}$ and $I_0(p) = \int d\tilde{q}\, \dfrac{1}{\left[m^2 - (p-q)^2\right]\left(m^2 - q^2\right)}$.

b) Pseudoscalar-pseudoscalar function is defined as

$$S_{\pi}^{ab}(x-x') \equiv \mathrm{tr}\left[S_{2}^{(1)}\begin{pmatrix} x & x \\ x' & x' \end{pmatrix} \gamma_5 \frac{\tau^a}{2} \gamma_5 \frac{\tau^b}{2} \right] \sim \left\langle \overline{\psi}\gamma_5 \frac{\tau^a}{2}\psi(x)\overline{\psi}\gamma_5 \frac{\tau^b}{2}\psi(x') \right\rangle$$

From the equation for two-particle function we obtain (in momentum space)

$$S_{\pi}^{ab}(p) = -\frac{1}{ig}\left(\delta^{ab} - i\Delta_{\pi}^{ab}(p)\right) \quad .$$

Here we define the pion propagator

$$\Delta_{\pi}^{ab}(p) = -\frac{\delta^{ab} Z_{\pi}(p)}{p^2},$$

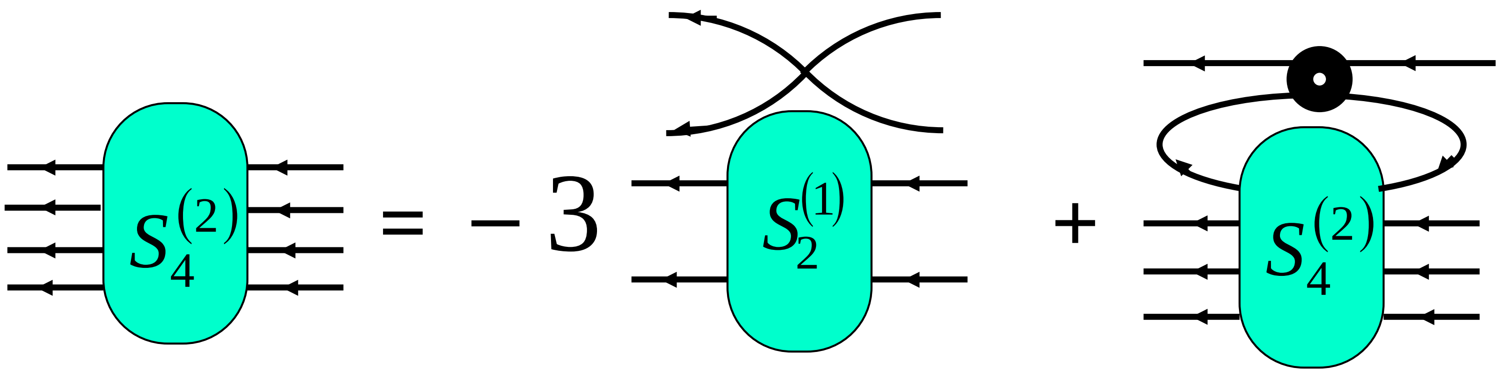
$$Z_{\pi}(p) = \frac{I_0(0)}{I_0(p^2)}.$$

Second order equations

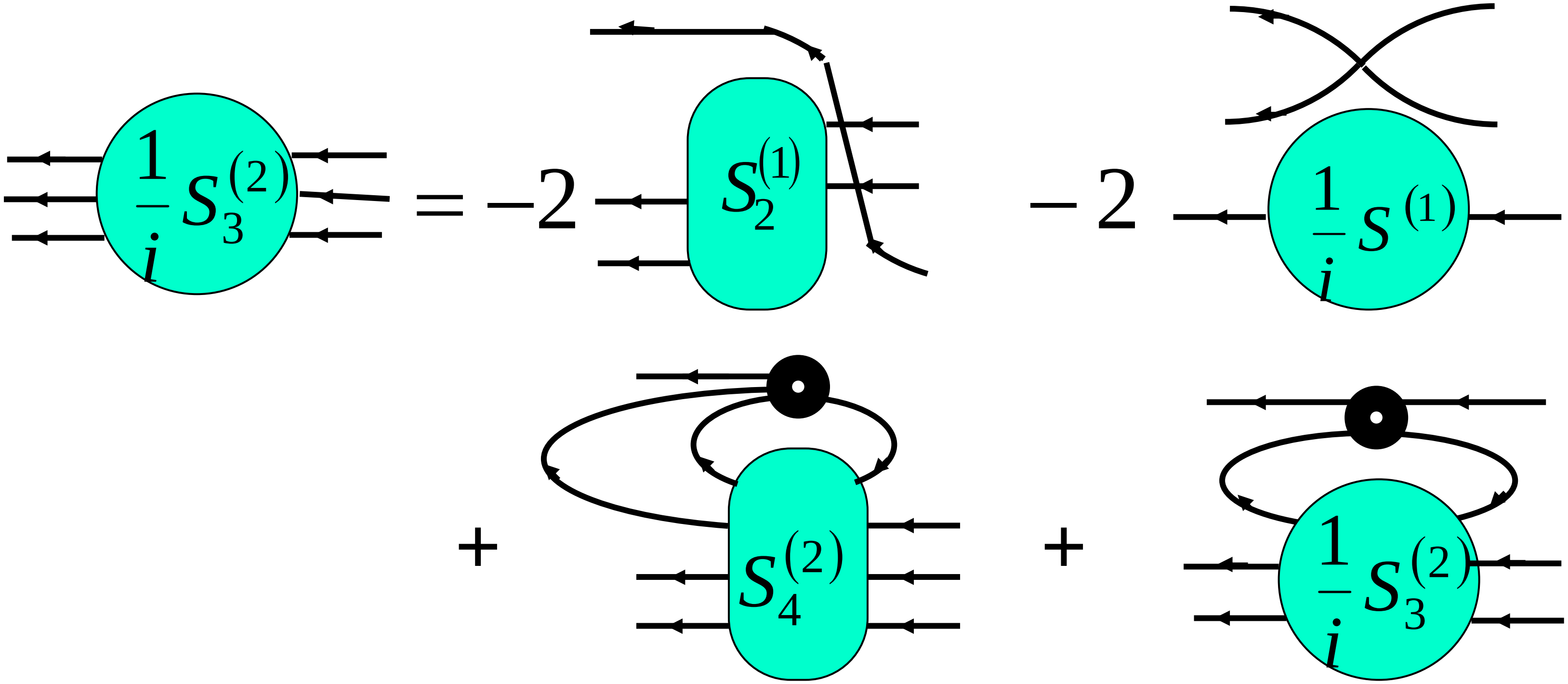
Second-order generating functional is

$$G^{(2)} = \left\{ \frac{1}{4!} \mathrm{Tr}\left(S_4^{(2)} * \eta^4 \right) + \frac{1}{3!} \mathrm{Tr}\left(S_3^{(2)} * \eta^3 \right) + \frac{1}{2} \mathrm{Tr}\left(S_2^{(2)} * \eta^2 \right) + \mathrm{Tr}\left(S^{(2)} * \eta \right) \right\} G^{(0)}.$$

The equations for four-quark and three-quark functions are

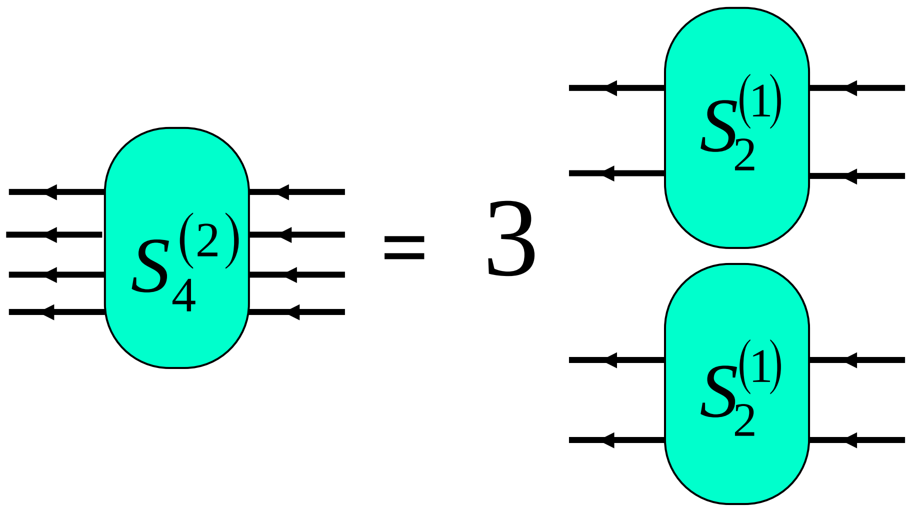


Four particle functions equation



Three particle functions equation

The equation for the four-quark function has a simple exact solution which is the product of first-order two-quark functions:



As it seen from this solution, the $\pi\pi-$scattering in NJL model is suppressed, i.e. in the second order of MFE this scattering is absent, and it can be arise in the third order only.

Vertex $\sigma\pi\pi$

The existence of above exact solution for the four-quark function gives us a possibility to obtain a closed equation for the three-quark function. As a first step in an investigation of this rather complicated equation we shall solve a problem of definition of $\sigma\pi\pi -$ vertex with composite sigma-meson and pions.

Let us introduce a function:

$$W^{ab}_{\sigma\pi\pi}(xx'\,x'') \equiv \mathrm{tr}\left[S_3^{(2)}\begin{pmatrix} x & x \\ x' & x' \\ x'' & x'' \end{pmatrix} \gamma_5 \frac{\tau^a}{2} \gamma_5 \frac{\tau^b}{2} \right] \sim \left\langle \bar{\psi}\psi(x)\bar{\psi}\gamma_5 \frac{\tau^a}{2}\psi(x')\bar{\psi}\gamma_5 \frac{\tau^b}{2}\psi(x'') \right\rangle$$

and define:

a) scalar vertex

$$V_\sigma(xx'x'') \equiv \mathrm{tr}\left(S^{(0)}(x-x') S_2^{(1)} \begin{pmatrix} x' & x \\ x'' & x'' \end{pmatrix} \right) = 2in_c \int dx_1 v_S(xx'x_1) \Delta_\sigma(x_1 - x'')$$

(here $v_s(xx'x'') = \mathrm{tr}\big(S_0(x-x') S_0(x'-x'') S_0(x''-x)\big)$, is the triangle diagram).

b) pseudoscalar vertex

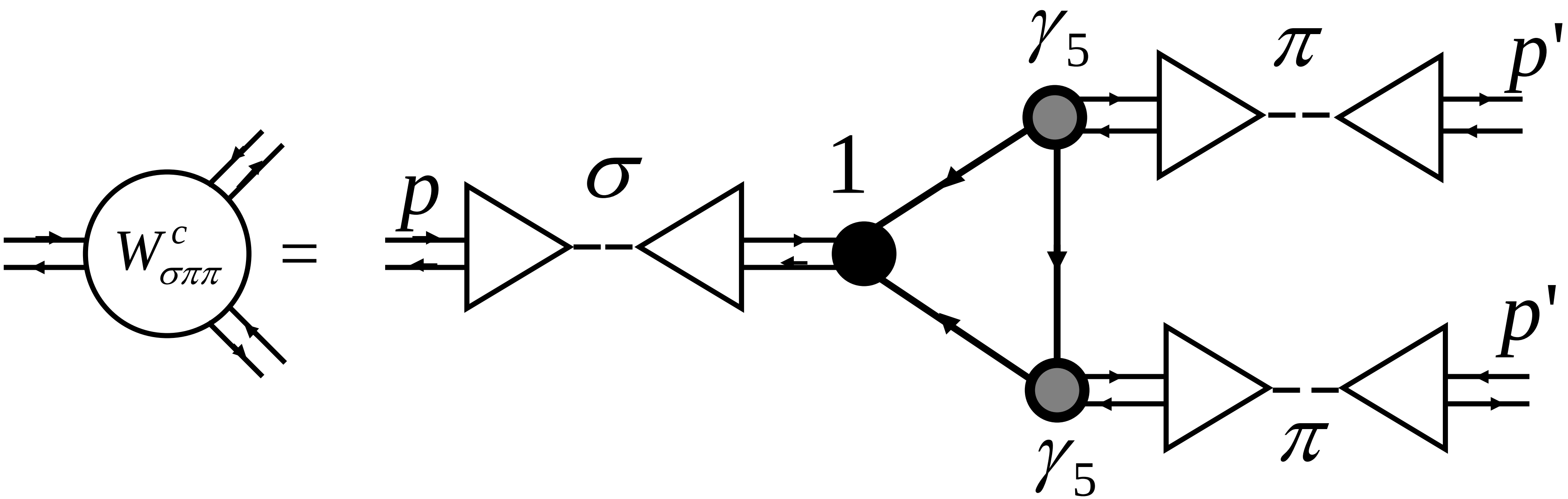
$$V_\pi^{ab}(xx'x'') \equiv \mathrm{tr}\left(S^{(0)}(x-x') \gamma_5 \frac{\tau^a}{2} S_2^{(1)} \begin{pmatrix} x' & x \\ x'' & x'' \end{pmatrix} \gamma_5 \frac{\tau^b}{2} \right) = 2in_c \int dx_1 v_P(xx'x_1) \Delta_\pi^{ab}(x_1 - x'')$$

(here $v_P(xx'x'') = \mathrm{tr}\big(S^{(0)}(x-x') \gamma_5 S^{(0)}(x-x'') \gamma_5 S^{(0)}(x''-x)\big)$).

With these definitions we obtain from the second-order equations for $W^{ab}_{\sigma\pi\pi}$, which can be easy solved in the momentum space. The connected part of $W^{ab}_{\sigma\pi\pi}$ is an amplitude of decay $\sigma \to 2\pi$.
It has a following form:

$$\left[W^{ab}_{\sigma\pi\pi}(pp'\,p'')\right]^{con} = \frac{2n_c}{i}\Delta_\sigma(p)\left[v_P(pp'\,p'') + v_P(pp''\,p')\right]\Delta^{aa_1}_\pi(p')\Delta^{a_1 b}_\pi(p'')$$

or, graphically:



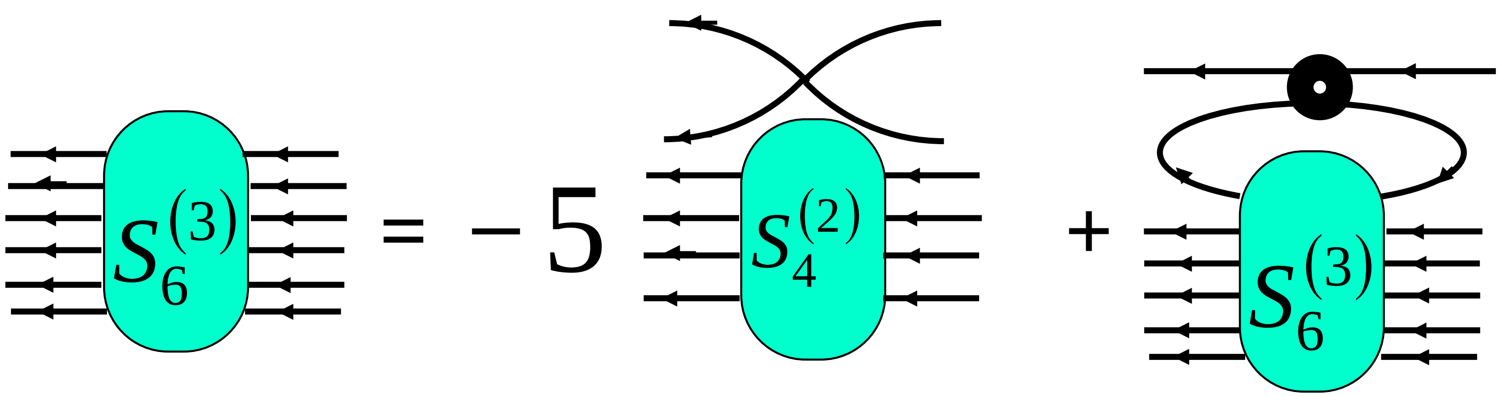
Third order

The third-order generating functional is

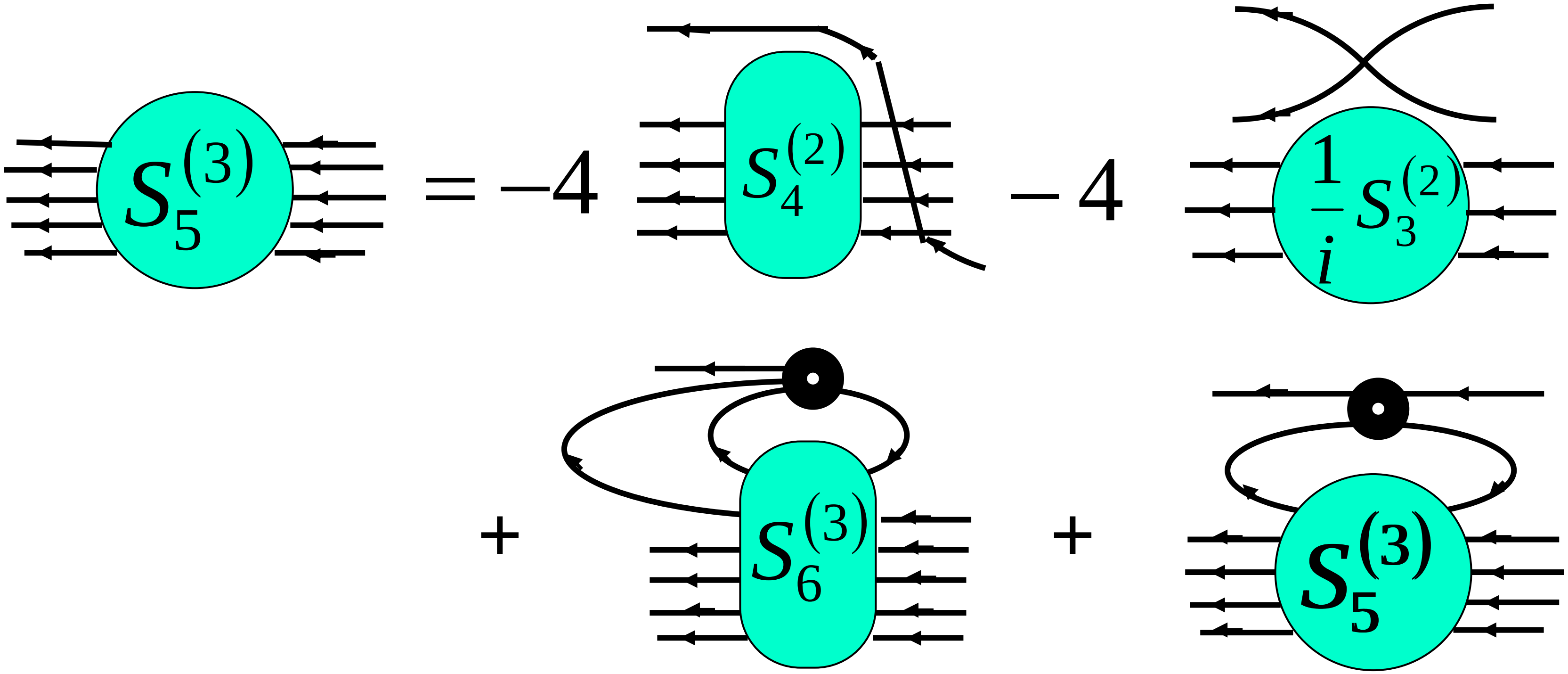
$$G^{(3)}[\eta]=\left\{\frac{1}{6!}\mathrm{Tr}\left(S_6^{(3)}*\eta^6\right)+\frac{1}{5!}\mathrm{Tr}\left(S_5^{(3)}*\eta^5\right)+\frac{1}{4!}\mathrm{Tr}\left(S_4^{(3)}*\eta^4\right)+\right.$$

$$\left.+\frac{1}{3!}\mathrm{Tr}\left(S_3^{(3)}*\eta^3\right)+\frac{1}{2}\mathrm{Tr}\left(S_2^{(3)}*\eta^2\right)+\mathrm{Tr}\left(S^{(3)}*\eta\right)\right\}G^{(0)}$$

The equation for six-quark functions is



And equation for five-quark function is finded as following

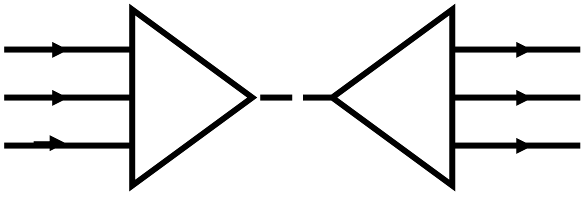


The equations for the six-quark function and for the five-quark function in our iteration scheme are new, and the equations for four-quark function $S_4^{(3)}$, three-quark function $S_3^{(3)}$, two-quark function $S_2^{(3)}$ and quark propagator $S^{(3)}$ have the same form as the second-order equations except of the inhomogeneous term, which contains the six-quark function and the five-quark function.

Conclusion

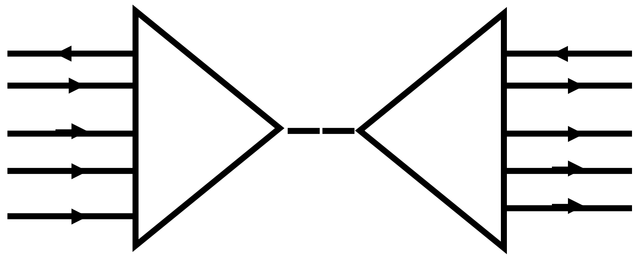
IN conclusion we list some possible physical application of the multi-quark equations.

1. Vector meson decays (in generalized NJL model).
2. Nucleons as the bound states in three-quark function (?)



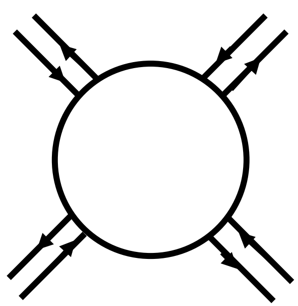
$$S_3^{(2)}$$

3. Pentaquarks as the bound states in five-quark function (??)



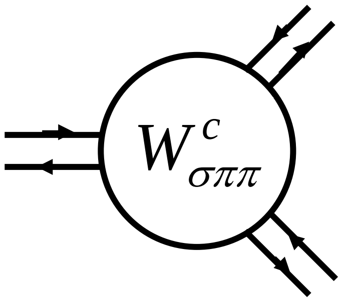
$S_5^{(3)}$

4. $\pi\pi-$ scattering in four-quark function of third order.



$S_4^{(3)}$

Appendix



$$W^{con}_{\sigma\pi\pi}(pp'\,p'') = i\Delta_{\sigma}(p)\left[v_P(pp'\,p'') \underset{p'}{+} v_P(pp''\,p')\right]\Delta_{\pi}(p')\Delta_{\pi}(p'')$$

Let's result of calculation of the triangular diagram.

$$v_P(pp'\,p'') = \int d\tilde{q}\,tr\left[S^{(0)}(p+q)\gamma_5 S^{(0)}(p''+q)\gamma_5 S^{(0)}(q)\right]$$

Here p - 4-momenta of sigma-meson, a p' , p'' - 4-momenta of pions.

$$v_P = 4m\left[I_0(p) + (pp'')I_3(p, p'')\right]$$

here I_0 is two loop integral, which we calculated in different regularizations (in dimensional-analytical regularization and in 4-cutoff regularization), and

$$I_3(p, p'') = \int \frac{d\tilde{q}}{\left(m^2 - (p+q)^2\right)\left(m^2 - q^2\right)\left(m^2 - (p''+q)^2\right)},$$

The converging integral, which on a mass shell

$$p^2 = 4m^2,\ p'^2 = p''^2 = 0,\ (pp') = (p'p'') = 2m^2$$

is

$$I_3^{ms} = \frac{1}{256m^2}.$$

Finally we have the result for decay width

$$\boxed{A_{\sigma\to\pi\pi} = \frac{8g^3}{g_{\sigma qq}\, g_{\pi qq}^2} \cdot \nu_p}$$

$$\Gamma_{\sigma\to\pi\pi} \cong m_\sigma \cong 700 MeV$$