

A teleparallel representation of the Weyl Lagrangian

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The main result of the paper is a new representation of the Weyl Lagrangian (massless Dirac Lagrangian). As the dynamical variable we use the coframe, i.e. an orthonormal tetrad of covector fields. We write down a simple Lagrangian – wedge product of axial torsion with a lightlike element of the coframe – and show that variation of the resulting action with respect to the coframe produces the Weyl equation. The advantage of our approach is that it does not require the use of spinors, Pauli matrices or covariant differentiation. The only geometric concepts we use are those of a metric, differential form, wedge product and exterior derivative. Our result assigns a variational meaning to the tetrad representation of the Weyl equation suggested by J. B. Griffiths and R. A. Newing.

1. Traditional model for the neutrino

Throughout this paper we work on a 4-manifold M equipped with prescribed Lorentzian metric g .

The accepted mathematical model for a neutrino field is the following linear partial differential equation on M known as the *Weyl equation*:

$$i\sigma^\alpha_{ab}\{\nabla\}_\alpha\xi^a = 0. \quad (1)$$

The corresponding Lagrangian is

$$L_{\text{Weyl}}(\xi) := \frac{i}{2}(\bar{\xi}^b\sigma^\alpha_{ab}\{\nabla\}_\alpha\xi^a - \xi^a\sigma^\alpha_{ab}\{\nabla\}_\alpha\bar{\xi}^b) * 1. \quad (2)$$

Here σ^α , $\alpha = 0, 1, 2, 3$, are Pauli matrices, ξ is the unknown spinor field, and $\{\nabla\}$ is the covariant derivative with respect to the Levi-Civita connection: $\{\nabla\}_\alpha\xi^a := \partial_\alpha\xi^a + \frac{1}{4}\sigma_\beta^{a\dot{c}}(\partial_\alpha\sigma^\beta_{b\dot{c}} + \{\Gamma\}^\beta_{\alpha\gamma}\sigma^\gamma_{b\dot{c}})\xi^b$ where $\{\Gamma\}^\beta_{\alpha\gamma}$ are Christoffel symbols uniquely determined by the metric.

2. Teleparallel model for the neutrino

The purpose of our paper is to give an alternative representation of the Weyl equation (1) and the Weyl Lagrangian (2). To this end, we follow¹ in introducing instead of the spinor field a different unknown – the so-called *coframe*. A coframe is a quartet of real covector fields ϑ^j , $j = 0, 1, 2, 3$, satisfying the constraint

$$g = o_{jk}\vartheta^j \otimes \vartheta^k \quad (3)$$

where $o_{jk} = o^{jk} := \text{diag}(1, -1, -1, -1)$. In other words, the coframe is a field of orthonormal bases with orthonormality understood in the Lorentzian sense.

We define an affine connection and corresponding covariant derivative $|\nabla|$ from the conditions

$$|\nabla|\vartheta^j = 0. \quad (4)$$

Let us emphasize that we employ holonomic coordinates, so in explicit form conditions (4) read $\partial_\alpha \vartheta_\beta^j - |\Gamma|^\gamma_{\alpha\beta} \vartheta_\gamma^j = 0$ giving a system of linear algebraic equations for the unknown connection coefficients $|\Gamma|^\gamma_{\alpha\beta}$. The connection defined by the system of equations (4) is called the *teleparallel* or *Weitzenböck* connection.

Let l be a nonvanishing real lightlike teleparallel covector field ($l \cdot l = 0$, $|\nabla|l = 0$). Such a covector field can be written down explicitly as $l = l_j \vartheta^j$ where the l_j are real constants (components of the covector l in the basis ϑ^j), not all zero, satisfying

$$\vartheta^{jk} l_j l_k = 0. \quad (5)$$

We define our Lagrangian as

$$L(\vartheta^j, l_j) = l_i o_{jk} \vartheta^i \wedge \vartheta^j \wedge d\vartheta^k \quad (6)$$

where d stands for the exterior derivative. Note that $\frac{1}{3} o_{jk} \vartheta^j \wedge d\vartheta^k$ is the axial (totally antisymmetric) piece of torsion of the teleparallel connection. Let us emphasize that formula (6) does not explicitly involve connections or covariant derivatives.

The Lagrangian (6) is a rank 4 covariant antisymmetric tensor so it can be viewed as a 4-form and integrated over the manifold M to give the action $S(\vartheta^j, l_j) := \int L(\vartheta^j, l_j)$. Variation with respect to the coframe ϑ^j subject to the constraint (3) produces an Euler–Lagrange equation which we write symbolically as

$$\partial S(\vartheta^j, l_j) / \partial \vartheta^j = 0. \quad (7)$$

The explicit form of the field equation (7) is given in Griffiths' and Newing's paper.¹

3. Equivalence of the two models

Let us define the spinor field ξ as the solution of the system of equations

$$|\nabla|\xi = 0, \quad \sigma_{\alpha ab} \xi^a \bar{\xi}^b = \pm l_\alpha = \pm l_j \vartheta_\alpha^j$$

where $|\nabla|_\alpha \xi^a := \partial_\alpha \xi^a + \frac{1}{4} \sigma_\beta^{ac} (\partial_\alpha \sigma^\beta_{bc} + |\Gamma|^\beta_{\alpha\gamma} \sigma^\gamma_{bc}) \xi^b$. The above system determines the spinor field ξ uniquely up to a complex constant factor of modulus 1. This non-uniqueness is acceptable because we will be substituting ξ into the Weyl equation (1) and Weyl Lagrangian (2) which are both U(1)-invariant. We will call ξ the spinor field *associated* with the coframe ϑ^j .

The main result of our paper is the following

Theorem 3.1. *For any coframe ϑ^j we have $L(\vartheta^j, l_j) = \pm 4L_{\text{Weyl}}(\xi)$ where ξ is the associated spinor field. The coframe satisfies the field equation (7) if and only if the associated spinor field satisfies the Weyl equation (1).*

The proof² of Theorem 3.1 is based on the observation that our Lagrangian (6) is invariant under the action of a certain class of local (i.e. with variable coefficients) transformations of the coframe, the class in question being the subgroup B^2 of the Lorentz group.³ This means that coframes come in equivalence classes (cosets) and the nature of these cosets is such that they can be identified with spinors.

4. Discussion

Our Lagrangian (6) has the unusual feature that it depends on a quartet of real parameters l_j which can be chosen arbitrarily as long as they satisfy the condition (5). This parameter dependence requires an explanation. One possible explanation of the physical nature of the l_j 's is sketched out below.

Consider the Lagrangian

$$L(\vartheta^j) := \|o_{jk} \vartheta^j \wedge d\vartheta^k\|^2 * 1 \quad (8)$$

(“axial torsion squared”). Putting $S(\vartheta^j) := \int L(\vartheta^j)$ and varying with respect to the coframe ϑ^j subject to the constraint (3) we get an Euler–Lagrange equation which we write symbolically as

$$\partial S(\vartheta^j)/\partial \vartheta^j = 0. \quad (9)$$

It turns out that in the case of *Minkowski* metric one can construct an explicit solution of (9) as follows. Let $\tilde{\vartheta}^j$ be a constant reference coframe and let $l \neq 0$ be a constant real lightlike covector; here “constant” means “parallel with respect to the Levi-Civita connection induced by the Minkowski metric”. Of course,

$$l = l_j \tilde{\vartheta}^j \quad (10)$$

for some real constants l_j satisfying the condition (5). Perform a rigid Lorentz transformation $\tilde{\vartheta}^j \mapsto \tilde{\vartheta}^j = \Lambda^j_k \tilde{\vartheta}^k$ so that $l = \omega(\tilde{\vartheta}^0 + \tilde{\vartheta}^3)$ for some $\omega \neq 0$, put

$$\begin{pmatrix} \tilde{\vartheta}^0 \\ \tilde{\vartheta}^1 \\ \tilde{\vartheta}^2 \\ \tilde{\vartheta}^3 \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \omega(x^0 + x^3) & \sin \omega(x^0 + x^3) & 0 \\ 0 & -\sin \omega(x^0 + x^3) & \cos \omega(x^0 + x^3) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\vartheta}^0 \\ \tilde{\vartheta}^1 \\ \tilde{\vartheta}^2 \\ \tilde{\vartheta}^3 \end{pmatrix}$$

where $x^j := \int \tilde{\vartheta}^j \cdot dy$ (here y^α are arbitrary local coordinates on the manifold M and x^j are Minkowskian coordinates on M associated with the constant coframe $\tilde{\vartheta}^j$), and, finally, set $\vartheta^j := (\Lambda^{-1})^j_k \tilde{\vartheta}^k$. Straightforward calculations show that the coframe ϑ^j is indeed a solution of (9). We call this solution *plane wave* with *momentum* l . Note that for a plane wave $o_{jk} \vartheta^j \wedge d\vartheta^k = \pm 2 * l$ and $l = l_j \vartheta^j$ where the l_j are the original constants from (10).

Now consider a perturbation of a plane wave. This perturbation can be the result of either a) us looking for a wider class of solutions or b) the metric ceasing to be Minkowski. Application of a formal perturbation argument to the Lagrangian (8) with $o_{jk} \vartheta^j \wedge d\vartheta^k \mp 2l_j * \vartheta^j$ as small parameter gives the Lagrangian (6).

References

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3. A. L. Besse, *Einstein manifolds* (Springer, 1987).