The Variational Approach to Alternative Theories of Gravity

Gianluca ALLEMANDI[1,2,*] and Mauro FRANCAVIGLIA[1,2,3,†]

[1] *Dipartimento di Matematica, Università di Torino*

Via C. Alberto 10, 10123 TORINO (Italy)

[2] *INFN, Section of Torino, Iniziativa Specifica NA12*

[3] *E.S.G. (Evolutionary Systems Group) of the University of Calabria (Italy)*

(Dated: June 22, 2006)

Abstract

This paper is based on lectures given in Ladek-Zdroj, Poland during the 42nd edition of the Karpac winter school on Theoretical Physics. The aim of these lectures was to give an introduction to a rigorous, mathematically well based, variational approach to alternative theories of Gravity, in view of their very interesting recent applications in the framework of Cosmology. We discuss alternative theories of Gravity both in the metric, purely affine and Palatini formalism, stressing differences and analogies both from a physical and mathematical viewpoint. Moreover we give and introduction to almost all the alternative theories of Gravity, that have been recently studied in view of their physical interest (higher order theories of Gravity, including Lovelock and Chern-Simons Lagrangians). Finally we skip to one of the most striking recent applications of alternative theories of Gravity in the framework of cosmological models: alternative theories of Gravity have been proven to provide a consistent model for accelerating universes.

PACS numbers:

[*] Electronic address: `allemandi@dm.unito.it`

[†] Electronic address: `francaviglia@dm.unito.it`

I. INTRODUCTION

Some alternative variational principles, based on different choices of the gravitational field (metric tensor, affine connection, or both) are known to reproduce more or less directly Einsteins gravitational equations (as it was proven for example in [20] and [35]), and should therefore be regarded as equivalent descriptions of the same physical model; while other variational principles (Scalar-tensor theories and Higher-derivative theories) yield pictures of the gravitational interaction which appear to be, a priori, physically distinct from General Relativity. Moreover this different *scenario*, which in any case is mathematically interesting, assumes a deeper physical meaning when investigating experimental results which cannot be understood in the framework of General Relativity, as much as the accelerating behavior of the universe or the rotational curves of galaxies, the cosmological constant problem or the coincidence problem, and so on ...

We first discuss Higher Order Theories of Gravitation in the higher order metric formalism. We discuss the variational derivation of field equations for general Lagrangians f (R) depending in an arbitrary way from the scalar curvature of a metric, including Hilbert Lagrangian as a particular case. Fourth order field equations that arise are hence discussed in their general structure. We also discuss briefly the divergence problems and the structure of these equations. We explain in which sense the Hilbert Lagrangian is degenerate and why. First order equivalents of the Hilbert Lagrangian are introduced by different techniques as an evidence of the degeneration of the Hilbert Lagrangian itself and in view of their applications to conserved quantities.

We then introduce scalar tensor theories and we discuss the equivalence between these theories and Einstein theory, when a conformal transformation is suitably introduced. We consider again the case of alternative $f(R)$ theories of Gravity and the equivalence of these theories with scalar tensor theories, obtained by means of a suitably chosen conformal transformation. Purely affine theories á la Eddington are moreover shortly introduced to explain the Legendre transformation technique, which is very useful to interpret conformal transformations. These theories are proven to be somehow equivalent to the standard General Relativity.

The Legendre transformation can be applied to Lagrangians f(R) to show that they are equivalent to scalar-tensor theories by a general conformal transformation [21]. The conformal field is interpreted as a curvature effect due to non-linear terms in the Lagrangian. As a matter of facts $f(R)$ theories are known to admit a reformulation (in a different set of variables) which is formally identical to General Relativity (with auxiliary fields having a nonlinear self-interaction). The physical signicance of this change of variables has been questioned by several authors in recent

years and up to now there is no accordance in literature about their interpretation [20].
We then pass to Metric-Affine Theories of Gravitation (in a first order á la Palatini formalism). We discuss the variational derivation of field equations for general Lagrangians depending on both a metric and a torsionless connection. The so-called *Palatini method* is reviewed and discussed. We show equivalence in the case of the Hilbert-Palatini Lagrangian in vacuum [35]. We discuss the geometric bi-metric structure of spacetime in the Palatini formalism and properties of field equations together with their solutions, which are governed by the so called *structural equation*. Lagrangians depending on non-linear curvature invariants are then introduced and we discuss universality of Einstein equations for a large class of Lagrangians depending on the scalar curvature or higher order invariants. Both cases with or without matter are considered. Bimetric structures equivalent to metric-affine structures are considered for some specific classes of Lagrangians, including f(R) and f(Ricci squared).

In this paper we try moreover to better understand and to analyze possible cosmological applications of alternative theories of Gravity in relation with their capability to explain the cosmological acceleration of the Universe, both in early times (inflation) and in present time universes. Nevertheless we will focus our attention on the possible theoretical explanations of the present cosmological acceleration. Recent astronomical observations have shown in fact that the universe is accelerating at present time (see [1] and [2] for supernova observation results; see [3] for the observations about the anisotropy spectrum of the cosmic microwave background (CMBR); see [4] for the results about the power spectrum of large-scale structure). Physicists have thus to face the evidence of the acceleration of the Universe and should give a coherent theoretical explanation to those experimental results. The first attempts to explain accelerating models of the Universe where made in the context of *dark energy theories*. The real nature of dark energy, which is required by General Relativity in this cosmological context, is unknown but it is fairly well accepted that dark energy should behave like a fluid with a large negative pressure. The dark energy models with effective equation of state w_{eff} (which determines the relation between pressure p and density of matter ρ) smaller than $w_{eff} < -1$ are currently preferrable, owing to the experimental results of [3].
Another possibility to explain the physical evidence is to assume that we do not yet understand Gravity at large scales, which suggests to us that General Relativity should be modified or replaced by alternative gravitational theories of Gravity when the curvature of spacetime is small (see for example [8], [9], [10] and references therein), thus providing modified Friedmann

equations. Hints in this direction are suggested moreover from the quantization on curved spacetimes, when interactions among the quantum fields and the background geometry or the self interaction of the gravitational field are considered. It follows that the standard Hilbert-Einstein Lagrangian has to be suitably modified by means of corrective terms, which are essential in order to remove divergences [8]. These corrective terms turn out to be higher-order terms in the curvature invariants such as R, $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$, $R\Box^l R$, or non-minimally coupled terms between scalar fields and the gravitational field. It is moreover interesting that such corrective terms to the standard Hilbert-Einstein Lagrangian can be predicted in higher dimensions by some time-dependent compactification in string/M-theory (see [9]) and corrective terms of this type arise surely in brane-world models with large spatial extra dimensions [10]. As a matter of facts, if these brane models are the low energy limit of string theory, it is likely that the field equations should include in particular the Gauss-Bonnet term, which in five dimensions is the only non-linear term in the curvature which yields second order field equations. In this framework Gauss-Bonnet corrections should be seriously taken into account and cosmological models deriving from the Gauss-Bonnet term have been recently studied; see [12] and references therein.

As an alternative to extra dimensions it is also possible to explain the modification to Friedmann equations (which could provide a theoretical explanation for the acceleration of the Universe) by means of a modified theory of four-dimensional gravity. The first attempts in this direction were performed by adding to the standard Hilbert-Einstein Lagrangian analytical terms in the Ricci scalar curvature invariant [11]. A simple task to modify General Relativity, when the curvature is very small, is hence to add to the Lagrangian of the theory a piece which is proportional to the inverse of the scalar curvature $\frac{1}{R}$ or to replace the standard Hilbert-Einstein action by means of polynomial-like Lagrangians, containing both positive and negative powers of the Ricci scalar R and logarithmic-like terms. Such theories have been analyzed and studied both in the metric [13] and the Palatini formalisms [14], [16]. It turns out hat both in the metric and the Palatini formalism they can provide a possible theoretical explanation to the present time acceleration of the universe. In these paper we also discuss briefly cosmological applications of $f(R)$ and $f(S)$ theories (where S is the Ricci squared curvature invariant) referring to [14] and [15] for further references and discussions.

Moreover we treat some cases of very interesting theories that are from a physical viewpoint: Lovelock and Chern-Simons Lagrangians in Gravitation. These theories, as already explained before, can be considered as low-energy limits of more fundamental theories. We discuss in larger detail both Gauss-Bonnet Lagrangians and Lovelock Lagrangians; these are in turn related with

so-called Chern-Simons Lagrangians. We shortly discuss the general framework for Chern-Simons sheafs of (local) Lagrangians and a way of globalizing them. We introduce and discuss field equations for all of the above theories and we discuss the Witten transformation (decomposition) which allows to interpret a Chern-Simons theory as a gravitational theory, mostly equivalent to General Relativity or generalizing it in a suitable way.

II. ALTERNATIVE THEORIES OF GRAVITY IN THE METRIC FORMALISM

A. Higher-order gravity: the $f(R)$ case

We start considering the case of $f(R)$ gravitational theories in the metric formalism. In general, fourth-order theories of gravity are given by the action

$$\mathcal{A} = \int \sqrt{g} f(R), \tag{1}$$

where $f(R)$ is an arbitrary (analytic) function of the Ricci curvature scalar R and $\sqrt{g}$ denotes $|det||g_{\mu\nu}||^{\frac{1}{2}}$. We are here considering the simplest case of fourth-order gravity but we could construct such kind of theories also using other invariants in $R_{\mu\nu}$ or $R^{\alpha}_{\gamma\mu\nu}$. The standard Hilbert–Einstein action is of course recovered for $f(R) = R$. Varying with respect to $g_{\alpha\beta}$, we get the field equations

$$f'(R)R_{\alpha\beta} - \frac{1}{2}f(R)g_{\alpha\beta} = f'(R)^{;\mu\nu}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}\right) , \tag{2}$$

which are fourth-order equations due to the term $f'(R)^{;\mu\nu}$; the prime indicates the derivative with respect to R. By a suitable manipulation the above equation can be rewritten under the form:

$$G_{\alpha\beta} = \frac{1}{f'(R)}\left\{\frac{1}{2}g_{\alpha\beta}\left[f(R) - Rf'(R)\right] + f'(R)_{;\alpha\beta} - g_{\alpha\beta}\Box f'(R)\right\} , \tag{3}$$

where the gravitational contribution due to higher-order terms can be simply reinterpreted as a stress-energy tensor contribution. This means that additional and higher order terms in the gravitational action produce in the theory the same effects of a stress-energy tensor of matter, related to the chosen form of $f(R)$. In the more general case of matter theories, where a minimal interaction between the gravitational field and matter fields is present, i.e.

$$\mathcal{A} = \int \sqrt{g} f(R) + L_{\rm mat}(\Psi) , \tag{4}$$

than the stress-energy tensor due to higher order gravitational terms in the Lagrangian adds to the true stress-energy tensor $T_{\alpha\beta}$ of matter, giving:

$$G_{\alpha\beta} = \frac{1}{f'(R)} \left\{ \frac{1}{2} g_{\alpha\beta} \left[f(R) - R f'(R) \right] + f'(R)_{;\alpha\beta} - g_{\alpha\beta} \Box f'(R) \right\} + T_{\alpha\beta} \,, \tag{5}$$

where $T_{\alpha\beta} = -\frac{2}{\sqrt{g}} \frac{\delta L_{\rm mat}}{\delta g_{\alpha\beta}}$. The case of standard General Relativity is of course reproduced for $f(R) = R$. In this particular case, in fact, the higher order terms in field equations vanish identically, due to the fact that the derivatives of $f(R)$ vanish. This is related with the degeneration of the Hilbert-Einstein Lagrangian. The degeneration of the Hilbert-Einstein Lagrangian is well known in literature and it is related to the very particular form of the Lagrangian itself which, even thoug it is a second order Lagrangian, can be non-covariantly rewritten as the sum of a first order Lagrangian plus a pure divergence term. The Hilbert's Lagrangian can be in fact recasted as follows:

$$L = \mathcal{L}\sqrt{g} = \Big[p^{\alpha\beta} (\Gamma^{\rho}_{\alpha\sigma} \Gamma^{\sigma}_{\rho\beta} - \Gamma^{\rho}_{\rho\sigma} \Gamma^{\sigma}_{\alpha\beta}) + d_{\sigma} (p^{\alpha\beta} u^{\sigma}{}_{\alpha\beta}) \Big] ds \tag{6}$$

where:

$$p^{\alpha\beta} = \sqrt{g} g^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial R_{\alpha\beta}} \tag{7}$$

and Γ is the Levi-Civita connection of g. Since $u^{\sigma}_{\alpha\beta}$ is not a tensor the above expression is not covariant; however a standard procedure have been studied to recast covariance in first order theories [22]. This clearly shows that field equations should consequently be second order field equations and the Hilbert-Einstein Lagrangian is thus degenerate.

B. The Scalar-Tensor case

In four dimensions, a general non-minimally coupled scalar-tensor theory of gravity is given by the effective (purely metric) action

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[F(\phi) R + \frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} - V(\phi) \right] \tag{8}$$

where R is the Ricci scalar, $V(\phi)$ and $F(\phi)$ are generic functions describing respectively the potential and the coupling of a scalar field ϕ. We shall adopt Planck units. The Brans-Dicke theory of gravity is a particular case of the action (8) for $V(\phi)$=0 [29]. The variation with respect to $g_{\mu\nu}$ gives the field equations

$$F(\phi) G_{\mu\nu} = F(\phi) \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right] = -\frac{1}{2} T_{\mu\nu} - g_{\mu\nu} \Box_g F(\phi) + F(\phi)_{;\mu\nu} \tag{9}$$

that are generalized Einstein equations; here $\Box_\Gamma$ is the d'Alembert operator with respect to the metric g, and $G_{\mu\nu}$ is the Einstein tensor. Here and below, semicolon denotes metric covariant derivative with respect to g. The energy-momentum tensor relative to the scalar field is

$$T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{\alpha}_{;} + g_{\mu\nu}V(\phi) \tag{10}$$

The variation with respect to ϕ provides the Klein–Gordon equation, i.e. field equations for the scalar field:

$$\Box_g\phi - RF_\phi(\phi) + V_\phi(\phi) = 0 \tag{11}$$

where $F_\phi = dF(\phi)/d\phi$, $V_\phi = dV(\phi)/d\phi$. This last equation is equivalent to the Bianchi contracted identity [30].

We will introduce now a conformal transformation to show that every non-minimally coupled scalar-tensor theory, in absence of ordinary matter, e.g. a perfect fluid, is conformally equivalent to an Einstein theory. The conformal transformation on the metric $g_{\mu\nu}$ is

$$\bar{g}_{\mu\nu} = e^{2\omega}g_{\mu\nu} \tag{12}$$

in which $e^{2\omega}$ is the conformal factor. Under this transformation the Lagrangian density in (8) becomes

$$\begin{aligned}\sqrt{-g}\left(FR + \frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V\right) &= \sqrt{-\bar{g}}e^{-2\omega}\left(F\bar{R} - 6F\Box_{\bar{g}}\omega + \right.\\ &\left. -6F\omega_{;\alpha}\omega^{\alpha}_{;} + \frac{1}{2}\bar{g}^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - e^{-2\omega}V\right)\end{aligned} \tag{13}$$

in which $\bar{R}$ and $\Box_{\bar{g}}$ are the Ricci scalar and the d'Alembert operator respectively relative to the metric $\bar{g}$. Requiring the theory in the metric $\bar{g}_{\mu\nu}$ to appear as a standard Einstein theory [27], the conformal factor has to be related to F, that is

$$e^{2\omega} = -2F. \tag{14}$$

where F must be negative in order to restore physical coupling. Using this relation and introducing a new scalar field $\bar{\phi}$ and a new potential $\bar{V}$, defined respectively by

$$\bar{\phi}_{;\alpha} = \sqrt{\frac{3{F_\phi}^2 - F}{2F^2}}\,\phi_{;\alpha}, \quad \bar{V}(\bar{\phi}(\phi)) = \frac{V(\phi)}{4F^2(\phi)}, \tag{15}$$

we see that the Lagrangian density (13) becomes

$$\sqrt{-g}\left(FR + \frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V\right) = \sqrt{-\bar{g}}\left(-\frac{1}{2}\bar{R} + \frac{1}{2}\bar{\phi}_{;\alpha}\bar{\phi}^{\alpha}_{;} - \bar{V}\right) \tag{16}$$

which is the usual Hilbert-Einstein Lagrangian density plus the standard Lagrangian density relative to the scalar field $\bar{\phi}$. Therefore, every non-minimally coupled scalar-tensor theory, in absence of ordinary matter, e.g. perfect fluid, is conformally equivalent to an Einstein theory, being the conformal transformation and the potential suitably defined by (14) and (15). The converse is also true: for a given $F(\phi)$, such that $3{F_\phi}^2 - F > 0$, we can transform a standard Einstein theory into a non-minimally coupled scalar-tensor theory. This means that, in principle, if we are able to solve the field equations in the framework of the Einstein theory in presence of a scalar field with a given potential, we should be able to get the solutions for the scalar-tensor theories, assigned by the coupling $F(\phi)$, via the conformal transformation (14) with the constraints given by (15). Following the standard terminology, the "Einstein frame" is the framework of the Einstein theory with the minimal coupling and the "Jordan frame" is the framework of the non-minimally coupled theory [20].

In the framework of alternative theories of Gravity, as previously discussed, the gravitational contribution to the stress-energy tensor of the theory can be reinterpreted by means of a conformal transformation as the stress-energy tensor of a suitable scalar field and then as "matter" like terms. Performing the conformal transformation (12) in the field equations (3) we get (see [20] for detailed calculations):

$$\bar{G}_{\alpha\beta} = \frac{1}{f'(R)}\left\{\frac{1}{2}g_{\alpha\beta}\left[f(R) - Rf'(R)\right] + f'(R)_{;\alpha\beta} - g_{\alpha\beta}\Box f'(R)\right\} + \tag{17}$$

$$+2\left(\omega_{;\alpha;\beta} + g_{\alpha\beta}\Box\omega - \omega_{;\alpha\beta} + \frac{1}{2}g_{\alpha\beta}\omega_{;\gamma}\omega^{;\gamma}\right) .$$

We can then choose the conformal factor to be

$$\omega = \frac{1}{2}\ln|f'(R)| , \tag{18}$$

which has now to be substituted into (5). Rescaling ω in such a way that

$$k\phi = \omega , \tag{19}$$

and $k = \sqrt{1/6}$, we obtain the Lagrangian equivalence

$$\sqrt{-g}f(R) = \sqrt{-\bar{g}}\left(-\frac{1}{2}\bar{R} + \frac{1}{2}\bar{\phi}_{;\alpha}\bar{\phi}^{\alpha}_{;} - \bar{V}\right) \tag{20}$$

and the Einstein equations in standard form

$$\bar{G}_{\alpha\beta} = \phi_{;\alpha}\phi_{;\beta} - \frac{1}{2}\bar{g}_{\alpha\beta}\phi_{;\gamma}\phi^{;\gamma} + \bar{g}_{\alpha\beta}V(\phi) , \tag{21}$$

with the potential

$$V(\phi) = \frac{e^{-4k\phi}}{2}\left[f(\phi) - \mathcal{F}\left(e^{2k\phi}\right)e^{2k\phi}\right] = \frac{1}{2}\frac{f(R) - Rf'(R)}{f'(R)^2}\,. \tag{22}$$

There $\mathcal{F}$ is the inverse function of $f'(\phi)$ and $f(\phi) = \int \exp(2k\phi)d\mathcal{F}$. However, the problem is completely solved if $f'(\phi)$ can be analytically inverted. In summary, a fourth-order theory is conformally equivalent to the standard second-order Einstein theory plus a scalar field (see also [24, 26]).

This procedure can be extended to more general theories. If the theory is assumed to be higher than fourth order, we may have Lagrangian densities of the form [25, 28],

$$\mathcal{L} = \mathcal{L}(R, \Box R, ...\Box^k R)\,. \tag{23}$$

Every $\Box$ operator introduces two further terms of derivation into the field equations. For example a theory like

$$\mathcal{L} = R\Box R\,, \tag{24}$$

is a sixth-order theory and the above approach can be pursued by considering a conformal factor of the form

$$\omega = \frac{1}{2}\ln\left|\frac{\partial\mathcal{L}}{\partial R} + \Box\frac{\partial\mathcal{L}}{\partial\Box R}\right|\,. \tag{25}$$

In general, increasing two orders of derivation in the field equations (*i.e.* for every term $\Box R$), corresponds to adding a scalar field in the conformally transformed frame [25]. A sixth-order theory can be reduced to an Einstein theory with two minimally coupled scalar fields; a $2n$-order theory can be, in principle, reduced to an Einstein theory $(n-1)$-scalar fields. On the other hand, these considerations can be directly generalized to higher-order-scalar-tensor theories in any number of dimensions as shown in [23].

As concluding remarks, we can say that conformal transformations work at three levels: *i*) on the Lagrangian of the given theory; *ii*) on the field equations; *iii*) on the solutions. The table below summarizes the situation for fourth-order gravity (FOG), non-minimally coupled scalar-tensor theories (NMC) and standard Hilbert-Einstein (HE) theory. Clearly, direct and inverse transformations correlate all the steps of the table but no absolute criterion, at this point of the discussion, is able to select which is the "physical" framework since, at least from a mathematical point of view, all the frames are equivalent (see also [20] for a detailed discussion). This point is up to now unsolved even if a deep discussion is present in literature.

$\mathcal{L}_{FOG}$	$\longleftrightarrow$	$\mathcal{L}_{NMC}$	$\longleftrightarrow$	$\mathcal{L}_{HE}$
$\updownarrow$		$\updownarrow$		$\updownarrow$
FOG Eqs.	$\longleftrightarrow$	NMC Eqs.	$\longleftrightarrow$	Einstein Eqs.
$\updownarrow$		$\updownarrow$		$\updownarrow$
FOG Solutions	$\longleftrightarrow$	NMC Solutions	$\longleftrightarrow$	Einstein Solutions

III. PURELY AFFINE GRAVITATIONAL THEORIES

We want hereafter to briefly discuss purely affine theories of gravity, first introduced by Einstein and Eddington. The starting example is in fact the Einstein-Eddington Lagrangian is:

$$L_{EE}(\Gamma, \partial\Gamma) = \sqrt{det(R_{\mu\nu})} \tag{26}$$

which is a first order Lagrangian in the connection Γ. The metric structure associated to each solution is obtained by the prescription:

$$\pi_{\mu\nu} = g_{\mu\nu}\sqrt{det(g)} = \frac{\partial L_{EE}}{\partial R_{(\mu\nu)}} \tag{27}$$

that amounts to consider the metric as a momentum canonically conjugated to the affine structure itself. The second order Euler-Lagrange equations ensuing from the above Lagrangian are consequently:

$$\nabla_\alpha(g_{\mu\nu}\sqrt{det(g)}) = 0 \tag{28}$$

so that the connection Γ is forced to be the Levi-Civita connection of the metric g, if it is also assumed to be torsionless. The vacuum gravitational equations generated by the purely affine Lagrangian are consequently equivalent to the standard Einstein equations with a cosmological constant so that the theory is in fact fully equivalent to the Einstein theory. Purely affine theories are much more complicated for more general Lagrangians and/or in the case of interaction with matter, see [21] for details, however calculations to obtain field equations can be performed in the same way.

The technique used above, to obtain field equations, is quite general and it is usually called a Legendre transformation: in the case treated here it allows to transform a purely affine Lagrangian into a dynamically equivalent one where the fields (π, Γ) are regarded as independent variables. This new Lagrangian is called the Helmholtz Lagrangian. The same technique can be used to define conformally related theories in the purely metric formalism. Equivalences and transformations

between the Einstein and the Jordan frame can be discussed as *Legendre transformed* theories, see [20, 21] for details.

IV. FIRST ORDER $f(R)$ AND RICCI SQUARED GRAVITY

We study hereafter the so-called Palatini formalism for alternative theories of Gravity, that we deeply investigated in view of cosmological applications [14, 15]. The action for $f(R)$ Gravity is introduced to be:

$$A = A_{\text{grav}} + A_{\text{mat}} = \int (\sqrt{\det g} f(R) + 2\kappa L_{\text{mat}}(\Psi)) d^4x \tag{29}$$

where $R \equiv R(g, \Gamma) = g^{\alpha\beta} R_{\alpha\beta}(\Gamma)$ is the *generalized Ricci scalar* and $R_{\mu\nu}(\Gamma)$ is the Ricci tensor of a torsionless connection Γ. The gravitational part of the Lagrangian is controlled by a given real analytic function $f(R)$ of one real variable, while $\sqrt{g}$ denotes the scalar density $\mid \det \parallel g_{\mu\nu} \parallel \mid^{\frac{1}{2}}$ of weight 1. The total Lagrangian contains also a matter part L_{mat} in minimal interaction with the gravitational field, depending on matter fields Ψ together with their first derivatives and equipped with a gravitational coupling constant $\kappa = 8\pi G$.

Equations of motion, ensuing from the first order á la Palatini formalism are (we assume the spacetime manifold to be a Lorentzian manifold M with $\dim M = 4$; see [39]):

$$f'(R) R_{(\mu\nu)}(\Gamma) - \frac{1}{2} f(R) g_{\mu\nu} = \kappa T_{\mu\nu} \tag{30}$$

$$\nabla^{\Gamma}_{\alpha}(\sqrt{\det g} f'(R) g^{\mu\nu}) = 0 \tag{31}$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta L_{\text{mat}}}{\delta g_{\mu\nu}}$ denotes again the matter source stress-energy tensor and ∇^{Γ} means now covariant derivative with respect to Γ.

We shall use the standard notation denoting by $R_{(\mu\nu)}$ the symmetric part of $R_{\mu\nu}$, i.e. $R_{(\mu\nu)} \equiv \frac{1}{2}(R_{\mu\nu} + R_{\nu\mu})$; notice that for an arbitrary torsionless Γ the Ricci tensor is not a priori symmetric. In order to get (31) one has to additionally assume that L_{mat} is functionally independent of Γ; however it may contain metric covariant derivatives $\nabla^g \Psi$ of fields. This means that the matter stress-energy tensor $T_{\mu\nu} = T_{\mu\nu}(g, \Psi)$ depends on the metric g and some matter fields denoted here by Ψ, together with their derivatives. From (31) one sees that $\sqrt{\det g} f'(R) g^{\mu\nu}$ is a symmetric twice contravariant tensor density of weight 1, so that if not degenerate one can use it to define a metric $h_{\mu\nu}$ such that the following holds true

$$\sqrt{\det g} f'(R) g^{\mu\nu} = \sqrt{\det h} h^{\mu\nu} \tag{32}$$

This means that both metrics h and g are conformally equivalent. The corresponding conformal factor can be easily found to be $f'(R)$ (in $\dim M = 4$) and the conformal transformation results to be:

$$h_{\mu\nu} = f'(R) g_{\mu\nu} \tag{33}$$

Therefore, as it is well known, equation (31) implies that $\Gamma = \Gamma_{LC}(h)$, i.e. coincides with the Levi-Civita connection of the metric h defined by (32) and $R_{(\mu\nu)}(\Gamma) = R_{\mu\nu}(h) \equiv R_{\mu\nu}$. Let us now introduce a (1,1)-tensorfield P by

$$P^{\mu}_{\nu} = g^{\mu\alpha} R_{\alpha\nu}(h) \tag{34}$$

so that (30) re-writes as

$$f'(R) P^{\nu}_{\mu} - \frac{1}{2} f(R) \delta^{\nu}_{\mu} = \kappa \widehat{T}^{\nu}_{\mu} \tag{35}$$

where, with an abuse of notation. We set $\hat{T} = \hat{T}^{\nu}_{\mu} = g^{\mu\alpha} T_{\alpha\nu}$ and from (34) we obtain that $R = \mathrm{tr} P$. Equation (35) can be supplemented by the scalar-valued equation obtained by taking the trace of (35)

$$f'(R) R - 2 f(R) = \kappa g^{\alpha\beta} T_{\alpha\beta} \equiv \kappa \tau \tag{36}$$

which controls solutions of (35); (we define $\tau = \mathrm{tr} \hat{T}$). We shall refer to this scalar-valued equation as the *structural equation* of spacetime. The structural equation (33), if explicitly solvable, provides an expression of $R = F(\tau)$ and consequently both $f(R)$ and $f'(R)$ can be expressed in terms of τ. More precisely, for any real solution $R = F(\tau)$ of (36) one has that the operator P can obtained from the matrix equation (35):

$$P = \frac{f(F(\tau))}{2 f'(F(\tau))} I + \frac{\kappa}{f'(F(\tau))} \hat{T} \tag{37}$$

Now we are in position to introduce generalized Einstein equations under the form

$$g_{\mu\alpha} P^{\alpha}_{\nu} = R_{\mu\nu}(h) \tag{38}$$

where $h_{\mu\nu}$ is given by (33) and P^{μ}_{ν} is obtained from the algebraic equations (36) and (37) (for a given $g_{\mu\nu}$ and $T_{\mu\nu}$); see also [14] and [39]. For the matter-free case we find that $R = F(0)$ becomes a constant, thus implying that the two metrics are proportional and the operator P is proportional to the identity (i.e. to Kronecker delta). Equation (38) is hence nothing but Einstein equation for the metric g, almost independently on the choice of the function $f(R)$, as already obtained in [39].

Also the standard Einstein equation with a cosmological constant Λ can be recasted into the form (38). It corresponds to the choice $f(R) = R - \Lambda$. These properties justify the name of generalized Einstein equation given to (38). In the presence of matter, equation (38) expresses a deviation for the metric g to be an Einstein metric as it was discussed in [14]. It can be otherwise interpreted as an Einstein equation with additional stress-energy contributions deriving from the modified gravitational Lagrangian [16], or possibly as a modified theory of gravity with a time dependent cosmological constant.

We move now our attention to consider, as a more general case, the *Ricci squared theories of Gravity*, defined by means of the action functional:

$$A = A_{\text{grav}} + A_{\text{mat}} = \int (\sqrt{\det g} f(S) + 2\kappa L_{\text{mat}}(\Psi)) d^4x \tag{39}$$

where $S \equiv S(g, \Gamma) = g^{\mu\alpha} R_{(\alpha\nu)}(\Gamma) g^{\nu\beta} R_{(\beta\mu)}(\Gamma)$ and $R_{\mu\nu}(\Gamma)$ is, as above, the Ricci tensor of a torsionless connection Γ, see discussion after formula (29). The gravitational part of the Lagrangian is controlled by a given real function of one real variable $f(S)$; see [35]. Under the same assumptions of [35] and in 4-dimensional spacetimes M ($\dim M = 4$) equations of motion ensuing from the variational principle in the Palatini formalism are [35]:

$$2f'(S) g^{\alpha\beta} R_{(\mu\alpha)}(\Gamma) R_{(\beta\nu)}(\Gamma) - \frac{1}{2} f(S) g_{\mu\nu} = \kappa T_{\mu\nu} \tag{40}$$

$$\nabla^{\Gamma}_{\sigma}(\sqrt{\det g} f'(S) g^{\mu\alpha} R_{(\alpha\beta)}(\Gamma) g^{\beta\nu}) = 0 \tag{41}$$

The above system of equations splits, as before, into an algebraic part (40) and a differential one (41) for the unknown variables g (the metric) and Γ (the connection).
Following the general strategy elaborated for the matter-free case [35] and [39] (see also [14]) let us notice that $\sqrt{\det g}\, f'(S) g^{\mu\alpha} R_{(\alpha\beta)} g^{\beta\nu}$ is a symmetric $(2,0)$-rank tensor density of weight 1 which we additionally assume to be nondegenerate. This assumption entitles us to introduce a new metric $h_{\mu\nu}$ by the following definition

$$\sqrt{\det h}\, h^{\mu\nu} = \sqrt{\det g}\, f'(S) g^{\mu\alpha} R_{(\alpha\beta)}(\Gamma) g^{\beta\nu} \tag{42}$$

The metric h is hence called a Levi-Civita metric since the field equations (41) and imply that Γ is the Levi-Civita connection of h: $\Gamma = \Gamma_{LC}(h)$. The Ricci tensor of h can be simply defined as $R_{(\mu\nu)}(\Gamma) = R_{\mu\nu}(h) \equiv R_{\mu\nu}$. It should be easily recognized that eq. (41) defines $h_{\mu\nu}$ only up to a multiplicative constant. Therefore the metric h is not a good candidate for a physically meaningful

object.

The algebraic equation (40) can be easily converted into the matrix form

$$P^2 = \frac{1}{4f'(S)} f(S) I + \frac{\kappa}{2f'(S)} \hat{T} \tag{43}$$

by using the endomorphisms (i.e. $(1,1)$-tensorfields) P and $\hat{T}$ as defined before. Equation (43) can be supplemented by the scalar-valued equation obtained by taking the trace of (40) or of (43)[1]

$$f'(S)S - f(S) = \frac{\kappa}{2} g^{\alpha\beta} T_{\alpha\beta} \equiv \frac{\kappa}{2}\tau \tag{44}$$

which governs solutions of the matrix equations (40) and (43) will again define it to be the *structural equation* of the spacetime under analysis.

The structural equation (44) can be formally (and hopefully explicitly) solved expressing $S = F(\tau)$. This allows to reinterpret both $f(S)$ and $f'(S)$ as functions of τ. The tensorfield P turns out consequently to be a function of $P = P(\tau)$, due to (43) and consequently the operator P will be simply functions of τ.

On the other hand equation (42) tells us the the metric h is conformal to a symmetric bilinear form; i.e., in matrix notation:

$$h \simeq (g^{-1} R g^{-1})^{-1} = P^{-1} g \tag{45}$$

Once again to obtain explicitly this expression for $h_{\mu\nu}$ we should require that (43) can be solved analytically. It is then possible to calculate P^{μ}_{ν} and $\det P$, see [15] for details, so that the generalized Einstein equation ensuing from (40) take the simple form:

$$R_{\mu\nu}(h) = P^{\alpha}_{\nu} g_{\mu\alpha} \tag{46}$$

with $h_{\mu\nu}$ given by (42); now the physical metric g does no longer need to be Einstein. This expression for the generalized Einstein equations is formally the same obtained for non-linear Lagrangians in the generalized Ricci scalar in (38). Differences arise in the definition of the operator P (compare expressions (37) and (43)) and the metric h (compare expressions (33) and (42)), which in this last case turns out, in general, to be no longer conformal to g.

A. Cosmological applications of first-order non-linear gravity

We give here a brief summary of the results obtained in [14] where cosmological applications of $f(R)$ Gravity were deeply discussed. Owing to the cosmological principle we assume g to be a

[1] We remark that in this context $S = \mathrm{tr} P^2$

Friedmann-Robertson-Walker (FRW) metric which (in spherical coordinates) takes the standard form:

$$g = -dt^2 + a^2(t)\Big[\frac{1}{1-Kr^2}dr^2 + r^2\Big(d\theta^2 + \sin^2(\theta)d\varphi^2\Big)\Big] \tag{47}$$

where $a(t)$ is the so-called *scale factor* and K is the space curvature ($K = 0, 1, -1$). We further choose a perfect fluid stress-energy tensor for matter:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \tag{48}$$

where p is the pressure, ρ is the density of matter and u^μ is a co-moving fluid vector, which in a co-moving frame ($u^\mu = (1, 0, 0, 0)$) becomes simply:

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{pa^2(t)}{1-Kr^2} & 0 & 0 \\ 0 & 0 & pa^2(t)r^2 & 0 \\ 0 & 0 & 0 & pa^2(t)r^2\sin^2(\theta) \end{pmatrix} \tag{49}$$

The metric h turns out to be conformal to the FRW metric g by means of the conformal factor $f'(R)$, which can be moreover expressed in terms of τ by means of (36) and finally as a function of time

$$b(t) = f'(R(\tau)) \tag{50}$$

by an abuse of notation. From (38) we can obtain an analogue of the Friedmann equation under the form

$$\hat{H} = \Big(\frac{\dot{a}}{a} + \frac{\dot{b}}{2b}\Big)^2 = \frac{\kappa}{3b}\Big[\rho + \frac{f(\tau) + \kappa\tau}{2\kappa}\Big] - \frac{K}{a^2} \tag{51}$$

which can be seen as a generalized definition of a *modified Hubble constant* $\hat{H} = \Big(\frac{\dot{a}}{a} + \frac{\dot{b}}{2b}\Big)$, taking into account the presence of the conformal factor $b(t)$ which enters into the definition of the conformal metric h (see [14] for details). This equation reproduces, as expected, the standard Einstein equations in the case $f(R) = R$.

Considering the particular example $f(R) = \beta R^n$ of a pure-power Lagrangian we have obtained that the Hubble constant for the metric g can be (locally) calculated to be:

$$H^2 = \varepsilon r(n, w)a^{\frac{-3(w+1)}{n}} - s(n, w)\frac{K}{a^2} \tag{52}$$

where:

$$\begin{cases} r(n, w) = \frac{2n}{3(3w-1)[3w(n-1)+(n-3)]}\left[\frac{-\kappa(3w-1)}{\beta(2-n)}\right]^{\frac{1}{n}} \\ s(n, w) = \left[\frac{2n}{3w(n-1)+(n-3)}\right]^2 \end{cases}$$

are functions of the exponent n and of the equation of state for matter, through w. We remark that the values of can be $\varepsilon = \pm 1$, in particular $\varepsilon = \text{sign}R = 1$ for odd values of n and $\varepsilon = \pm 1$ for even values of n; see [14] for details. The *deceleration parameter* can be obtained from the Hubble constant by means of the following relation:

$$q(t) := -\left(1 + \frac{\dot{H}(t)}{H^2(t)}\right) = -\left(\frac{\ddot{a}(t)}{a(t)H^2(t)}\right) \tag{53}$$

and from (52) it turns out to be formally equal to:

$$q(t,w,n) = -1 + \frac{\frac{3(1+w)}{2n}\varepsilon r(n,w)a^{-\frac{3(1+w)}{n}} - s(n,w)Ka^{-2}}{\varepsilon r(n,w)a^{-\frac{3(1+w)}{n}} - s(n,w)Ka^{-2}} \tag{54}$$

It follows that when the term a^{-2} dominates over $a^{-\frac{3(1+w)}{2n}}$ the deceleration parameter turns out to be positive, i.e. $q(t,w,n) \to 0$. On the contrary, when the term $a^{-\frac{3(1+w)}{n}}$ dominates over a^{-2} (or in the case $K = 0$ corresponding to spatially flat spacetimes) the deceleration parameter turns out to be:

$$q(w,n) = -1 + \frac{3(1+w)}{2n} \tag{55}$$

which is negative for $n < 0$ or $n > \frac{3(1+w)}{2} > 0$ owing to the positivity of $(1+w)$ for standard matter; see [14]. This implies that the accelerated behavior of the Universe is predicted in a suitable limit. In particular it follows that super-acceleration ($q < -1$) can be achieved only for $n < 0$. The effective w_{eff} can be obtained (as in [11]) by means of simple calculations from (52) and (55). It turns out to be, for this theory:

$$w_{eff} = \frac{2}{3}q(n,w) - \frac{1}{3} = -1 + \frac{(w+1)}{n} \tag{56}$$

We remark that the range of $-1.45 < w_{eff} < -0.74$ for dark energy, stated in [3], can be easily recovered in this theory by choosing suitable and admissible values[2] of n. We refer to [14] for physical considerations and for more detailed discussions and examples concerning polynomial-like Lagrangians in the generalized Ricci scalar.

B. Cosmological applications of Ricci squared Gravity

For cosmological applications (as already explained in the previous Section) one has first to choose the physical metric, which is assumed to be g for the moment, to be the Friedmann-Robertson-

[2] As already explained in [14] the parameter n should not be an integer, it can be any real number satisfying some reliability conditions; see [14] for further discussions and details.

Walker (FRW) metric, as well as to choose the perfect fluid stress-energy tensor for matter, introduced in (48) and in a co-moving frame in (49). All diagonal solutions of (43) can be thus calculated, using expressions (47) and (48):

$$P_{\mu}^{\nu} = \frac{1}{2}\sqrt{\frac{f(\tau)+2\kappa p}{f'(\tau)}}\mathrm{Diag}\left(\varepsilon_0\sqrt{\frac{f\left(\tau\right)-2\kappa\rho}{f(\tau)+2\kappa p}}, \varepsilon_1, \varepsilon_2, \varepsilon_3\right) \tag{57}$$

where we have formally expressed $S = F(\tau)$ from (43), with $\tau = 3p - \rho$. We introduce moreover $\varepsilon_\mu = \pm 1; \quad \mu = 0, \ldots, 3$, ensuing from the square root of the operator P^2. This exhibits a phenomenon of signature change in $f(S)$ theories (see below; [15] and [38]). Reality condition forces us to assume that all three terms

$$f'(\tau) \neq 0, \qquad f(\tau) - 2\kappa\rho \neq 0 \quad \text{and} \quad f(\tau) + 2\kappa p \neq 0 \tag{58}$$

have to have at the same time the same (negative or positive) sign. In what follows we denote $\varepsilon = sign\, f'(\tau) = sign\,(f(\tau) - 2\kappa\rho) = sign\,(f(\tau) + 2\kappa p)$.

Neglecting an irrelevent multiplicative constant factor (which can be in general complex or imaginary) the metric h can be suitably written as:

$$h_{\mu\nu} = b(\tau)\mathrm{Diag}\left(-\varepsilon_0 c(\tau), \frac{\varepsilon_1 a^2}{1-Kr^2}, \varepsilon_2 r^2 a^2, \varepsilon_3 r^2 a^2 \sin^2(\theta)\right) \tag{59}$$

where:

$$\begin{cases} b(\tau) = \sqrt{|f'(\tau)|}[(f(\tau)+2\kappa p)(f(\tau)-2\kappa\rho)]^{\frac{1}{4}} \\ c(\tau) = \sqrt{\frac{f(\tau)+2\kappa p}{f(\tau)-2\kappa\rho}} \end{cases} \tag{60}$$

and $b(t)$ turns out to be a *generalized conformal factor*[3] between the two metrics g and h, while $c(t)$ describes a *rescaling factor* for the cosmological time t. We notice that both the generalized conformal factor $b(t)$ and the rescaling factor $c(t)$ are positive definite by definition. The change of signature is related with coefficients $\varepsilon = \pm 1$ and the freedom in their choice produces a multiplying of the Friedmann-Robertson-Walker manifold, which could be related with quantum cosmology phenomena.

To obtain modified Friedmann equation, we have to take into account the relevant generalized Einstein equations; see [15]. We obtain the modified Friedmann equations under the form:

$$\left[\frac{\dot{a}}{a} + \frac{\dot{b}}{2b}\right]^2 = \frac{c\varepsilon_0}{4}\sqrt{\frac{f(\tau)+2\kappa p}{f'(\tau)}} - \frac{\varepsilon_0}{12}\sqrt{\frac{f\left(\tau\right)-2\kappa\rho}{f'(\tau)}} - \varepsilon_0\varepsilon_1\frac{Kc}{a^2} \tag{61}$$

[3] It is evident from the above expression that the two metrics h and g are no more conformal as they were in the case of the $f(R)$ Lagrangians, apart from the very particular case of $c(t) = \text{const}$. However a suitable redefinition of the cosmic time variable restores the conformal relation between h and g

where the expression on the l.h.s. can be defined as a modified Hubble constant (which is moreover analogue to (51); see also [14]), which rules the dynamical evolution of the Universe:

$$\hat{H}^2 = \left[\frac{\dot{a}}{a} + \frac{\dot{b}}{2b}\right]^2 \tag{62}$$

The r.h.s. of the modified Friedmann equations for Ricci squared theories differs however from the r.h.s. of (51) for Ricci scalar theories, as it should be reasonably expected. The evolution of the model is just dependent on the evolution of the scale factor $a(t)$ and of the modified conformal factor $b(t)$ (i.e. on $\hat{H}$), while derivatives of the factor $c(t)$ disappear. This fact is strongly analogous with the case of Ricci scalar theories.

We choose, as a relevant example to deal with, polynomial Lagrangians in S or S^{-1}, described by means of sums of simple Lagrangians:

$$f(S) = \beta S^n \tag{63}$$

As a matter of facts polynomial Lagrangians can be approximated to pure power Lagrangians if the asymptotical behavior is considered and just the first leading term is taken into account.

Skipping all calculations (see [15] for details) we obtain that the Hubble constant H^2 for the physical metric g is:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \varepsilon_0 P(n,w) a^{-\frac{3(1+w)}{2n}} - \frac{\varepsilon_0}{\varepsilon_1} Q(n,w) K a^{-2} \tag{64}$$

where we have defined:

$$\begin{cases} P(n,w) = \frac{8n^2}{3\sqrt{n(3w-1)(3w+3-4n)}[(3w+1)(2n-1)-2]} \left[\frac{\kappa(3w-1)}{2\beta(n-1)}\eta\right]^{\frac{1}{2n}} \\ Q(n,w) = \sqrt{\frac{4nw-(w+1)}{3(1+w)-4n}} \left[\frac{4n}{(3w+1)(2n-1)-2}\right]^2 \end{cases} \tag{65}$$

From the above expression the deceleration parameter can be calculated by means of the standard formula already introduced in (53) and it can be formally calculated from (64) under the form:

$$q(t,w,n) = -1 + \frac{\frac{3(1+w)}{4n}\varepsilon_1 P(n,w) a^{-\frac{3(1+w)}{2n}} - Q(n,w) K a^{-2}}{\varepsilon_1 P(n,w) \eta^{\frac{1}{2n}} a^{-\frac{3(1+w)}{2n}} - Q(n,w) K a^{-2}} \tag{66}$$

We obtain consequently that when the a^{-2} term dominates over $a^{-\frac{3(1+w)}{2n}}$ the deceleration parameter turns out to be positive, i.e. $q(t,w,n) \to 0^+$, while when the term $a^{-\frac{3(1+w)}{2n}}$ dominates over a^{-2} or in the physically important case $K = 0$, the deceleration parameter will be:

$$q(w,n) = -1 + \frac{3(1+w)}{4n} \tag{67}$$

This implies that $q(w,n)$ is negative for $n<0$ or $n>\frac{3(1+w)}{4}>0$, owing to the positivity of $(1+w)>0$ for standard matter. Comparing expression (64) and the standard relation which derives from General Relativity (see [11] and [14]) it is easy to obtain that the effective value of w_{eff} deriving from Ricci squared alternative theories of Gravity is:

$$w_{eff}(w,n)=-1+\frac{(1+w)}{2n} \tag{68}$$

which depends on the theoretical parameters (w,n). Suitable choices of the above parameter allow to reproduce the experimentally expected values for $w_{eff}(w,n)$, as already remarked for $f(R)$ theories; see discussion after formula (56) .

V. CHERN-SIMONS THEORIES

As we said before, Chern-Simons theories are very important as models for alternative theories of Gravity in higher-dimensions. We discuss now the 3-dimensional Chern-Simons Lagrangian and particular its gravitational equivalent Lagrangian. The 3–dimensional Chern–Simons Lagrangian can be written as:

$$L_{\mathcal{CS}}(A)=\frac{\kappa}{4\pi}\varepsilon^{\mu\nu\rho}Tr\left(A_\mu d_\nu A_\rho+\frac{2}{3}A_\mu\,A_\nu\,A_\rho\right)\,d^3x \tag{69}$$

where κ is a constant which will be fixed later, while $A_\mu=A^i_\mu\,J_i$ are the coefficients of the connection 1–form $A=A_\mu\,dx^\mu$ taking their values in any Lie algebra $\mathfrak{g}$ with generators J_i. By fixing $\mathfrak{g}=sl(2,\Re)$ and choosing the generators

$$J_0=\frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad J_1=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad J_2=\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{70}$$

we have $[J_i,J_j]=\eta^{lk}\varepsilon_{kij}\,J_l$ and $Tr(J_iJ_j)=1/2\,\eta_{ij}$, with $\eta=\mathrm{diag}(-1,1,1)$ and $\varepsilon_{012}=1$. In this case the Lagrangian (69) can be explicitly written as:

$$\begin{aligned}L_{\mathcal{CS}}(A)&=\tfrac{\kappa}{8\pi}\varepsilon^{\mu\nu\rho}\left(\eta_{ij}A^i_\mu d_\nu A^j_\rho+\tfrac{1}{3}\varepsilon_{ijk}A^i_\mu\,A^j_\nu\,A^k_\rho\right)\,d^3x\\ &=\tfrac{\kappa}{16\pi}\varepsilon^{\mu\nu\rho}\left(\eta_{ij}F^i_{\mu\nu}A^j_\rho-\tfrac{1}{3}\varepsilon_{ijk}A^i_\mu\,A^j_\nu\,A^k_\rho\right)\,d^3x\end{aligned} \tag{71}$$

where $F^i_{\mu\nu}=d_\mu A^i_\nu-d_\nu A^i_\mu+\varepsilon^i_{jk}A^j_\mu A^k_\nu$ is the field strength. We then consider two independent $sl(2,\Re)$ dynamical connections A and $\bar{A}$, the evolution of which is dictated by the Lagrangian

$$L_{\mathcal{CS}}(A,\bar{A})=L_{\mathcal{CS}}(A)-L_{\mathcal{CS}}(\bar{A}) \tag{72}$$

which is nothing but the difference of two Chern–Simons Lagrangians (71), one for each dynamical connection. Field equations ensuing from (72) are of course:

$$\begin{cases} \eta_{ij}\, \varepsilon^{\mu\nu\rho}\, F^i_{\mu\nu} = 0 \\ \\ \eta_{ij}\, \varepsilon^{\mu\nu\rho}\, \bar{F}^i_{\mu\nu} = 0 \end{cases} \tag{73}$$

Starting from the fields $A, \bar{A}$ it is then possible (see [43]) to define two new dynamical fields, e^i and ω^i, through the rule:

$$A^i = \omega^i + \frac{1}{l}\, e^i \qquad \bar{A}^i = \omega^i - \frac{1}{l}\, e^i \quad (l = \text{constant}) \tag{74}$$

In terms of the new (e, w) variables field equations (73) become:

$$R^{ij} = -\frac{1}{l^2} e^i \wedge e^j, \qquad T^i = de^i + \omega^i_j \wedge e^j = 0 \tag{75}$$

where $\omega^i = 1/2\, \eta^{ij}\, \varepsilon_{jkl}\, \omega^{kl}$ and $R^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j$. Equations (75) are nothing but Einstein's equations with cosmological constant $\Lambda = -1/l^2$ written in terms of the triad field e^i and the torsion–free spin connection ω^i_j. Moreover the Lagrangian (72), in the new variables, reads as:

$$\begin{aligned} L_{\mathcal{CS}}(A(w,e), \bar{A}(w,e)) &= \tfrac{\kappa}{4\pi\, l}\sqrt{g}(g^{\mu\nu}\, R_{\mu\nu} - 2\Lambda) + d_\mu \left\{ \tfrac{k}{8\pi} \eta_{ij} \varepsilon^{\mu\nu\rho} A^i_\nu\, \bar{A}^j_\rho \right\} \\ &= \tfrac{\kappa}{4\pi\, l}\sqrt{g}(g^{\mu\nu}\, R_{\mu\nu} - 2\Lambda) + d_\mu \left\{ \tfrac{k}{4\pi l} \eta_{ij} \varepsilon^{\mu\nu\rho} e^i_\nu\, \omega^j_\rho \right\} \end{aligned} \tag{76}$$

with $g_{\mu\nu} = \eta_{ij}\, e^i_\mu\, e^j_\nu$ and $R_{\mu\nu} = R^i_{j\rho\nu}\, e^j_\mu\, e^\rho_i$ being the Ricci tensor of the metric g. Notice, however, that the transition from Chern–Simons theory to General Relativity displays some theoretical undesiderable features. Indeed, while Chern–Simons equations of motion are manifestly covariant with respect to spacetime diffeomorphism as well as with respect to gauge transformations, the Chern–Simons Lagrangian (71) is not gauge invariant. The simplest way to overcome this drawback is to push the divergence term appearing in the right hand side of (76) into the left hand side, defining in this way a global covariant Chern–Simons Lagrangian:

$$L_{CCS}(A, \bar{A}) = \frac{k}{8\pi} \varepsilon^{\mu\nu\rho} \left\{ \eta_{ij}\, \bar{F}^i_{\mu\nu}\, B^j_\rho + \eta_{ij}\, \bar{D}_\mu B^i_\nu\, B^j_\rho + \frac{1}{3} \varepsilon_{ijk} B^i_\mu\, B^j_\nu\, B^k_\rho \right\} d^3x \tag{77}$$

where $\bar{D}_\mu$ is the covariant derivative with respect to the connection $\bar{A}$ and we set $B^i_\mu = A^i_\mu - \bar{A}^i_\mu$. Being B^i_μ tensorial, expression (77) tranforms as a scalar function under gauge transformations (and as a scalar density under diffeomorphisms). Taking into account definition (74) we now obtain:

$$L_{CCS}(A(\omega, e), \bar{A}(\omega, e)) = \frac{\kappa}{4\pi\, l} \sqrt{g}\, (g^{\mu\nu}\, R_{\mu\nu} - 2\Lambda) \tag{78}$$

meaning that the Chern–Simons Lagrangian is mapped exactly (i.e. without undesiderable non-invariant boundary terms) into the Hilbert-Einstein Lagrangian for General Relativity with a negative cosmological constant, provided we set

$$\kappa = \frac{l}{4G} \tag{79}$$

being G Newton's constant (and setting $c = 1$) for details see [46].

A. 5-dimensional Euler-Chern-Simons Lagrangian theories

The 5-dimensional Euler–Chern-Simons Lagrangian for the anti-de Sitter gauge group $SO(4,2)$ is written in terms of a gauge connection

$$A = A_\mu^{AB} dx^\mu J_{AB} \tag{80}$$

where $\mu = 0, \ldots, 4$, $A, B = 0 \ldots 5$, and J_{AB} are the Lie algebra generators:

$$[J_{AB}, J_{CD}] = \frac{1}{2}\{-J_{AC}\eta_{BD} + J_{AD}\eta_{BC} + J_{BC}\eta_{AD} - J_{BD}\eta_{AC}\} \tag{81}$$

with $\eta_{AB} = (-1, 1, 1, 1, 1, -1)$. The Lagrangian turns out to be [44, 45]:

$$L_1(A) = \tfrac{\kappa}{3} \ < F \wedge F \wedge A - \tfrac{1}{2} F \wedge A^3 + \tfrac{1}{10} A^5 > \tag{82}$$

where $F^{AB} = dA^{AB} + A^A{}_C \wedge A^{CB}$ is the field strength of the gauge connection and $< \mid >$ denotes the invariant form:

$$< J_{AB} J_{CD} J_{EF} >= \varepsilon_{ABCDEF} \tag{83}$$

According to [44], the constant k in (82) is fixed as $\kappa = l/12\pi^2$ ($l = const$). The Euler–Lagrange field equations ensuing from (82) turns out to be:

$$\frac{\delta L_1}{\delta A_\lambda^{EF}} = \kappa \varepsilon^{\mu\nu\rho\sigma\lambda} \varepsilon_{ABCDEF} (F_{\mu\nu}^{AB} F_{\rho\sigma}^{CD}) = 0 \tag{84}$$

Furthermore, the set of generators J_{AB} can be split as $J_{AB} = (J_{ab}, P_a)$, $a = 0, \ldots, 4$, namely: into the generators J_{ab} of Lorentz rotations in 5 dimensions and the generators $P_a := J_{A5}$ of inner translations. Accordingly, the $SO(4,2)$ gauge connection splits as follows:

$$A = \omega^{ab} J_{ab} + \frac{1}{l} e^a P_a \tag{85}$$

This assumption implies that the field strength decomposes as:

$$F = (R^{ab} + \frac{1}{l^2} e^a \wedge e^b) J_{ab} + \frac{1}{l} T^a \, P_a \tag{86}$$

where

$$R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega^{b}_{c} \tag{87}$$

is the field strength of the $SO(1,4)$ gauge connection ω^{ab} while

$$T^{a} = de^{a} + \omega^{a}{}_{b} \wedge e^{b} \tag{88}$$

is the torsion 2–form. Substituting expressions (85) and (86) into (82) and taking into account that the only terms which are not vanishing are those of the kind $< J_{ab} J_{cd} P_{e} >= \varepsilon_{abcde}$, we can rewrite the Euler–Chern–Simons Lagrangian in terms of the new dynamical fields (ω, e). Apart from boundary terms we obtain:

$$\begin{aligned} L_2(\omega, e) \;=\; \kappa \varepsilon_{abcde} \Bigg(& \frac{1}{l} R^{ab} \wedge R^{cd} \wedge e^{e} + \frac{2}{3l^3} R^{ab} \wedge e^{c} \wedge e^{d} \wedge e^{e} \\ & \frac{1}{5l^5} e^{a} \wedge e^{b} \wedge e^{c} \wedge e^{d} \wedge e^{e} \Bigg) d^5 x \end{aligned} \tag{89}$$

Notice that, in the latter expression, the "vielbein" e^a appears without derivatives and the gauge connection ω^{ab} enters only through its field strength R. This remarkable fact is due to the fact that all the terms involving explicitly the torsion and/or the gauge connection ω are pushed, through integrations by parts, into boundary terms which are discarded in the expression (89). Notice also that, until field equations are solved, no rule exists a priori relating the fields e^a and ω^{ab}: they are completely independent.

Varying the Lagrangian (89) with respect to the independent fields (e, ω) we obtain, respectively, the field equations:

$$\begin{cases} \frac{\delta L_2}{\delta e^{e}_{\lambda}} = \frac{\kappa}{4} \varepsilon^{\mu\nu\rho\sigma\lambda} \varepsilon_{abcde} \hat{R}^{ab}_{\mu\nu} \hat{R}^{cd}_{\rho\sigma} = 0 \\ \\ \frac{\delta L_2}{\delta \omega^{ab}_{\lambda}} = \frac{\kappa}{2} \varepsilon^{\mu\nu\rho\sigma\lambda} \varepsilon_{abcde} \hat{R}^{cd}_{\mu\nu} T^{e}_{\rho\sigma} = 0 \end{cases} \tag{90}$$

where

$$\hat{R}^{ab} = R^{ab} + \frac{1}{l^2} e^{a} \wedge e^{b} \tag{91}$$

Notice that field equations (90) are dynamically equivalent to field equations (84) once the substitution (85) is taken into account. Namely, the difference between the two sets of equations is merely a matter of notation.

If we now consider the particular solution $T^a = 0$ of the second set of field equations (90), it turns out that the $SO(1,4)$ connection ω^a is in this case the spin connection of the vielbein e^a:

$$\omega^{a}{}_{b\mu} = e^{a}_{\alpha} (\Sigma^{\alpha}{}_{\beta\mu} - \Sigma_{\beta}{}^{\alpha}{}_{\mu} + \Sigma_{\mu\beta}{}^{\alpha}) e^{\beta}_{b} \,, \qquad \Sigma^{\alpha}{}_{\beta\mu} = e^{\alpha}_{c} \, \partial_{[\beta} \, e^{c}_{\mu]} \tag{92}$$

and

$$R^{ab}_{\mu\nu} = e^a_\rho e^b_\sigma R^{\rho\sigma}{}_{\mu\nu}(j^2 g) \tag{93}$$

where $R^{\rho\sigma}{}_{\mu\nu}(j^2 g)$ is the Riemann tensor of the metric $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$. Inserting (93) back into the Lagrangian (89), we end up with the new Lagrangian

$$L_3(j^2 g) := L_2|_{T^a=0} = \frac{4\kappa}{l^3}\sqrt{g}\{R + 6\Lambda\} + \frac{\sqrt{g}}{12\pi^2}(R^2 - 4R^\alpha_\beta R^\beta_\alpha + R^{\mu\nu}_{\alpha\beta} R^{\alpha\beta}_{\mu\nu}) \tag{94}$$

which is nothing but the sum of the Hilbert Lagrangian with cosmological constant $\Lambda = 1/l^2$ and the Gauss–Bonnet Lagrangian. Notice that the coefficient in front of the Gauss–Bonnet term is uniquely fixed and is dictated by the initial Euler–Chern–Simons form (82). So far we have introduced three different Lagrangians: see expressions (82), (89) and (94). The transition among them can be schematically drawn as follows:

$$L_1(A) \overset{A=A(\omega,e)}{\Longrightarrow} L_2(\omega, e) \overset{T=0}{\Longrightarrow} L_3(j^2 g) \tag{95}$$

Notice however that the equivalence between the first two Lagrangians, as already explained, holds true off–shell. On the contrary the equivalence between L_2 and L_3 is an on–shell equivalence since it holds true only along the sub-space of solutions $T = 0$.

Gauss-Bonnet Lagrangians have been recently studied in view of their cosmological applications in four dimensions [47]. It we be surely of great interest to study cosmological applications of these theories also in the Palatini formalism [48].

VI. ACKNOWLEDGEMENTS

We are very grateful to the Physical Department of the University of Wroclaw for the kind invitation and support in the 42th Karpac Winter school.

This work is partially supported by GNFM–INdAM research project "*Metodi geometrici in meccanica classica, teoria dei campi e termodinamica*" and by MIUR: PRIN 2003 on "*Conservation laws and thermodynamics in continuum mechanics and field theories*". Gianluca Allemandi is supported by the I.N.d.A.M. grant: "Assegno di collaborazione ad attivitá di ricerca a.a. 2002-2003".

[1] S. Perlmutter et al., Ap. J. 517, 565 (1999) (astro-ph/9812133); S. Perlmutter el al., Nature 404, 955 (2000).

[2] A.G. Riess et al., Ap. J., 116, 1009 (1998) (astro-ph/9805201).

[3] C.L. Bennett et al., Astrophys. J. Suppl. 148, 1 (2003) (astro-ph/0302207); C.B. Netterfield et al., Astrophys. J. 571, 604-614 (2002) (astro-ph/0104460); N.W. Halverson et al., Ap. J. 568, 38 (2002) (astro-ph/0104489); A.H. Jaffe et al., Phys. Rev. Lett. 86, 3475 (2001) (astro-ph/0007333); A.E. Lange et al., Phys. Rev. D63, 042001 (2001); A. Melchiorri et al., Ap. J. 563, L63 (2000) (astro-ph/9911445); D.N. Spergel et al., Ap. J. 148, 175 (2003); R.R. Caldwell, Phys. Lett. B 545, 23 (2002).

[4] L. Verde et al., MNRAS, 335, 432 (2002).

[5] S.M. Carroll, Measuring and Modeling the Universe, Carnegie Observatories Astrophysics Series Vol. 2, ed. W.L. Freedman (astro-ph/0310342); S.M. Carroll, Living Rev. Rel. 4, 1 (2001) (astro-ph/0004075).

[6] P.J.E. Peebles and Bharat Ratra, Rev. Mod. Phys. 75, 559-606 (2003) (astro-ph/0207347); V. Sahni, to appear in the Proceedings of the Second Aegean Summer School on the Early Universe, Syros, Greece, September 2003, (astro-ph/0403324); R.R. Caldwell, R. Dave and P.J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998); S.M. Carroll, M. Hoffman and M. Trodden, Phys.Rev. D68 023509 (2003) (astro-ph/0301273); S. Nojiri and S.D. Odintsov, Phys. Lett. B562 147-152 (2003) (hep-th/0303117); A. Kamenshchik, U. Moschella and V. Pasquier, Phys. Lett. B 511, 265 (2001); A. Frolov, L. Kofman and A. Starobinsky, Phys. Lett. B545, 8-16 (2002) (hep-th/020418); T. Inagaki, X. H. Meng and T. Murata, hep-ph/0306010; E. Elizalde, J.E. Lidsey, S. Nojiri and S.D. Odintsov, Phys. Lett. B574, 1-7 (2003) (hep-th/0307177).

[7] V. Sahni and A. Starobinsky, Int. J. Mod. Phys. D9, 373-444 (2000) (astro-ph/9904398).

[8] A. Starobinsky, Phys. Lett. B 91, 99 (1980).

[9] S. Nojiri and S.D. Odintsov, Phys. Lett. B576, 5-11 (2003) (hep-th/0307071); S. Nojiri and S.D. Odintsov, Mod. Phys. Lett. A, 19(8), 627-638 (2004) (hep-th/0310045).

[10] G. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B485, 208 (2000) (hep-th/0005016); G. Dvali, A. Gruzinov and M. Zaldarriaga, Phys. Rev. D68, 24012 (2003); G. Dvali and M.S. Turner, (astro-ph/0301510); G. Dvali and M. Zaldarriaga, Phys. Rev. Lett. 88, 91303 (2002); K. Freese and M. Lewis, Phys. Lett. B540, 1 (2002) (astro-ph/0201229); N. Ohta, Phys. Rev. Lett. 91 (2003) 061303, (hep-th/0303238); K. Maeda and N. Ohta (hep-th/0405205).

[11] S.M. Carroll, V. Duvvuri, M. Trodden and M.S. Turner, (astro-ph/0306438); S. Capozziello, S. Carloni and A. Troisi, "Recent Research Developments in Astronomy and Astrophysics" -RSP/AA/21-2003 (astro-ph/0303041); S. Capozziello, Int. J. Mod. Phys. D11, 483 (2002); S. Capozziello, V.F. Cardone, S. Carloni and A. Troisi, Int. J. Mod. Phys. D12, 1969 (2003).

[12] N. Deruelle and T. Dolezel, Phys. Rev. D62, 103502 (2000) (gr-qc/0004021); P. Horava and E. Witten, Nucl. Phys. 460, 506 (1996); D. Bailin and A. Love, Rep. Prog. Phys. 50, 1087 (1987); N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B429, 263 (1998); L. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B 436, 257 (1998); M.H. Dehghani (hep-th/0404118); A. Padilla (hep-th/0406157).

[13] S. Nojiri and S.D. Odintsov, Phys. Rev. D68, 123512 (2003) (hep-th/0307288); S. Nojiri and S.D.

Odintsov, to appear in GRG (hep-th/0308176); T. Chiba, Phys. Lett. B575, 1-3 (2003) (astro-ph/0307338); D. Barraco, V.H. Hamity and H. Vucetich, Gen. Rel. Grav. 34 (4), 533-547 (2002); R. Dick, (gr-qc/0307052).

[14] G. Allemandi, A. Borowiec and M. Francaviglia, Phys.Rev. D **70**, 043524 (2004).

[15] G. Allemandi, A. Borowiec and M. Francaviglia, Phys. Rev. D 70 (10), (hep-th/0407090), G. Allemandi, A. Borowiec, M. Francaviglia and S.D. Odintsov, Phys. Rev. D 72 (6), (gr-qc/0504057);

[16] D.N. Vollick, Phys. Rev. D68, 063510 (2003) (astro-ph/0306630); X.H. Meng and P. Wang, Class. Quant. Grav. 20, 4949-4962 (2003) (astro-ph/0307354); X.H. Meng and P. Wang, Class. Quant. Grav. 21, 951-960 (2004) (astro-ph/0308031); X.H. Meng and P. Wang, (astro-ph/0308284); X.H. Meng and P. Wang, (hep-th/0309062); G.M. Kremer and D.S.M. Alves, (gr-qc/0404082).

[17] B. Feng, X. Wang. and X. Zhang (astro-ph/0404224); P.S. Corasaniti, M. Kunz, D. Parkinson, E.J. Copeland and B.A. Bassett (astro-ph/0406608).

[18] S. Nojiri and S.D. Odintsov, (astro-ph/0403622).

[19] E.E. Flanagan, Phys. Rev. Lett. 92, 071101 (2004) (astro-ph/0308111); E.E. Flanagan, Class. Quant. Grav. 21, 417-426 (2003) (gr-qc/0309015); D.N. Vollick, (gr-qc/0312041).

[20] G. Magnano and L.M. SokoLowski, Phys. Rev. D50, 5039 (1994), (gr-qc/9312008); G.J. Olmo and W. Komp, (gr-qc/0403092); A.D. Dolgov and M. Kawasaki, Phys. Lett. B573, 1-4 (2003) (astro-ph/0307285).

[21] G. Magnano, *Proceedings of XI Italian Conference on General Relativity and Gravitation, Trieste*, (1994), (gr-qc/9511027).

[22] M. Ferraris, M.Francaviglia, in: *Mechanics, Analysis and Geometry: 200 Years after Lagrange*; Editor: M. Francaviglia, Elsevier Science Publishers B:V:, (1991)

[23] K. Maeda, *Phys. Rev.* **D 39**, 3159 (1989).

[24] G. Magnano, M. Ferraris, and M. Francaviglia, *Gen. Relativ. Grav.* **19**, 465 (1987).

[25] S. Gottlöber, H.-J. Schmidt, and A.A. Starobinsky, *Class. Quantum Grav.* **7**, 893 (1990).

[26] Ferraris M, Francaviglia M and Magnano G 1988 *Class. Quantum Grav.* **5** L95.

[27] Damour T and Esposito-Farèse G 1992 *Class. Quantum Grav.* **9** 2093.

[28] H.A. Buchdahl, J. Phys. A **12** (8) (1979), 1229;

[29] S. Capozziello, R.de Ritis, C. Rubano, P. Scudellaro, *Int. Journ. Mod. Phys.* **D 5**, 85 (1996).

[30] S. Capozziello and R. de Ritis, *Class. Quantum Grav.* **11** (1994) 107.

[31] J.P. Duruisseau and R. Kerner, Gen. Rel. Grav. 15, 797-807 (1983).

[32] I. Brevik, S. Nojiri, S.D. Odintsov and L. Vanzo, (hep-th/0401073).

[33] J.W. Moffat, (astro-ph/0403266).

[34] D. Comelli and A. Dolgov, (gr-qc/0404065).

[35] A. Borowiec, M. Ferraris, M. Francaviglia and I. Volovich, Class. Quant. Grav. 15, 43-55 (1998).

[36] A. Borowiec, M. Ferraris, M. Francaviglia and I. Volovich, J. Math. Phys. 40, 3446-3464 (1999) (dg-ga/9612009); A. Borowiec, M. Francaviglia and I. Volovich, Differ. Geom. Appl. 12, 281 (2000) (math-

ph/9906012).

[37] A. Borowiec, M. Francaviglia and V. Smirichinski, Proceedings of XXIII International Colloquium on Group Theoretical Methods in Physics (Group 23), Ed.: A.N. Sissakian, G.S. Pogosyan, L. G. Mordoyan, Vol.1,2, 209-212, JINR, Dubna 2002 (gr-qc/0011103); A. Borowiec, CS 173: GROUP 24: Physical and Mathematical Aspects of Symmetries: Proceedings of the 24th International Colloquium on Group Theoretical Methods in Physics, Paris, 15-20 July 2002, Ed.: J-P. Gazeau, R. Kerner, J-P. Antoine, S. Métens, J-Y. Thibon, Institute of Physics Conference Series 173 (2003) (gr-qc/9906043)

[38] A. Borowiec, M. Francaviglia and I. Volovich, in preparation.

[39] M. Ferraris, M. Francaviglia and I. Volovich, Nuovo Cim. B108, 1313 (1993) (gr-qc/9303007); M. Ferraris, M. Francaviglia and I. Volovich, Class. Quant. Grav. 11, 1505 (1994).

[40] S. Weinberg, *Gravitation and Cosmology*, John Wiley and Sons (1972), ISBN 0-471-92567-5.

[41] M.B. Mijic, M.S. Morris and W. Suen, Phys. Rev. D34 (10), 2934 (1986).

[42] E.E. Flanagan, (gr-qc/0403063).

[43] E. Witten, *Nucl. Phys.* **B 311** (1988), 46; E. Witten, *Commun. Math. Phys.* **121** (1989), 351.

[44] M. Bañados, C. Teitelboim, J. Zanelli, Phys.Rev. D **49** (2) (1994) 975.

[45] A. H. Chamseddine, *Nucl. Phys.* B 346 (1990), 213-234;
A. H. Chamseddine and J. Fröhlich, *Commun. Math. Phys.* **147** (1992), 549.

[46] G. Allemandi, M. Francaviglia, M. Raiteri, Class. Quantum Grav. **20** (2003), 483-506 (gr-qc/0211098).

[47] Guido Cognola, Emilio Elizalde, Shin'ichi Nojiri, Sergei D. Odintsov, Sergio Zerbini, Phys.Rev. D73 (2006) 084007; Shin'ichi Nojiri, Sergei D. Odintsov, O.G. Gorbunova, J.Phys. A39 (2006) 6627-6634; Shin'ichi Nojiri, Sergei D. Odintsov, Phys.Lett. B631 (2005) 1-6

[48] G. Allemandi, A. Borowiec and M. Francaviglia, in preparation.