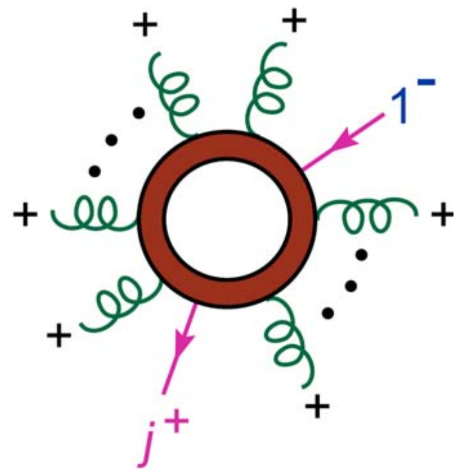
Twistoresque Methods for Perturbative QCD



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Introduction

- Need a flexible, efficient method to extend range of known tree, and particularly 1-loop QCD amplitudes with many external legs, for use in NLO corrections to many LHC processes, some ILC processes, etc.
- 1-loop not known beyond *n=5* legs, except for special helicity configurations
- Semi-numerical approaches to 1-loop amplitudes are one way to go, *e.g.*

Denner, Dittmaier, …, hep-ph/0212259,…; Nagy, Soper, hep-ph/0308127;
Giele, Glover, hep-ph/0402152; Andonov *et al.,* hep-ph/0411186;
van Hameren, Vollinga, Weinzierl, hep-ph/0502165; Binoth *et al.,* hep-ph/0504267;
Ellis, Giele, Zanderighi, hep-ph/0506196 [*Hgggg, Hqqgg, Hqqqq*]

Introduction (cont.)

- Another approach is to pay attention to the analytic properties of amplitudes
  - poles (factorization) at tree level
  - poles and branch cuts (unitarity) at loop level
- In this approach one can incorporate hidden symmetries of tree level QCD, due to its relation to N=4 super-Yang-Mills theory:
  - supersymmetry Ward identities — Grisaru, Pendleton, van Nieuwenhuizen (1977)
  - connection to twistor space — Penrose (1967)
    and to twistor string theory — Witten, hep-th/0312171
- These symmetries have loop-level implications for QCD *via* unitarity

Outline

- Motivation
- Role of N=4 super-Yang-Mills theory
- Color & helicity
- Supersymmetry Ward identities
- Twistor space, twistor strings, & MHV tree rules
- On-shell recursion relations at tree level
- (Generalized) unitarity and twistor structure of 1-loop amplitudes in N=4 super-Yang-Mills theory
- On-shell recursion relations at 1-loop, leading to new QCD amplitudes with 6 or more legs
- Conclusions

Role of N=4 super-Yang-Mills theory

- Essentially unique, maximally supersymmetric, conformal field theory
- Topological string in twistor space is most directly for N=4 SYM — Witten, hep-th/0312171
- N=4 SYM ⇔ QCD at tree level;

  can be thought of as 1 component of QCD at 1 loop
- Loop-level scattering amplitudes share many properties with those of QCD, but are simpler

  ⇒ "theoretical playground"

Bern, LD, Kosower, hep-ph/9403226, 9409265

N=4 super-Yang-Mills theory

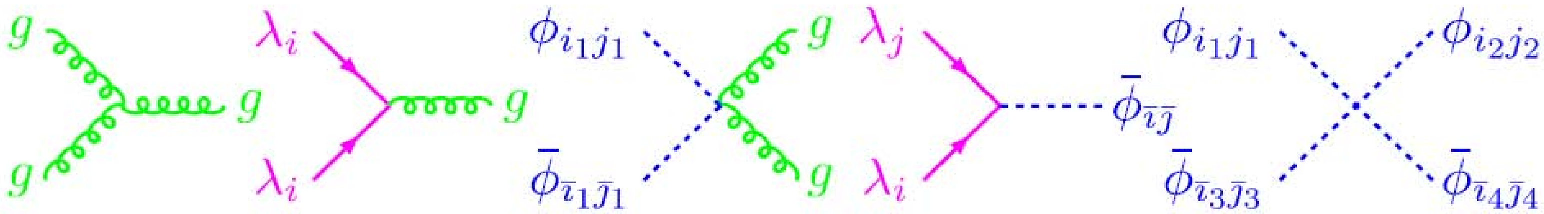
State multiplicities:

			1		
		1		1	
$\mathcal{N}=2$		1	2	1	
	1	3	3	1	

$\mathcal{N}=4$	1	4	6	4	1
	g^-	$\lambda_{\bar{i}}^-$	$\bar{\phi}_{\bar{i}\bar{j}}, \phi_{ij}$	λ_i^+	g^+
helicity	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1

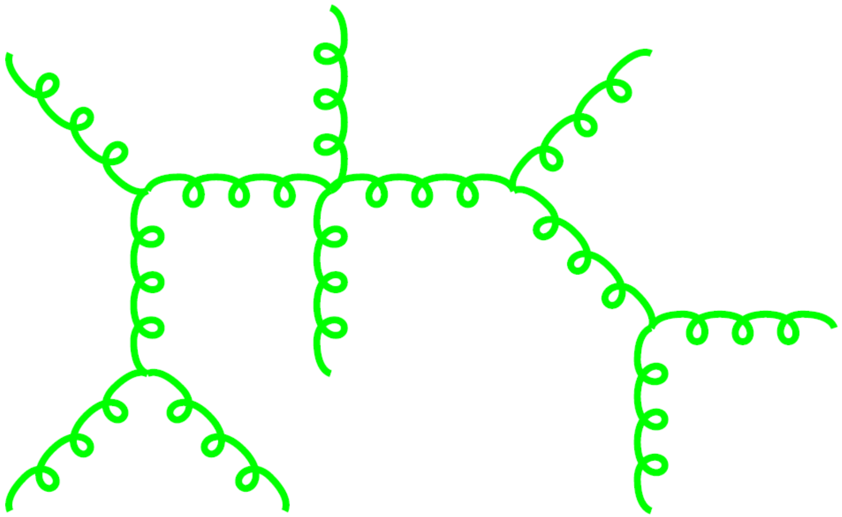
all in adjoint representation

Feynman rules: Usual gauge interactions, plus $W(\Phi)$



Tree level

- N=4 SYM ⇔ QCD at tree level for *n*-gluon amplitude because no fermions & scalars enter (as they must be pair-produced)

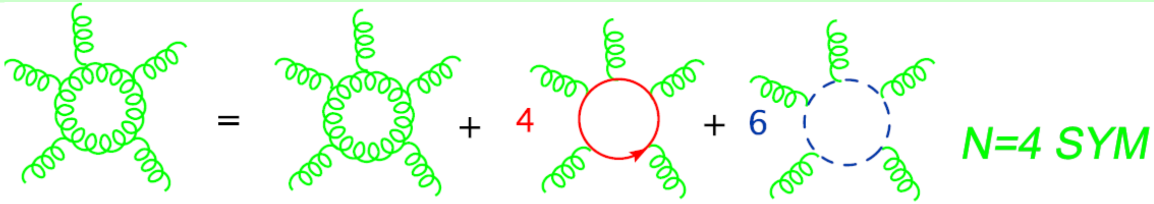


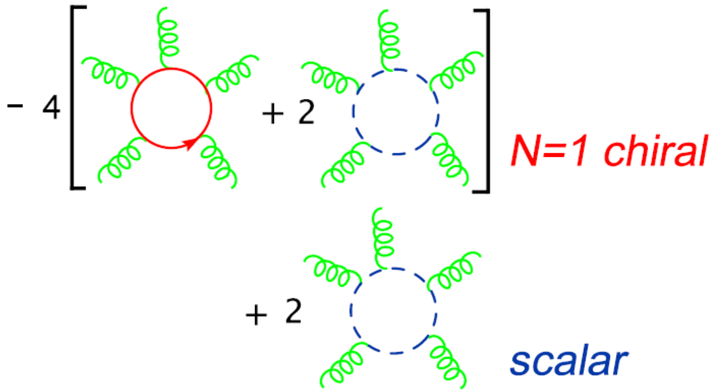
- Similar relations with external fermions too

One loop rearrangement

Can rewrite gluon (and fermion) loop for *n*-gluon QCD amplitude as linear combinations of:

- N=4 SYM (simplest)
- N=1 chiral matter multiplet (next simplest)
- scalar (non-supersymmetric, but no spin-tangling)





Similar relations with external fermions

Color-ordered amplitudes

Decompose tree-level n-gluon amplitudes as

$$\mathcal{A}_n^{\rm tree}(\{k_i,\lambda_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} {\rm Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}) \times A_n^{\rm tree}(\sigma(1^{\lambda_1}),\ldots,\sigma(n^{\lambda_n}))$$

$A_n^{\rm tree}$ color-ordered, only receive contributions from cyclicly-ordered Feynman diagrams, so poles in fewer kinematic variables

Mangano, Parke (1986)

Similarly decompose 1-loop n-gluon amplitudes as

$$\mathcal{A}_n^{\rm 1-loop}(\{k_i,\lambda_i,a_i\}) = g^{n} N_c \sum_{\sigma\in S_n/Z_n} {\rm Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}) \times A_{n;1}(\sigma(1^{\lambda_1}),\ldots,\sigma(n^{\lambda_n})) + \mathcal{O}(1/N_c)$$

Subleading-color terms, coeff's of Tr(..) Tr(..), not independent; sums of perm's of color-ordered $A_{n;1}$

Bern, Dunbar, LD, Kosower, hep-ph/9403226

Spinor variables

Use Dirac (Weyl) spinors $u_\alpha(k_i)$ (spin ½), **not** 4-vectors k_i^μ (spin 1)

right-handed: $(\lambda_i)_\alpha = u_+(k_i)$ left-handed: $(\tilde{\lambda}_i)_{\dot{\alpha}} = u_-(k_i)$

Reconstruct k_i^μ from $u_\alpha(k_i)$ using positive-energy Dirac projector:

$$k_i^\mu(\sigma_\mu)_{\alpha\dot{\alpha}} = (\not{k}_i)_{\alpha\dot{\alpha}} = u_+(k_i)\bar{u}_+(k_i) = (\lambda_i)_\alpha(\tilde{\lambda}_i)_{\dot{\alpha}}$$

Singular 2 x 2 matrix:

$$\det(\not{k}_i) = \begin{vmatrix} k_t + k_z & k_x - ik_y \\ k_x + ik_y & k_t - k_z \end{vmatrix}$$
$$= k_t^2 - k_x^2 - k_y^2 - k_z^2 = 0$$

also shows $(\not{k}_i)_{\alpha\dot{\alpha}} = (\lambda_i)_\alpha(\tilde{\lambda}'_i)_{\dot{\alpha}}$

even for complex momenta

Gluon polarizations also in terms of spinors: $\varepsilon_\mu^\pm(k,\eta) = \pm\frac{\langle k^\pm|\gamma_\mu|\eta^\pm\rangle}{\sqrt{2}\langle k^\mp|\eta^\pm\rangle}$

Spinor products

Instead of Lorentz products: $s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$

Use spinor products:

$$\bar{u}_-(k_i)u_+(k_j) = \varepsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_j)_\beta = \langle i\, j\rangle$$

$$\bar{u}_+(k_i)u_-(k_j) = \varepsilon^{\dot\alpha\dot\beta}(\tilde\lambda_i)_{\dot\alpha}(\tilde\lambda_j)_{\dot\beta} = [i\, j]$$

These are **complex square roots** of Lorentz products:

$$\langle i\, j\rangle[j\, i] = \tfrac{1}{2}\mathrm{Tr}\left[\not{k}_i\ \not{k}_j\right] = 2k_i \cdot k_j = s_{ij}$$

$$\boxed{\langle i\, j\rangle = \sqrt{s_{ij}}\, e^{i\phi_{ij}} \qquad [j\, i] = \sqrt{s_{ij}}\, e^{-i\phi_{ij}}}$$

Supersymmetry Ward identities

Grisaru, Pendleton, van Nieuwenhuizen (1977)

In any unbroken supersymmetric theory, $Q|0\rangle = 0$, so

$$0 = \langle 0|[Q, \Phi_1 \Phi_2 \cdots \Phi_n]|0\rangle = \sum_{i=1}^{n} \langle 0|\Phi_1 \cdots [Q, \Phi_i] \cdots \Phi_n|0\rangle$$

Leads to powerful S-matrix identities:

$$A_n^{\rm SUSY}(1^\pm, 2^+, 3^+, 4^+, \ldots, n^+) = 0$$

$$A_n^{\rm SUSY}(1_f^-, 2_f^+, 3^-, 4^+, \ldots, n^+) = \frac{\langle 2\,3\rangle}{\langle 1\,3\rangle} \times A_n^{\rm SUSY}(1^-, 2^+, 3^-, 4^+, \ldots, n^+)$$

$\dfrac{A_n^{\mathcal{N}=4\ \rm SUSY}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)}{\langle i\,j\rangle^4}$ indep. of i, j *etc.*

- Results hold order by order in perturbation theory.
- At tree-level, can be applied directly to QCD.

Twistor Space

Start in **spinor space**: Amplitudes $A(k_i) \;\Rightarrow\; A(\lambda_i, \tilde{\lambda}_j)$

Twistor transform = "half Fourier transform":

Fourier transform $\tilde{\lambda}_i$, but not λ_i, for each leg i

$$\tilde{\lambda}_{\dot{a}} = i\frac{\partial}{\partial \mu^{\dot{a}}} \qquad\qquad \mu^{\dot{a}} = -i\frac{\partial}{\partial \tilde{\lambda}_{\dot{a}}}$$

Twistor space coordinates: $(\lambda_1, \lambda_2, \mu^{\dot{1}}, \mu^{\dot{2}})$ for each i

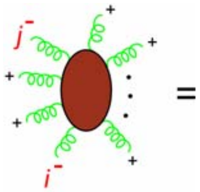
$$\sim (\xi\lambda_1, \xi\lambda_2, \xi\mu^{\dot{1}}, \xi\mu^{\dot{2}})$$

Amplitudes $A(k_i) \;\Rightarrow\; A(\lambda_i, \tilde{\lambda}_i) \;\Rightarrow\; A(\lambda_i, \mu_i)$

Twistor Transform in QCD

Witten, hep-th/0312171

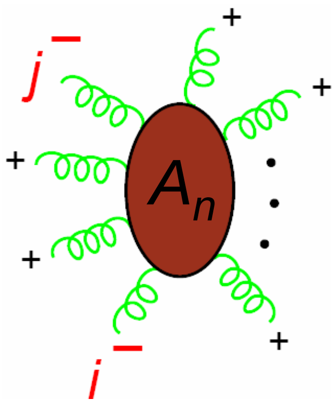
Parke-Taylor (1986) $\quad n_- = 2$ (MHV)



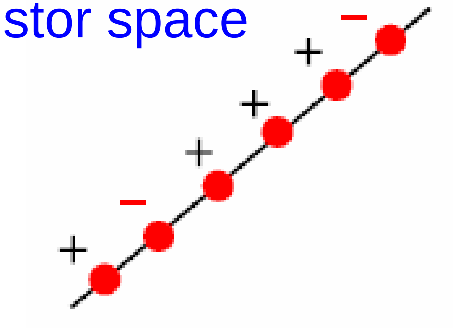
$$= \frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,\delta\left(\textstyle\sum_i k_i\right) \;=\; \int d^4x\, A(\lambda_i)\, \exp(i x \lambda_i \tilde{\lambda}_i)$$

$$\int d\tilde{\lambda}\, \exp(i\mu\tilde{\lambda})\, \exp(i x \lambda\tilde{\lambda}) \;\Rightarrow\; A(\lambda,\mu) \;\propto\; \delta(\mu + x\lambda)$$

linear constraints for each ***i*** imply
all points lie on single line in twistor space



$\Rightarrow$



Twistor implications in spinor space

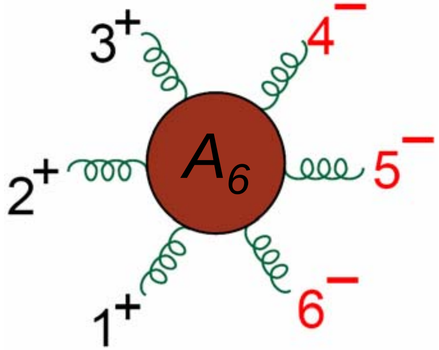
Witten, hep-th/0312171

- Vanishing relations on curves in twistor space $\Longrightarrow$ differential equations in $(\lambda_i, \tilde{\lambda}_j)$ space.
- i, j, k have collinear support if A annihilated by
  $C_{ijkL} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K \to \langle i\, j \rangle \frac{\partial}{\partial \tilde{\lambda}_k^{\dot{a}}} + \langle j\, k \rangle \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{a}}} + \langle k\, i \rangle \frac{\partial}{\partial \tilde{\lambda}_j^{\dot{a}}}$
  for $L = \dot{a}$.
- i, j, k, l are coplanar if A annihilated by
  $K_{ijkl} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L \to \langle i\, j \rangle\, \epsilon^{\dot{a}\dot{b}} \frac{\partial^2}{\partial \tilde{\lambda}_k^{\dot{a}} \partial \tilde{\lambda}_l^{\dot{b}}}$ + 5 perms

More Twistor Magic

Using **collinear**/**coplanar** differential operators, find:

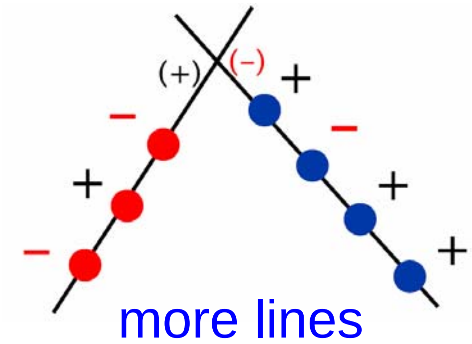
Witten,
hep-th/0312171



Mangano, Parke, Xu (1988) $\quad n_- = 3$ (NMHV)

$$= \frac{([1\,2]\,\langle 4\,5\rangle\,\langle 6^-|(1+2)|3^-\rangle)^2}{s_{61}s_{12}s_{34}s_{45}s_{612}} + \frac{([2\,3]\,\langle 5\,6\rangle\,\langle 4^-|(2+3)|1^-\rangle)^2}{s_{23}s_{34}s_{56}s_{61}s_{561}} + \frac{s_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6^-|(1+2)|3^-\rangle\langle 4^-|(2+3)|1^-\rangle}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}}$$

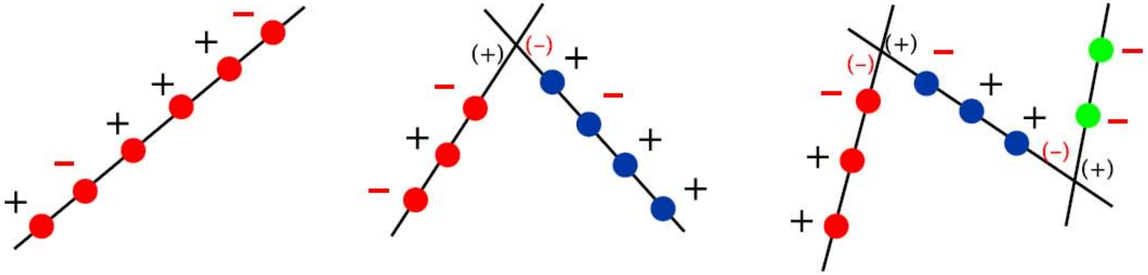
$\Rightarrow$



Twistor magic from twistor strings

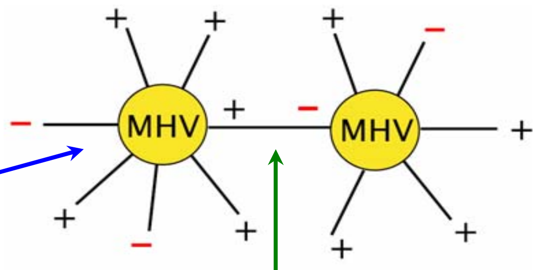
Original intuition from topological string:
L-loop amplitude with n_- negative-helicity gluons should be supported on curve in twistor space with degree $d = n_- - 1 + L$, genus $g \leq L$.
MHV case: $n_- = 2$, $g = 0 \Rightarrow d = 1$, a straight line.

"Experimentation" showed situation actually better than that for tree amplitudes: Cachazo, Svrček, Witten (2004)
supported on $n_- - 1$ intersecting straight lines
(degenerate limit of the degree d curve)



MHV rules

Based on the "experimental" results, and an interpretation of the twistor string path integral, Cachazo, Svrcek, Witten, hep-th/0403047 proposed "MHV rules" for *n*-gluon scattering:



off-shell MHV (Parke-Taylor) amplitudes

scalar propagator, $1/p^2$

For example, if a + leg goes off-shell, use:

$$A_n^{\mathrm{tree,MHV},ij}(1^*) = \frac{\langle i\, j\rangle^4}{\langle 1^*\, 2\rangle \ldots \langle n\, 1^*\rangle}$$

$$= \frac{\langle i\, j\rangle^4}{\langle \eta^+ |\, 1\, |2^+\rangle \ldots \langle n^- |\, 1\, |\eta^-\rangle}$$

where $\eta^2 = 0$ is arbitrary. Results independent of η, agree numerically with Feynman diagram computations

MHV rules for trees

Rules quite efficient, extended to many collider applications

• massless quarks

Georgiou, Khoze, hep-th/0404072;
Wu, Zhu, hep-th/0406146;
Georgiou, Glover, Khoze, hep-th/0407027

• Higgs bosons (*Hgg* coupling)

LD, Glover, Khoze, hep-th/0411092;
Badger, Glover, Khoze, hep-th/0412275

• vector bosons (W,Z,γ^*)

Bern, Forde, Kosower,
Mastrolia, hep-th/0412167

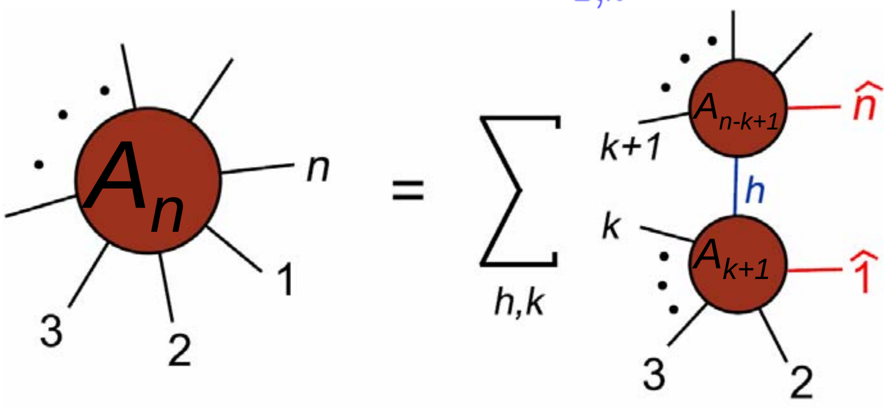
• Related approach to QCD + massive quarks
more directly from field theory

Schwinn, Weinzierl,
hep-th/0503015

Even better than MHV rules

On-shell recursion relations Britto, Cachazo, Feng, hep-th/0412308

$$A_n(1,2,\ldots,n) \;=\; \sum_{h=\pm}\sum_{k=2}^{n-2} A_{k+1}(\hat{1},2,\ldots,k,-\hat{K}_{1,k}^{-h}) \times \frac{i}{K_{1,k}^2} A_{n-k+1}(\hat{K}_{1,k}^{h},k+1,\ldots,n-1,\hat{n})$$

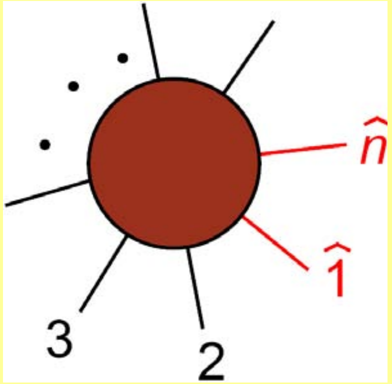


A_{k+1} and A_{n-k+1} are on-shell tree amplitudes with fewer legs, evaluated with 2 momenta shifted by a **complex** amount

Proof of on-shell tree recursion

Britto, Cachazo, Feng, Witten, hep-th/0501052

- Consider a family of on-shell amplitudes $A_n(z)$ depending on a complex parameter z which shifts the momenta.
- Best described using spinor variables.
- For example, the (n,1) shift:



$$\lambda_1 \to \hat{\lambda}_1 = \lambda_1 + z\lambda_n \qquad \tilde{\lambda}_1 \to \tilde{\lambda}_1$$

$$\lambda_n \to \lambda_n \qquad \tilde{\lambda}_n \to \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1$$

- On-shell condition: $$\begin{aligned}(\hat{k}_1)^\mu(\hat{k}_1)_\mu &= (\hat{k}_1)^{\alpha\dot{\alpha}}(\hat{k}_1)_{\dot{\alpha}\alpha} \\ &= \langle(\lambda_1 + z\lambda_n)(\lambda_1 + z\lambda_n)\rangle[1\,1] = 0\end{aligned}$$

  similarly, $\hat{k}_n^2 = 0$
- Momentum conservation:

$$\hat{k}_1 + \hat{k}_n = (\lambda_1 + z\lambda_n)\tilde{\lambda}_1 + \lambda_n(\tilde{\lambda}_n - z\tilde{\lambda}_1) = k_1 + k_n$$

MHV example

- Apply this shift to the Parke-Taylor (MHV) amplitudes:

$$A_n(z=0) = A_n^{jn,\,\text{MHV}} \;=\; \frac{\langle j\,n\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

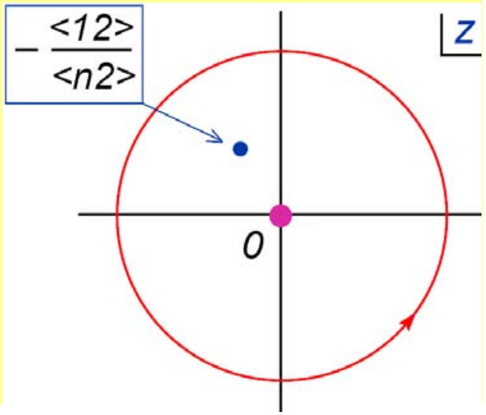
- Under the (n,1) shift: $\lambda_1 \to \lambda_1 + z\lambda_n \qquad \tilde\lambda_n \to \tilde\lambda_n - z\tilde\lambda_1$

$$\begin{aligned}\langle n\,1\rangle &= \lambda_n\lambda_1 \to \lambda_n(\lambda_1 + z\lambda_n) = \langle n\,1\rangle + z\langle n\,n\rangle = \langle n\,1\rangle \\ \langle 1\,2\rangle &= \lambda_1\lambda_2 \to (\lambda_1 + z\lambda_n)\lambda_2 = \langle 1\,2\rangle + z\langle n\,2\rangle\end{aligned}$$

- So $A_n(z) = \dfrac{\langle j\,n\rangle^4}{(\langle 1\,2\rangle + z\langle n\,2\rangle)\langle 2\,3\rangle\cdots\langle n\,1\rangle}$

- Consider: $\dfrac{1}{2\pi i}\oint_C dz\,\dfrac{A_n(z)}{z}$

- 2 poles, opposite residues



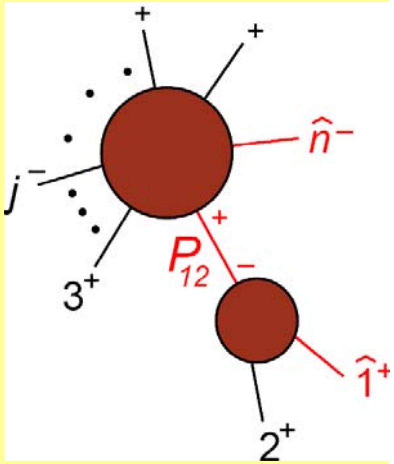
MHV example (cont.)

- MHV amplitude obeys: $$A_n(0) \;=\; -\underset{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}}{\text{Res}} \frac{A_n(z)}{z}$$

- Compute residue using factorization
- At $$z \;=\; -\frac{\langle 1\,2\rangle}{\langle n\,2\rangle} = -\frac{\langle 1\,2\rangle[2\,1]}{\langle n\,2\rangle[2\,1]} = -\frac{s_{12}}{\langle n^-|(1+2)|1^-\rangle}$$

  kinematics are complex collinear

$$\langle \hat{1}\,2\rangle \;=\; \langle 1\,2\rangle + z\langle n\,2\rangle = 0 \qquad [\hat{1}\,2] = [1\,2] \neq 0$$
$$s_{\hat{1}2} \;=\; \langle \hat{1}\,2\rangle[2\,\hat{1}] = 0$$



- so $$-\underset{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}}{\text{Res}} \frac{A_n(z)}{z} \;=\; A_{n-1}(\hat{P}_{12}^+, 3^+, \ldots, j^-, \ldots, n^-)$$
$$\times \left[-\underset{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}}{\text{Res}} \frac{1}{z}\frac{1}{\hat{P}_{12}^2(z)} \right] A_3(\hat{1}^+, 2^+, -\hat{P}_{12}^-)$$

note
$A_3(+,+,+) = 0$

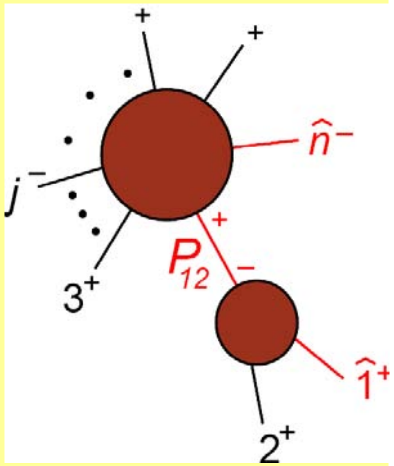
Evaluate ingredients

- Since $\hat{P}_{12}^2(z) \;=\; (k_1 + k_2 + z\lambda_n\tilde{\lambda}_1)^2 = s_{12} + z\langle n^-|(1+2)|1^-\rangle$

$$-\mathop{\mathrm{Res}}_{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}} \frac{1}{z}\frac{1}{\hat{P}_{12}^2(z)} \;=\; -\frac{\langle n^-|(1+2)|1^-\rangle}{s_{12}}\frac{1}{\langle n^-|(1+2)|1^-\rangle} = \frac{1}{s_{12}}$$

- So

$$\boxed{A_n(0) \;=\; A_{n-1}(\hat{P}_{12}^+, 3^+, \ldots, j^-, \ldots, n^-)\frac{1}{s_{12}}A_3(\hat{1}^+, 2^+, -\hat{P}_{12}^-)}$$



- Check this explicitly:

$$\begin{aligned} A_n(0) \;&=\; \frac{\langle j\,\hat{n}\rangle^4}{\langle \hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,\hat{n}\rangle\langle \hat{n}\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[\hat{1}\,2]^3}{[2\,\hat{P}][\hat{P}\,\hat{1}]} \\ &=\; \frac{\langle j\,n\rangle^4}{\langle \hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,n\rangle\langle n\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[1\,2]^3}{[2\,\hat{P}][\hat{P}\,1]} \end{aligned}$$

MHV check (cont.)

- Using
$$\begin{aligned}\langle n\,\hat{P}\rangle[\hat{P}\,2] &= \langle n^-|(1+2)|2^-\rangle + z\langle n\,n\rangle[1\,2] = \langle n\,1\rangle[1\,2]\\ \langle 3\,\hat{P}\rangle[\hat{P}\,1] &= \langle 3^-|(1+2)|1^-\rangle + z\langle 3\,n\rangle[1\,1] = \langle 3\,2\rangle[2\,1]\end{aligned}$$

one confirms

$$\begin{aligned}A_n(0) &= \frac{\langle j\,n\rangle^4}{\langle\hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,n\rangle\langle n\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[1\,2]^3}{[2\,\hat{P}][\hat{P}\,1]}\\ &= \frac{\langle j\,n\rangle^4[1\,2]^3}{(\langle 1\,2\rangle[2\,1])([1\,2]\langle 2\,3\rangle)(\langle n\,1\rangle[1\,2])\langle 3\,4\rangle\cdots\langle n-1,n\rangle}\\ &= \frac{\langle j\,n\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n-1,n\rangle\langle n\,1\rangle}\\ &= A_n^{jn,\,\mathrm{MHV}}\end{aligned}$$

So MHV amplitudes from *n=4* on are derived recursively

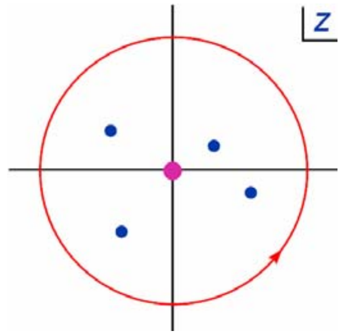
The general case

Same analysis as above – Cauchy's theorem + amplitude factorization

Let complex momentum shift depend on *z*. Use analyticity in *z*.

$$\hat{\lambda}_1 = \lambda_1 + z\lambda_n \qquad \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 \qquad \Rightarrow \quad A(0) \;\to\; A(z)$$

$$\hat{\lambda}_n = \lambda_n \qquad \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1$$

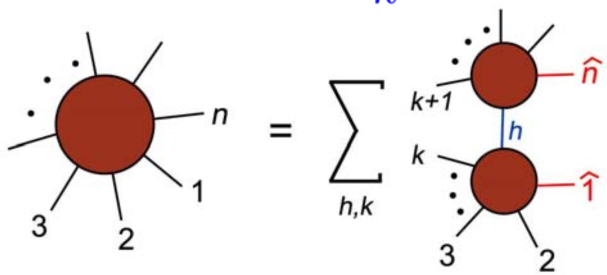


Cauchy: If $A(\infty) = 0$ then

$$0 = \frac{1}{2\pi i}\oint dz\,\frac{A(z)}{z} = A(0) + \sum_k \text{Res}\Big[\frac{A(z)}{z}\Big]\Big|_{z=z_k}$$

poles in *z*: physical factorizations $\hat{K}^2_{1,k} = 0$

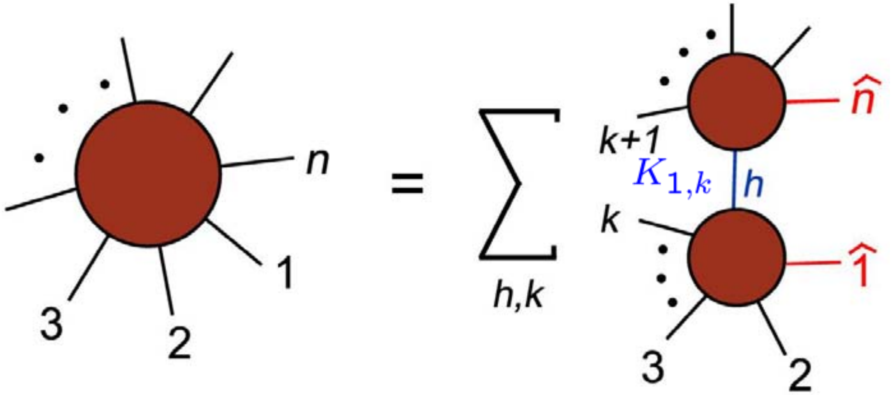
residue at $z_k = -\frac{K^2_{1,k}}{\langle n^-|\not{K}_{1,k}|1^-\rangle}$ = [$k^{\rm th}$ term]



Momentum shift

Shift for $k^{\rm th}$ term comes from setting $z = z_k$, where

$$z_k = -\frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle}$$



is the solution to

$$\hat{K}_{1,k}^2(z) = 0 = (K_{1,k} + z\lambda_n\tilde{\lambda}_1)^2 = K_{1,k}^2 + z\lambda_n^a (K_{1,k})_{a\dot{a}} \tilde{\lambda}_1^{\dot{a}}$$

plugging in, shift is:

$$\hat{\lambda}_1 = \lambda_1 - \frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle} \lambda_n \qquad \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1$$

$$\hat{\lambda}_n = \lambda_n \qquad \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n + \frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle} \tilde{\lambda}_1$$

To show: $A(\infty) = 0$

Britto, Cachazo, Feng, Witten, hep-th/0501052

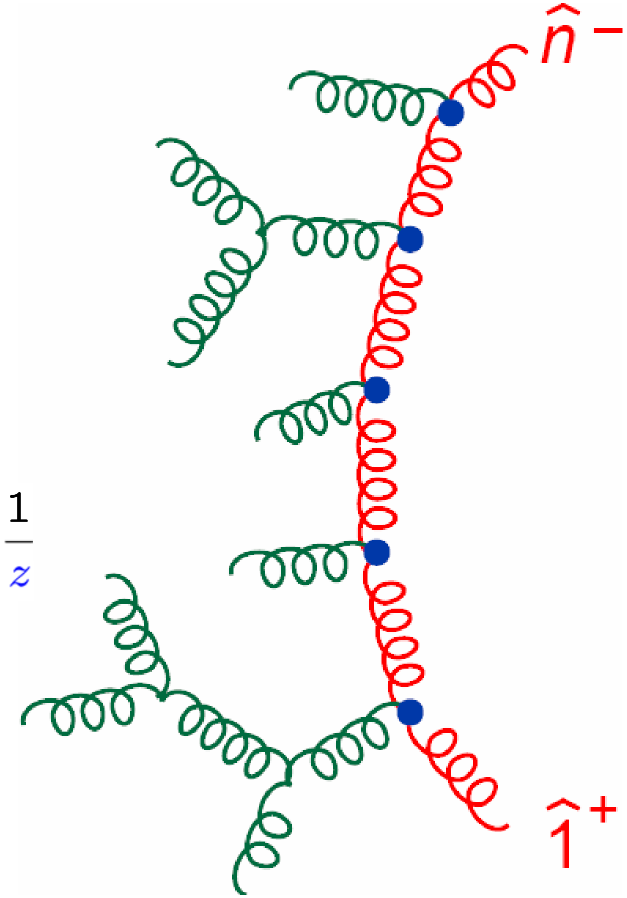
Propagators:

$$\frac{1}{\hat{K}^2_{1,k}(z)} = \frac{1}{K^2_{1,k} + z\lambda^a_n (K_{1,k})_{a\dot{a}} \tilde{\lambda}^{\dot{a}}_1} \sim \frac{1}{z}$$

3-point vertices: $\propto \ \hat{k}^\mu(z) \propto z$

Polarization vectors:

$$\not{\epsilon}^+_1 \ \propto \ \frac{\tilde{\lambda}_1 \lambda_q}{\langle \lambda_1 \, \lambda_q \rangle} \ \propto \ \frac{1}{z} \qquad \not{\epsilon}^-_n \ \propto \ \frac{\lambda_n \tilde{\lambda}_q}{\langle \tilde{\lambda}_n \, \tilde{\lambda}_q \rangle} \ \propto \ \frac{1}{z}$$



Total:

$$\frac{1}{z} \times \left(z \frac{1}{z} \right)^r z \times \frac{1}{z} = \frac{1}{z}$$

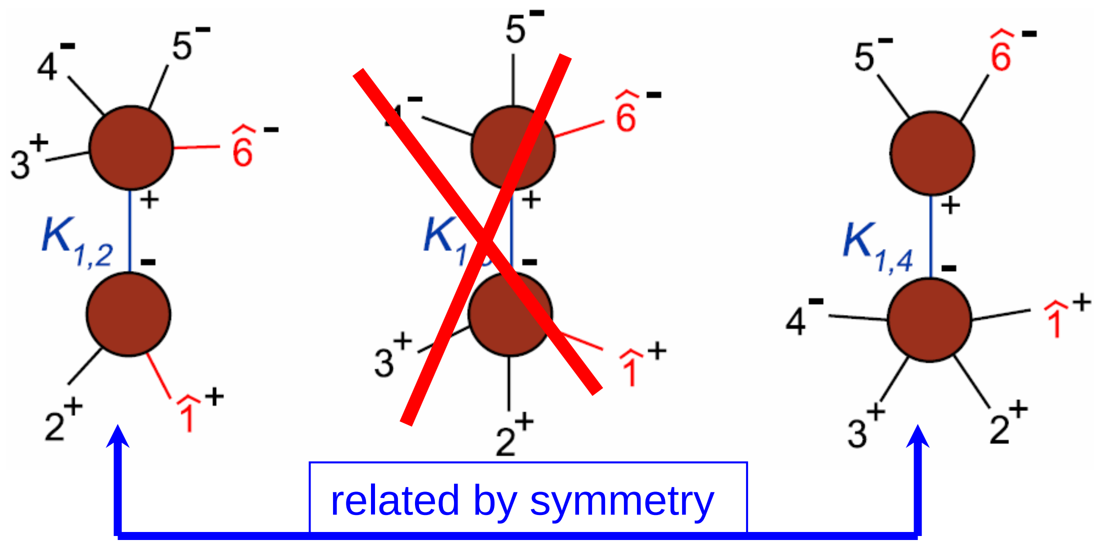
A 6-gluon example

220 Feynman diagrams for *gggggg*

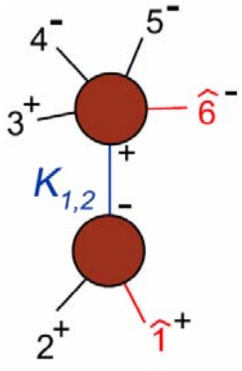
Helicity + color + MHV results + symmetries

$\Rightarrow$ only $\boxed{A_6(1^+,2^+,3^+,4^-,5^-,6^-)}$, $A_6(1^+,2^+,3^-,4^+,5^-,6^-)$

3 BCF diagrams $\quad\rightarrow$ 2 $\quad\rightarrow$ 1



The one $A_6(1^+,2^+,3^+,4^-,5^-,6^-)$ diagram



$$= -\frac{i}{s_{12}} \frac{[\hat{1}\,2]^3}{[2\,\hat{K}][\hat{K}\,\hat{1}]} \frac{[\hat{K}\,3]^3}{[3\,4][4\,5][5\,\hat{6}][\hat{6}\,\hat{K}]}$$

$$= -\frac{i}{s_{12}} \frac{[1\,2]^3}{([2\,\hat{K}]\langle\hat{K}\,6\rangle)(\langle 6\,\hat{K}\rangle[\hat{K}\,1])} \frac{(\langle 6\,\hat{K}\rangle[\hat{K}\,3])^3}{[3\,4][4\,5][5\,\hat{6}]([\hat{6}\,\hat{K}]\langle\hat{K}\,6\rangle)}$$

$$= i\,\frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\langle 1\,2\rangle[3\,4][4\,5]s_{612}\langle 2^-|(6+1)|5^-\rangle}$$

$$\begin{aligned}
\langle 6\,\hat{K}\rangle[\hat{K}\,a] &= \langle 6\,1\rangle\,[1\,a] + \langle 6\,2\rangle\,[2\,a] \\
&= \langle 6^-|(1+2)|a^-\rangle
\end{aligned}$$

$$[5\,\hat{6}] = [5\,6] + \frac{s_{12}[5\,1]}{\langle 6\,2\rangle[2\,1]} = \frac{\langle 5^+|(6+1)|2^+\rangle}{\langle 6\,2\rangle}$$

$$[\hat{6}\,\hat{K}]\langle\hat{K}\,6\rangle = \langle 6^+|(1+2)|6^+\rangle + s_{12} = s_{612}$$

Simple final form

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,s_{612}\langle 2^-|(6+1)|5^-\rangle} + \frac{\langle 4^-|(5+6)|1^-\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,[6\,1]\,s_{561}\langle 2^-|(6+1)|5^-\rangle}$$

Simpler than form found in 1980s — Mangano, Parke, Xu (1988)
despite (because of?) spurious singularities $\langle 2^-|(6+1)|5^-\rangle$

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{([1\,2]\,\langle 4\,5\rangle\,\langle 6^-|(1+2)|3^-\rangle)^2}{s_{61}s_{12}s_{34}s_{45}s_{612}} + \frac{([2\,3]\,\langle 5\,6\rangle\,\langle 4^-|(2+3)|1^-\rangle)^2}{s_{23}s_{34}s_{56}s_{61}s_{561}} + \frac{s_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6^-|(1+2)|3^-\rangle\langle 4^-|(2+3)|1^-\rangle}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}}$$

Relative simplicity even more striking for *n>6*

Bern, Del Duca, LD, Kosower (2004)

On-shell recursion at tree-level

Rules even more efficient, and easily extendable than MHV rules:

- massless quarks

Luo, Wen, hep-th/0501121, 0502009

- **massive** scalars

Badger, Glover, Khoze, Svrcek, hep-th/0504159;
Forde, Kosower, hep-th/0507292

- **massive vector bosons** and **fermions**

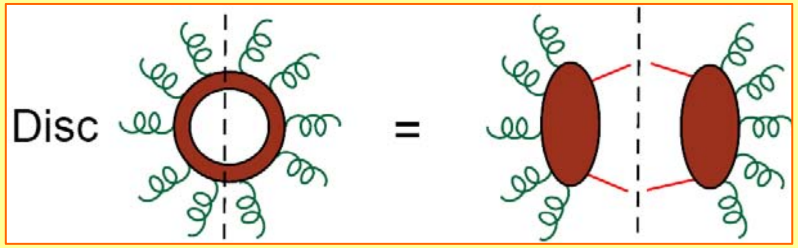
Badger, Glover, Khoze,
hep-th/0507161

Unitarity

- Unitarity is an efficient method for determining imaginary parts of loop amplitudes:

$$S = 1 + iA$$
$$S^\dagger S = 1 \quad \Rightarrow \quad 1 = (1 - iA^\dagger)(1 + iA)$$
$$\Rightarrow \quad -i(A - A^\dagger) = 2\,\mathrm{Im}\,A = \mathrm{Disc}\,A = A^\dagger A$$

- Efficient because it recycles trees into loops

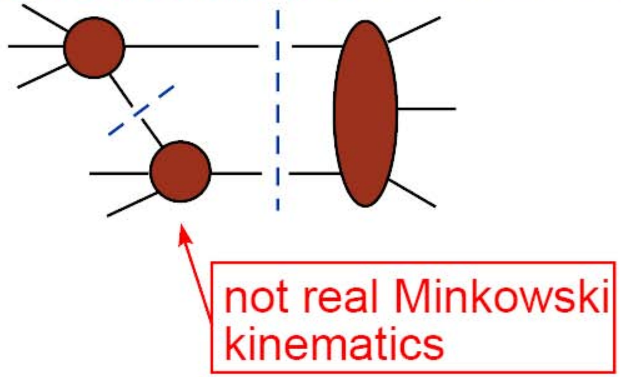


- Only thing missing: rational functions
- Can get these using on-shell recursion relations

Generalized unitarity

Eden, Landshoff, Olive, Polkinghorne (1966); Bern, LD, Kosower, hep-ph/9708239; Britto, Cachazo, Feng, hep-th/0412103

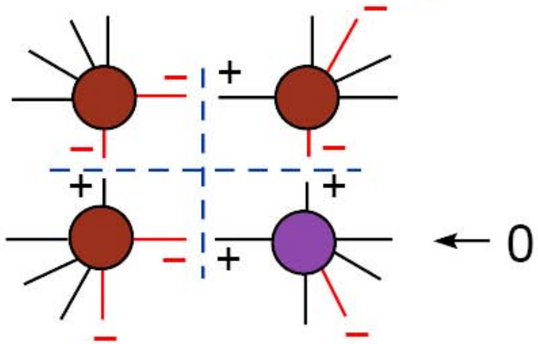
- Triangle and box integrals have 3 or 4 propagators "on shell". Can extract from more restrictive cut kinematics, such as:

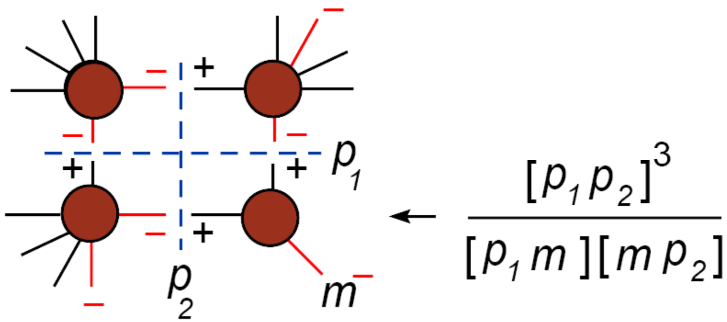


- Get a product of 3 or 4 simpler tree amplitudes, compared with the ordinary cut.

Generalized unitarity (cont.)

- For example, use quadruple cut to show all 4-mass box integrals vanish in all NMHV amplitudes. Bern, Del Duca, LD, Kosower, hep-th/0410224
  (Have $3+4=7$ negative helicities; need $2\times 4=8$.)





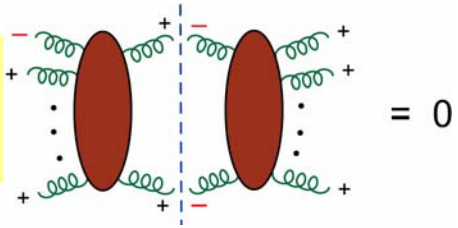
- 3-mass boxes do not vanish, because 3-point "amplitude" can be $(++-)$ (in complex kinematics).
- Computation of $c^{3\mathrm{m}}$ from quadruple cut can be done algebraically because all 4 components of loop momentum are frozen by the 4 on-shell constraints Britto, Cachazo, Feng, hep-th/0412103

On-shell recursion at one loop

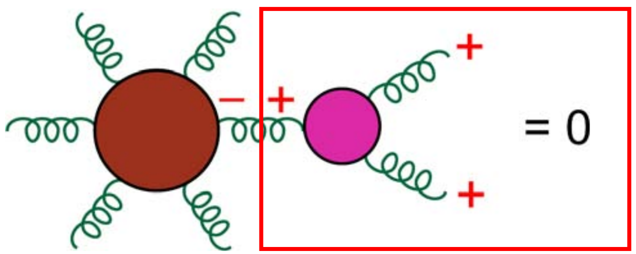
Bern, LD, Kosower, hep-th/0501240, hep-ph/0505055, hep-ph/0507005

- **Same techniques** can be used to compute one-loop amplitudes
  -- which are much harder to obtain by other methods than are trees.

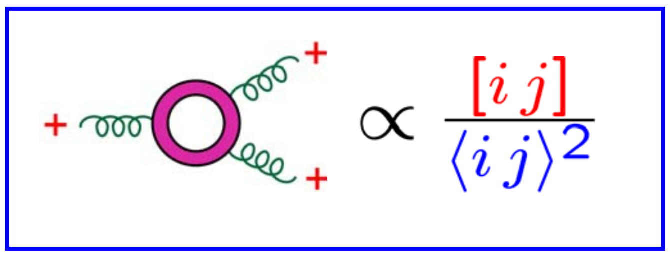
- First consider special tree-like one-loop amplitudes with no cuts, only poles: $A_n^{1-\text{loop}}(1^\pm, 2^+, 3^+, \ldots, n^+)$



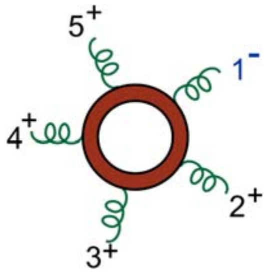
- **New features** arise compared with tree case due to different collinear behavior of loop amplitudes:



but



A one-loop pole analysis

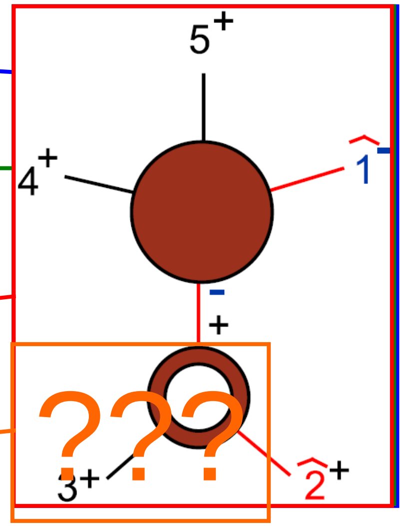


$$= -\frac{[2\,5]^3}{[5\,1][1\,2]\langle 3\,4\rangle^2} + \frac{\langle 1\,4\rangle^3[4\,5]\langle 3\,5\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle^2\langle 4\,5\rangle^2} + \frac{\langle 1\,3\rangle^3[2\,3]\langle 2\,4\rangle}{\langle 2\,3\rangle^2\langle 3\,4\rangle^2\langle 4\,5\rangle\langle 5\,1\rangle}$$

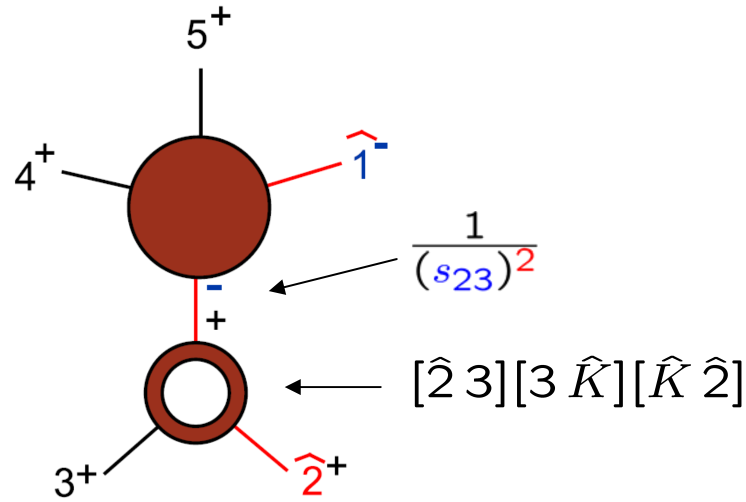
Bern, LD, Kosower, hep-ph/9302280

under shift $\hat{\tilde\lambda}_1 = \tilde\lambda_1 - z\tilde\lambda_2 \qquad \hat\lambda_2 = \lambda_2 + z\lambda_1$ plus partial fraction

$$\Rightarrow \quad -\frac{[2\,5]^3}{([5\,1]-z[5\,2])[1\,2]\langle 3\,4\rangle^2}$$
$$+\frac{\langle 1\,4\rangle^3[4\,5]\langle 3\,5\rangle}{\langle 1\,2\rangle(\langle 2\,3\rangle + z\langle 1\,3\rangle)\langle 3\,4\rangle^2\langle 4\,5\rangle^2}$$
$$-\frac{\langle 1\,3\rangle^2[2\,3]\langle 1\,2\rangle\langle 3\,4\rangle}{(\langle 2\,3\rangle + z\langle 1\,3\rangle)^2\langle 3\,4\rangle^2\langle 4\,5\rangle\langle 5\,1\rangle}$$
$$-\frac{\langle 1\,3\rangle^2[2\,3]\langle 1\,4\rangle}{(\langle 2\,3\rangle + z\langle 1\,3\rangle)\langle 3\,4\rangle^2\langle 4\,5\rangle\langle 5\,1\rangle}$$



Underneath the double pole



Missing diagram should be related, but suppressed by factor of s_{23}

Don't know collinear behavior at this level, must guess the correct suppression factor:

$$s_{23}\,\mathcal{S}(a,\hat{K}^+,b)\,\mathcal{S}(c,(-\hat{K})^-,d)$$

in terms of universal eikonal factors for soft gluon emission

$$\mathcal{S}(a,s^+,b) = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}$$

$$\mathcal{S}(a,s^-,b) = -\frac{[a\,b]}{[a\,s][s\,b]}$$

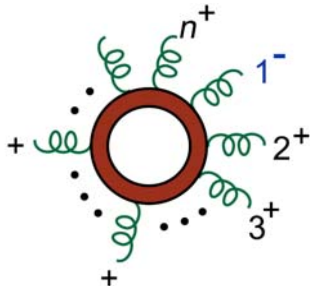
Here, multiplying 3rd diagram by

$$s_{23}\,\mathcal{S}(\hat{1},\hat{K}^+,4)\,\mathcal{S}(3,(-\hat{K})^-,\hat{2})$$

gives the correct missing term!

A one-loop all-*n* recursion relation

Same suppression factor works in the case of *n* external legs!



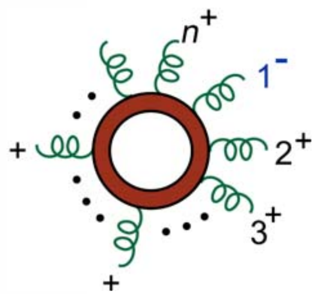
$$
\begin{aligned}
&A_n^{(1)}(1^-,2^+,\ldots,n^+) \\
&\quad = A_{n-1}^{(1)}(4^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{23}^+)\,\frac{i}{K_{23}^2}\,A_3^{(0)}(\hat{2}^+,3^+,-\hat{K}_{23}^-) \\
&\quad + \sum_{j=4}^{n-1} A_{n-j+2}^{(0)}((j+1)^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{2\ldots j}^-)\,\frac{i}{K_{2\ldots j}^2}\,A_j^{(1)}(\hat{2}^+,3^+,\ldots,j^+,-\hat{K}_{2\ldots j}^+) \\
&\quad + A_{n-1}^{(0)}(4^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{23}^-)\,\frac{i}{(K_{23}^2)^2}\,V_3^{(1)}(\hat{2}^+,3^+,-\hat{K}_{23}^+) \\
&\qquad\qquad \times \Big(1 + K_{23}^2\,\mathcal{S}^{(0)}(\hat{1},\hat{K}_{23}^+,4)\,\mathcal{S}^{(0)}(3,-\hat{K}_{23}^-,\hat{2})\Big)
\end{aligned}
$$

Know it works because results agree with Mahlon, hep-ph/9312276, though much shorter formulae are obtained from this relation

Solution to recursion relation

hep-ph/0505055

$$A_n^{(1)}(1^-,2^+,3^+,\ldots,n^+) = \frac{i}{3}\frac{T_1+T_2}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,,$$



where

$$T_1 = \sum_{l=2}^{n-1}\frac{\langle 1\,l\rangle\langle 1\,(l+1)\rangle\langle 1^-|\,\slashed{K}_{l,l+1}\slashed{K}_{(l+1)\cdots n}\,|1^+\rangle}{\langle l\,(l+1)\rangle}\,,$$

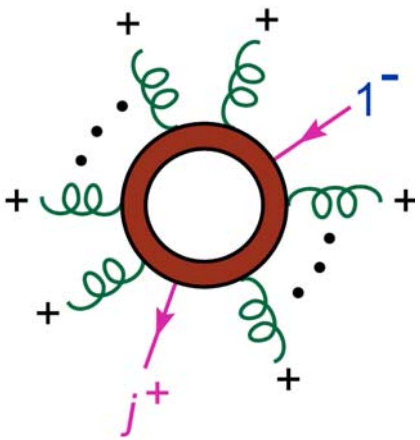
$$T_2 = \sum_{l=3}^{n-2}\sum_{p=l+1}^{n-1}\frac{\langle (l-1)\,l\rangle}{\langle 1^-|\,\slashed{K}_{(p+1)\cdots n}\slashed{K}_{l\cdots p}\,|(l-1)^+\rangle\,\langle 1^-|\,\slashed{K}_{(p+1)\cdots n}\slashed{K}_{l\cdots p}\,|l^+\rangle}$$
$$\times\frac{\langle p\,(p+1)\rangle}{\langle 1^-|\,\slashed{K}_{2\cdots(l-1)}\slashed{K}_{l\cdots p}\,|p^+\rangle\,\langle 1^-|\,\slashed{K}_{2\cdots(l-1)}\slashed{K}_{l\cdots p}\,|(p+1)^+\rangle}$$
$$\times\left\langle 1^-\right|\slashed{K}_{l\cdots p}\slashed{K}_{(p+1)\cdots n}\left|1^+\right\rangle^3$$
$$\times\frac{\langle 1^-|\,\slashed{K}_{2\cdots(l-1)}[\mathcal{F}(l,p)]^2\slashed{K}_{(p+1)\cdots n}\,|1^+\rangle}{s_{l\cdots p}}\,.$$

$$\mathcal{F}(l,p) = \sum_{i=l}^{p-1}\sum_{m=i+1}^{p}\slashed{k}_i\slashed{k}_m$$

External fermions too

hep-ph/0505055

Can similarly write down recursion relations
for the finite, cut-free amplitudes with 2 external fermions:



and the solutions are just as compact

Loop amplitudes with cuts

- Recently extended same recursive technique (combined with unitarity) to loop amplitudes with cuts (hep-ph/0507005)
- Here rational-function terms contain "spurious singularities", e.g. $\sim \frac{\ln(r)+1-r}{(1-r)^2}, \quad r = s_2/s_1$
- accounting for them properly yields simple "overlap diagrams" in addition to recursive diagrams
- No loop integrals required to bootstrap the rational functions from the cuts and lower-point amplitudes
- Tested method on 5-point amplitudes, used it to compute $A_6(1^-,2^-,3^+,4^+,5^+,6^+)$, $A_7(1^-,2^-,3^+,4^+,5^+,6^+,7^+)$

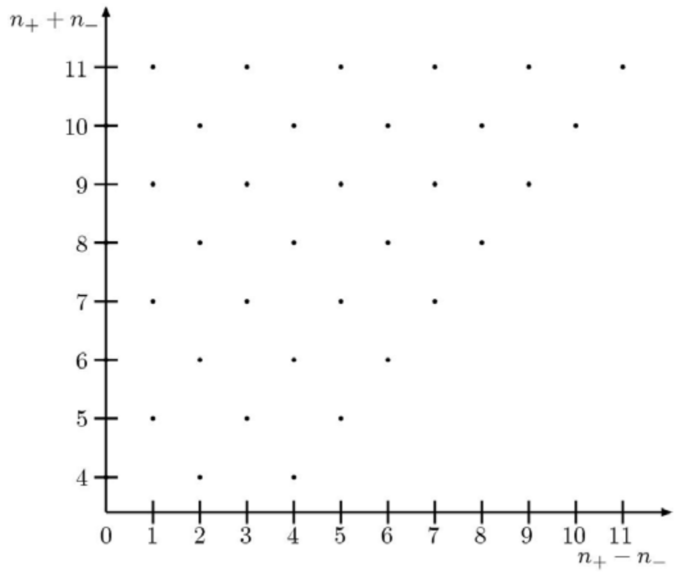
Conclusions

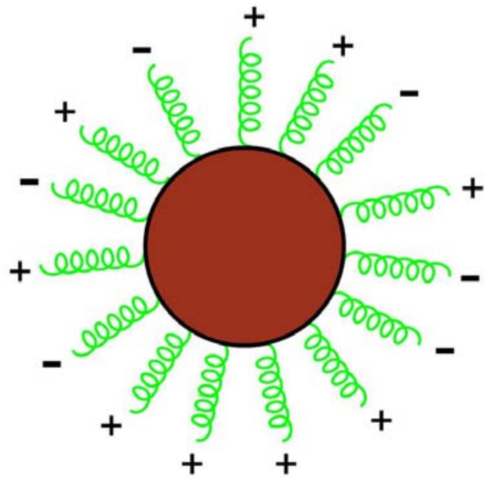
- MHV rules, and especially on-shell recursion relations a very efficient way to compute multi-leg tree amplitudes in gauge theory
- Development a spinoff from twistor string theory
- Also much progress on loops in supersymmetric theories using (generalized) unitarity
- Quite recently, new loop amplitudes in QCD, needed for colliders, are beginning to fall to twistor-inspired recursive approaches
- Prospects look very good for attacking a wide range of multi-parton processes in this way

Some other reviews

- V.V. Khoze, hep-th/0408233
- F. Cachazo, P. Svrcek, hep-th/0504194 (Trieste lectures)
- N. Glover, talk at SUSY2005
  http://susy-2005.dur.ac.uk/PLENARY/WED/GLOVERsusy.pdf

March of the *n*-gluon helicity amplitudes





n_+ positive-helicity gluons

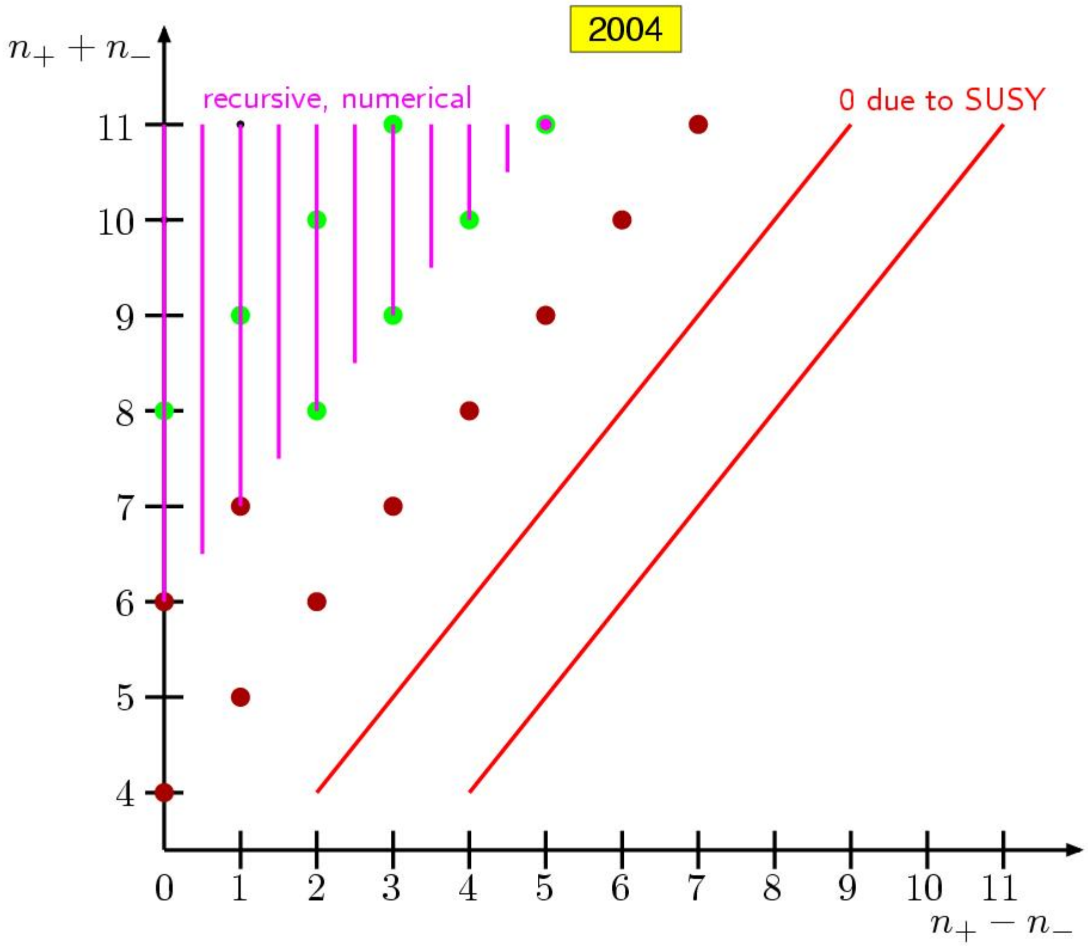
n_- negative-helicity gluons

$n = n_+ + n_- \geq 4$

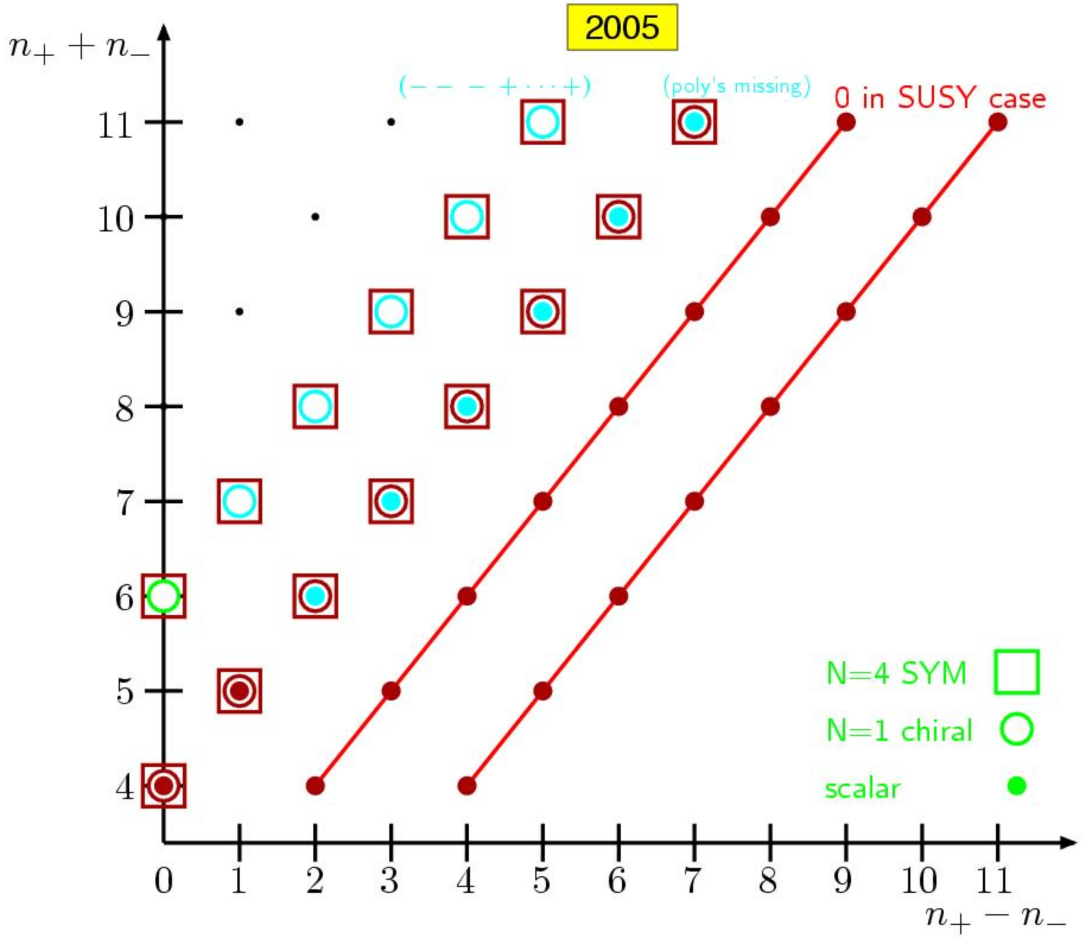
$n_+ \geq n_-$ by parity

At 1-loop, QCD decomposable into
N=4 SYM, N=1 chiral, scalar contributions

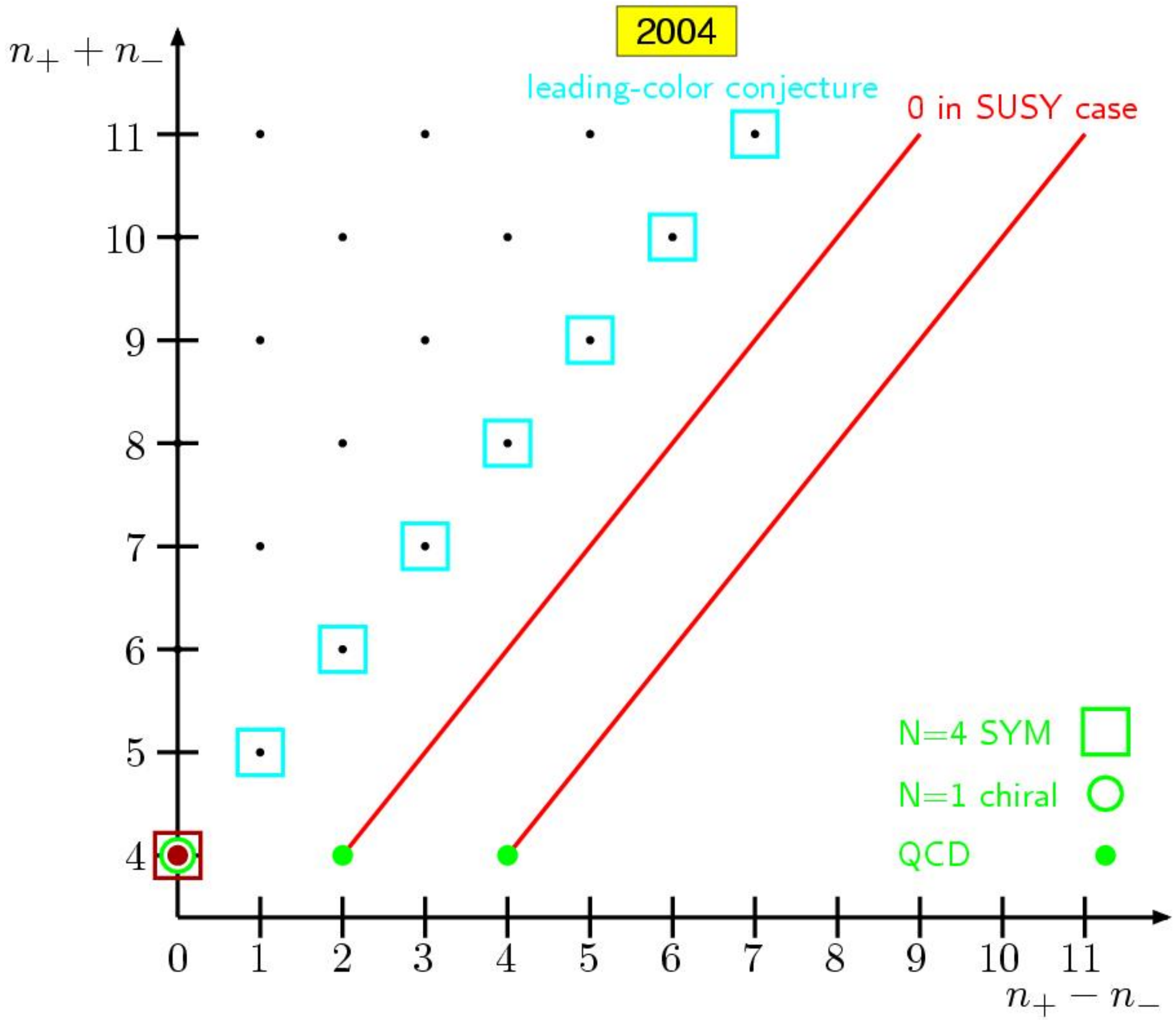
March of the tree amplitudes



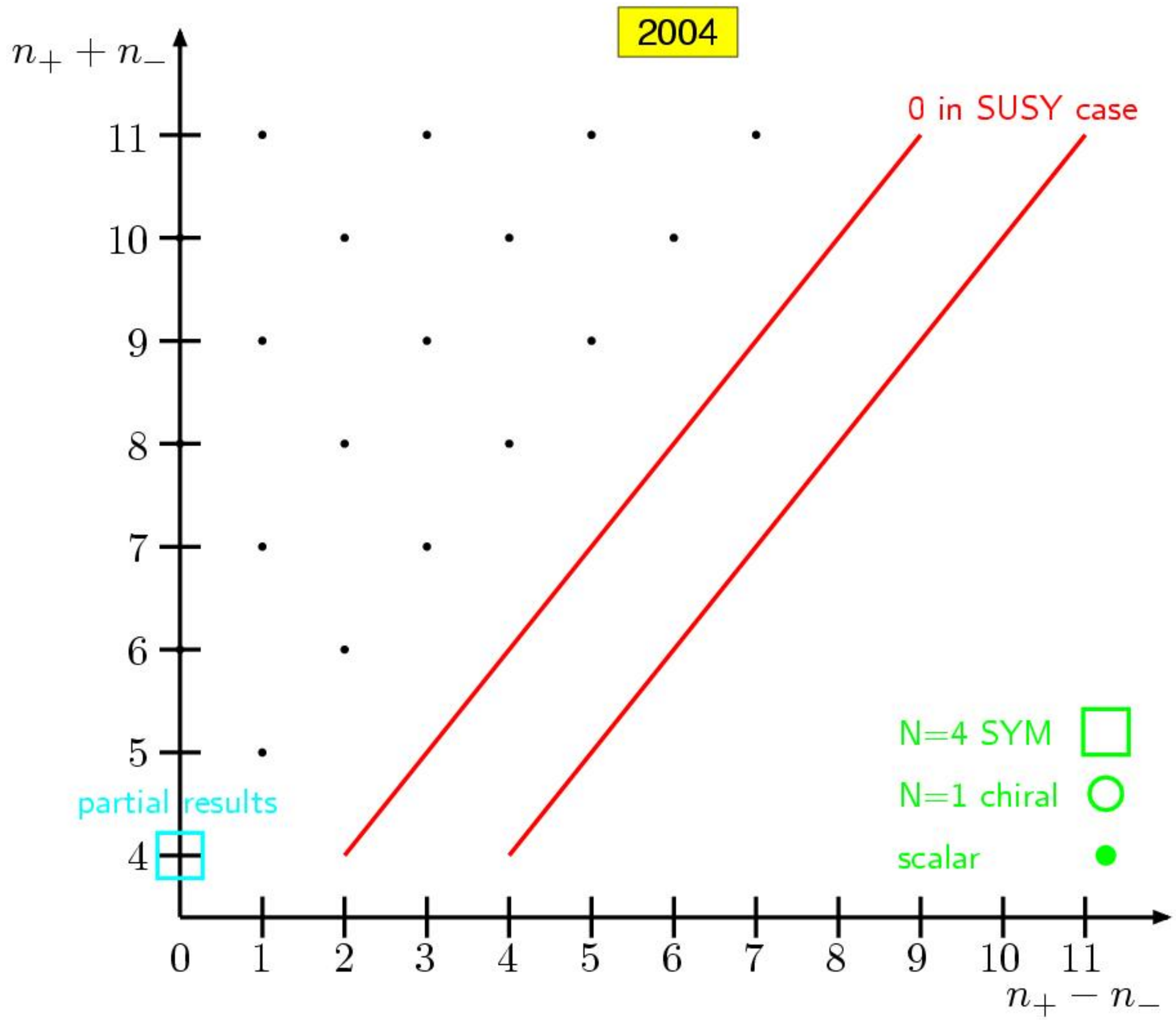
March of the 1-loop amplitudes



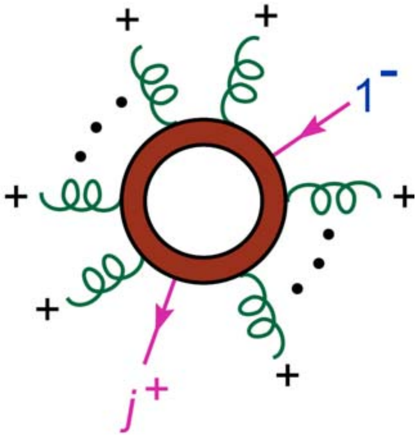
March of the 2-loop amplitudes



March of the 3-loop amplitudes



Fermionic solutions



$$A_n^{L-s}(1_f^-, 2^+, \ldots, j_f^+, \ldots, n^+) = \frac{i}{2} \frac{\langle 1\, j\rangle\, \Sigma_{l=3}^{n-1} \langle 1^- |\, \not{K}_{2\cdots l} \not{k}_l \,|1^+\rangle}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$$

and

$$A_n^s(j_f^+) = \frac{i}{3} \frac{S_1 + S_2}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle n\, 1\rangle},$$

where

$$S_1 = \sum_{l=j+1}^{n-1} \frac{\langle j\, l\rangle \langle 1\, (l+1)\rangle \langle 1^- |\, \not{K}_{l,l+1} \not{K}_{(l+1)\cdots n} \,|1^+\rangle}{\langle l\, (l+1)\rangle},$$

$$\begin{aligned} S_2 = \sum_{l=j+1}^{n-2} \sum_{p=l+1}^{n-1} & \frac{\langle (l-1)\, l\rangle}{\langle 1^- |\, \not{K}_{(p+1)\cdots n} \not{K}_{l\cdots p} \,|(l-1)^+\rangle \langle 1^- |\, \not{K}_{(p+1)\cdots n} \not{K}_{l\cdots p} \,|l^+\rangle} \\ & \times \frac{\langle p\, (p+1)\rangle}{\langle 1^- |\, \not{K}_{2\cdots(l-1)} \not{K}_{l\cdots p} \,|p^+\rangle \langle 1^- |\, \not{K}_{2\cdots(l-1)} \not{K}_{l\cdots p} \,|(p+1)^+\rangle} \\ & \times \langle 1^- \big|\, \not{K}_{l\cdots p} \not{K}_{(p+1)\cdots n} \,\big|1^+\rangle^2 \langle j^- \big|\, \not{K}_{l\cdots p} \not{K}_{(p+1)\cdots n} \,\big|1^+\rangle \\ & \times \frac{\langle 1^- |\, \not{K}_{2\cdots(l-1)} [\mathcal{F}(l,p)]^2 \not{K}_{(p+1)\cdots n} \,|1^+\rangle}{s_{l\cdots p}}, \end{aligned}$$

$$\mathcal{F}(l,p) = \sum_{i=l}^{p-1} \sum_{m=i+1}^{p} \not{k}_i \not{k}_m$$